Differential Graphical Games for Constrained Autonomous Vehicles Based on Viability Theory

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Abstract—This paper proposes an optimal distributed control protocol for multi-vehicle systems with an unknown switching communication graph. The optimal distributed control problem is formulated to differential graphical games, and the Pareto optimum to multi-player games is sought based on the viability theory and reinforcement learning techniques. The viability theory characterizes the controllability of a wide range of constrained nonlinear systems; and the viability kernel and the capture basin are the pillars of the viability theory. The capture basin is the set of all initial states, in which there exist control strategies that enable the states to reach the target in finite time, while remaining inside a set before reaching the target. In this regard, the feasible learning region is characterized for the reinforcement learner. In addition, the approximation of the capture basin provides the learner with prior knowledge. Unlike the existing works that employ the viability theory to solve control problems with only one agent and differential games with only two players, the viability theory, in this paper, is utilized to solve multi-agent control problems and multi-player differential games. The distributed control law is composed of two parts: the approximation of the capture basin and reinforcement learning, which are computed off-line and on-line respectively. The convergence properties of the parameters' estimation errors in reinforcement learning are proved, and the convergence of the control policy to the Pareto optimum of the differential graphical game is discussed. The guaranteed approximation results of the capture basin are provided, and the simulation results of the differential graphical game are provided for multi-vehicle systems with the proposed distributed control policy.

Index Terms—Cooperative control, multi-agent system, capture basin, differential graphical game, reinforcement learning, cooperative learning, viability theory.

I. INTRODUCTION

DISTRIBUTED control techniques and differential graphical games for constrained autonomous robotic systems have drawn considerable attention in the research field of applied engineering for years. The distributed control problem of large complex network systems is always one of the major barriers in industrial applications. Various studies on distributed control have pointed to air traffic control, sensor networks, satellite formation control, multi-robot docking, distributed learning, and so on [1], [2]. The aim of distributed control is to design a control policy for each connected agent such that the evolutions of all the agents will finally contribute to the desired behaviour of the network system

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[3]. Representative works for multi-vehicle systems in recent literature can be found in [4]–[7]. Among them, fading, switching and delay phenomenon of wireless channels were concerned in [4]–[6], and optimal distributed control problems were formulated to differential graphical games in [8].

Prior to 1944, in [9], game theory was proposed to build a framework for economic behaviours. Then, the interaction behaviours among players and their corresponding strategies were formulated as a game [10]. The game theory was extended to large-scale network games by introducing graphical models for multi-agent systems, which was referred to as graphical games in [11]. Then, the differential graphical game theory was linked to cooperative control in [8]; this study constructed the representations of local interactions among agents in a large network system based on the graphical game theory. By adopting the principle of dynamic programming, for nonlinear network systems, most papers investigated distributed Nash equilibrium solutions to the differential graphical games [12]–[15]. However, a Nash equilibrium may lead to a worse global performance, as it cannot guarantee the Pareto optimum. To our best knowledge, no comprehensive study has hitherto investigated the Pareto optimum to the differential graphical games for nonlinear systems with bounded control.

Two-player differential games for constrained nonlinear systems have been widely studied through the viability theory, as the viability theory characterizes the numerical solution to the game [16]. Based on the viability theory, qualitative and quantitative problems of both control and differential games ‘have been solved [17]. With regard to qualitative problems, the capture basin characterizes a set of initial states for constrained nonlinear systems with proper control such that at least one of their trajectories can hit the target in finite time; the capture basin also characterizes the victory domain of the winner in differential games [17]. With regard to quantitative problems, the approximations of the capture basins and the viability kernels determine the value functions in optimal control of constrained nonlinear systems; they also determine the viscosity solution of the Hamilton-Jacobi-Isaacs equation (HJIE) regarding the differential games [17]. The studies on two-player differential games based on viability theory can be found in the following literature [18]–[23]. Among them, in [18], [19], the intertemporal optimality theorem was associated with the viability theory in terms of differential games, which pioneered the fundamental viability approaches in solving the problems of differential games. Based on [18], [19], the barrier phenomenon of the victory domain was proved in [20], which provided a solution of the HJIE; in [21], the viability properties of differential games was investigated by seeking a viscosity solution of the HJIE; in [22], the optimal time function of the games for constrained nonlinear systems was characterized without any assumptions about controllability. The concepts of the conditional viability kernel and value function for ‘qualitative games’ were later developed in [23]. In the latest literature, based on the

viability theory and differential game theory, studies such as [24]–[27] were limited to two-player differential games modelled by constrained nonlinear differential equations. So far, in the field of the viability theory, no study has yet been carried out that adopts viability approaches to solve differential graphical games.

To fill this gap, we formulate multi-agent control problems to differential graphical games, and propose a novel distributed control policy based on the viability theory. Our algorithm is composed of off-line backward set propagation and on-line cooperative learning. We approximate the capture basin in a guaranteed way by using backward propagation on the target set, which improves the propagation approach proposed in [28], as the value function of the differential games can be approximated during propagation. Moreover, our propagation approach can be applied to the set with arbitrary shape. By combining our new proposed propagation methods with the reinforcement learning technique, our contributions can be mainly categorized into the following aspects: by using just local information of each vehicle, the proposed distributed control strategy converges the Pareto optimum to the differential graphical games for nonlinear system with bounded control despite the unknown switching communication graph; the feasible learning region of states, in which solutions exist, is characterized for the priori knowledge of reinforcement learner; the dimensionality issue is overcome when employing numerical tools through the viability theory; the computation redundancy of the on-line cooperative reinforcement learning is relatively low when compared with the distributed learning algorithms proposed in recent literature, and the computational complexity is not subject to the number of agents in the network system.

The remainder of this paper is organized as follows. Section II outlines the preliminary definitions and notations of the viability theory, differential games and multi-agent systems. Section III formulates the distributed control problem to the differential graphical game associated with the viability theory. Section IV discusses our proposed distributed control policy and provides convergence analysis; Section V provides the results of the approximated capture basin and the simulation results of the game. Section VI concludes this paper and discusses our future work.

II. PRELIMINARIES

A. Viability Theory

Consider a smooth nonlinear dynamic system F with bounded control:

$$\dot{x}(t) = f(x(t), u(x)), \quad u(x) \in [U], \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector; $u(x) \in \mathbb{R}^m$ is the control input; $[\cdot]$ stands for interval vector [29]; $[U] = [[U_1, \bar{U}_1], \dots, [U_m, \bar{U}_m]]^T$ is the control constraint. Denote $\varphi(\tau, x_0, u)$ as the state of the system F at time τ starting from $x_0 = x(0)$ with control input $u(x)$, and the corresponding trajectory can be written as [30], [31]:

$$\varphi([0, \tau]; x_0, u) = \bigcup_{t \in [0, \tau]} \varphi(t; x_0, u). \quad (2)$$

The communication zone is denoted as a closed set $K \subseteq \mathbb{R}^n$ and the target is denoted as a closed set $C \subseteq \mathbb{R}^n$, where $C \subset K$. The definitions of the viable set and the capture basin are indicated as follows:

Definition 1: As specified in [32], the viable set K is the subset of evolutions that satisfy

$$K = \{x(t) \mid \forall t \geq 0, x(t) \in K\}. \quad (3)$$

Definition 2: As specified in [31], [33], the capture basin $\text{Capt}_F(K, C)$ is the set of all initial states in K such that at least one of their evolutions reaches the target C in finite time τ while remaining viable in K for $\forall t \in [0, \tau]$.

B. Differential Games

Denote u_i as the strategy of the i th player and N as the number of players, where $u = [u_1^T, u_2^T, \dots, u_N^T]^T$. The following definitions distinguish the Nash equilibrium and the Pareto optimum:

Definition 3: As specified in [34], for multi-player games, for $\forall i \in \{1, \dots, N\}$, the Nash equilibrium $u^* = [u_1^{*T}, u_2^{*T}, \dots, u_N^{*T}]^T$ always fulfils $J_{\mathcal{N}_i}(x(t), u^*) \leq J_{\mathcal{N}_i}(x(t), u_i^*, \dots, u_i, \dots, u_N^*)$. In fact, Nash equilibrium may not guarantee a Pareto optimality of the game.

Definition 4: As specified in [35], the strategy $u^* = [u_1^{*T}, u_2^{*T}, \dots, u_N^{*T}]^T$ is said to be Pareto optimal for control strategies u if the following relation is fulfilled $\forall i \in \{1, \dots, N\}$, $J_{\mathcal{N}_i}(x(t), u^*) \leq J_{\mathcal{N}_i}(x(t), u)$ and $\exists i \in \{1, \dots, N\}$, $J_{\mathcal{N}_i}(x(t), u^*) < J_{\mathcal{N}_i}(x(t), u)$.

C. Multi-Agent Systems

Suppose the total number of nodes is N , the communication topology graph is formed as $\mathcal{G}_N = (\mathcal{V}_N, \mathcal{E}_N)$ where $\mathcal{V}_N = \{v_1, \dots, v_N\}$ represents nodes, and $\mathcal{E}_N \subseteq \mathcal{V}_N \times \mathcal{V}_N$ represents edges. Edge set $(v_i, v_j) \in \mathcal{E}_N$ denotes information sent to the i th node v_i from the j th node v_j , and correspondingly the i th node v_i has an in-degree edge; meanwhile, the j th node v_j has an out-degree edge. All the in-degree edges of node v_i are represented as $v_{\mathcal{N}_i} = \{v_j\} : j \in \mathcal{N}_i$, where $\mathcal{N}_i = \{j \mid (v_i, v_j) \in \mathcal{E}_N\}$. In this paper, except for especial denotation, we always denote $(\cdot)_{\mathcal{N}_i} = \{(\cdot)_{(i,j)}\} : j \in \mathcal{N}_i$, where $(\cdot)_{(i,j)}$ is shared by the i th and the j th agents. A sequence of edges is said to be a directed path from v_i to v_j , if they are connected in a graph through the nodes from v_i to v_j . An undirected graph is connected if it contains a directed path that connects all nodes. Adjacency matrix $A = \{a_{ij}\} \in \mathbb{R}^{N \times N}$ maps the whole communication network by weighing each edge on their corresponding entry a_{ij} [36]. Normally, if $(v_j, v_i) \in \mathcal{E}_N$, then we have $a_{ij} > 0$, and otherwise $a_{ij} = 0$, and all diagonal entries of adjacency matrix are equal to 0 i.e. $a_{ii} = 0$ [36].

III. PROBLEM FORMULATION

In this paper, each node represents an autonomous vehicle and we number the target vehicle by i and suppose there are N vehicles, denoting $x = [x_1^T, x_2^T, \dots, x_N^T]^T \in \mathbb{R}^{Nn}$ as the global state, $\theta = [\theta_1^T, \theta_2^T, \dots, \theta_N^T]^T \in \mathbb{R}^N$ as the global orientation where $\theta_i \in x_i$, and $u = [u_1^T, u_2^T, \dots, u_N^T]^T \in \mathbb{R}^{Nm}$ as the global control. The following assumptions always hold throughout this paper:

Assumption 1: The dynamics of all the autonomous vehicles are assumed to be the same.

Assumption 2: The communication topology is considered to be an unknown switching connected graph with $a_{ij} = a_{ji} = 1$ if $(v_j, v_i) \in \mathcal{E}_N$, otherwise $a_{ij} = a_{ji} = 0$. Moreover, $\mathcal{A} = \{\alpha_{ij}\}$ is denoted as the strictly upper triangular form of A , and $|\mathcal{A}| = \sum_{i,j=1}^N \alpha_{ij}$.

The network dynamic system for multiple vehicles is decomposed into subsystems in terms of relative dynamics. The relative state between v_i and v_j is denoted as e_{ij} regarding x_i as the origin of the local coordinate frame of v_i . Denoting $\mathcal{R}(\cdot) \in \mathbb{R}^{n \times n}$ as the clockwise rotation matrix and $\theta_i \in \mathbb{R}$ as the global orientation of v_i ,

then $e_{ij} = \mathcal{R}(\theta_{ij})(x_j - x_i) \in \mathbb{R}^n$ and $e_{ji} = -\mathcal{R}(\theta_{ij})e_{ij}$, where $\theta_{ij} \in e_{ij}$. For any connected vehicles in pairs, we have

$$\dot{e}_{ij} = f_e(e_{ij}) + g_e(e_{ij})u_{ij}, \quad (4a)$$

$$\dot{e}_{ji} = f_e(e_{ji}) + g_e(e_{ji})u_{ji}, \quad (4b)$$

$$e_{ij}, e_{ji} \in K, \quad u_{ij} \in [U_2],$$

where $u_{ij} = [u_i^T, u_j^T]^T \in \mathbb{R}^{2m}$; $U_n = \mathbf{1}_n[U]$; K is a closed set representing the communication zone of v_i and v_j . The above dynamic systems (4a) and (4b) are denoted as F^+ and F^- , respectively. Then, the following assumption holds:

Assumption 3: The functions $f_e(e_{ij})$ and $g_e(e_{ij})$ are uniformly continuous and differentiable. Moreover, both of them satisfy

$$\|f_e(e_{ij}(t))\| \leq C_f(1 + \|e_{ij}(t)\|), \quad \exists C_f \in \mathbb{R}, \quad \forall t \in [0, T], \quad (5a)$$

$$\|g_e(e_{ij}(t))\| \leq C_g(1 + \|e_{ij}(t)\|), \quad \exists C_g \in \mathbb{R}, \quad \forall t \in [0, T], \quad (5b)$$

Notably, either e_{ij} or e_{ji} can be transferred into the local coordinate frame of the other. For a pair of nodes (v_i, v_j) connecting by bilateral edges, we have

$$e_{(i,j)} = \begin{cases} e_{ij}, & \text{for } \theta_{ij} \geq 0, \\ -\mathcal{R}(\theta_{ij})e_{ij}, & \text{for } \theta_{ij} < 0, \end{cases} \quad (6a)$$

$$(6b)$$

such that $e_{(i,j)} = e_{(j,i)}$ always holds for both e_{ij} and e_{ji} regardless of whether the local coordinate frame is either on v_i or v_j . For higher dimension systems, when $\theta_i, \theta_j \in \mathbb{R}^3$, we transfer either e_{ij} or e_{ji} such that the sign of one of the elements in $\theta_{(i,j)}$ is always identical.

A. Differential Graphical Games

Let $g_e(e_{(i,j)}) = [g_e^\#(e_{(i,j)}), g_e^\flat(e_{(i,j)})]$ and $g_e^\pm(e_{(i,j)}) = [g_e^+(e_{(i,j)}), g_e^-(e_{(i,j)})]$, where

$$g_e^+(e_{(i,j)}) = \begin{cases} g_e^\#(e_{(i,j)}), & \text{for } \theta_{ij} \geq 0 \\ g_e^\flat(e_{(i,j)}), & \text{for } \theta_{ij} < 0, \end{cases} \quad (7a)$$

$$(7b)$$

$$g_e^-(e_{(i,j)}) = \begin{cases} g_e^\flat(e_{(i,j)}), & \text{for } \theta_{ij} \geq 0 \\ g_e^\#(e_{(i,j)}), & \text{for } \theta_{ij} < 0, \end{cases} \quad (7c)$$

$$(7d)$$

The error systems F^+ and F^- are unified to the error system F_e shown in (8)

$$\dot{e}_{(i,j)} = f_e(e_{(i,j)}) + g_e^\pm(e_{(i,j)})u_{ij}, \quad u_{ij} \in [U_2]. \quad (8)$$

Define a uniformly continuous and differentiable convex function $\psi_C(e_{(i,j)})$ which configures $C := \{e_{(i,j)} \in K \mid \psi_C(e_{(i,j)}) \leq 0\}$ as the target of system F_e , and the following assumption holds:

Assumption 4: Assume $e_{(i,j)}(T) \in \partial C$ and T is the terminal time which is determined by capture basin approximation, and C is a V-viable set satisfying [30]:

$$\begin{aligned} \forall e_{(i,j)} \in \partial C, \quad \exists u_{ij} \in [U_2], \\ \nabla \psi_C(e_{(i,j)})(f_e(e_{(i,j)}) + g_e^\pm(e_{(i,j)})u_{ij}) < 0. \end{aligned} \quad (9)$$

The corresponding cost function J_e is expressed as

$$J_e(e_{(i,j)}(\tau), u_{ij}) = \psi_C(e_{(i,j)}(T)) + \int_\tau^T L(e_{(i,j)}(t), u_{ij}(t)) dt, \quad (10a)$$

$$L(e_{(i,j)}(t), u_{ij}(t)) = Q(e_{(i,j)}(t)) + u_{ij}^T(t)R_2u_{ij}(t), \quad (10b)$$

where $Q(e_{(i,j)}(\tau)) : \mathbb{R}^n \rightarrow \mathbb{R}^+$ is a positive definite function; $R_2 = \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix}$ and R is a positive definite matrix. For the relative error system between v_i and v_j , the value function V_e is the viscosity solution of

$$H_e(e_{(i,j)}, \nabla V_e(e_{(i,j)})) = 0, \quad (11a)$$

$$V_e(e_{(i,j)}(T)) = \psi_C(e_{(i,j)}(T)), \quad (11b)$$

where

$$H_e(e_{(i,j)}, p) = \inf_{u_{ij} \in [U_2]} \{p(f_e(e_{(i,j)}) + g_e^\pm(e_{(i,j)})u_{ij}) + L(e_{(i,j)}, u_{ij})\}. \quad (12)$$

Express the global error system F_E as

$$\dot{e} = f_E(e) + g_E(e)u, \quad u \in [U_N], \quad (13a)$$

$$f_E(e) = [f_e^T(e_{(1,2)}), \dots, f_e^T(e_{(i,j)}), \dots, f_e^T(e_{(N-1,N)})]^T, \quad (13b)$$

$$g_E(e) = \text{diag}([g_e(e_{(1,2)}), \dots, g_e(e_{(i,j)}), \dots, g_e(e_{(N-1,N)})]), \quad (13c)$$

where the global error $e = [e_{(1,2)}^T, \dots, e_{(i,j)}^T, \dots, e_{(N-1,N)}^T]^T \in \mathbb{R}^{|\mathcal{A}|n}$ and its element $e_{(i,j)}$ is the relative error of v_i and v_j in connection, $f_E(e) \in \mathbb{R}^{|\mathcal{A}|n}$, $g_E(e) \in \mathbb{R}^{|\mathcal{A}|n \times |\mathcal{A}|m}$. The global cost function is formulated as

$$J(e(\tau), u) = \sum_{i,j=1}^N \alpha_{ij} J_e(e_{(i,j)}(\tau), u_{ij}), \quad (14)$$

and the value function V is the viscosity solution of

$$H(e, \nabla V(e)) = 0, \quad (15a)$$

$$V(e(T)) = \sum_{i,j=1}^N \alpha_{ij} \psi_C(e_{(i,j)}(T)). \quad (15b)$$

So far, the distributed control has been formulated as a differential game problem, and the formulation design from (5) to (8) aims to construct $H_{\mathcal{N}_i}$, such that the Pareto optimality can be achieved.

Theorem 1: For the game described in (13) and (14), there exists a Pareto solution $u_{\mathcal{N}_i}^*$ associated with the following decoupled Hamiltonian $H_{\mathcal{N}_i}$:

$$\begin{aligned} H_{\mathcal{N}_i}(e_{\mathcal{N}_i}, p_{\mathcal{N}_i}) = \inf_{u_i \in [U]} \left\{ \frac{1}{2} p_{\mathcal{N}_i} f_{e_{\mathcal{N}}}(e_{\mathcal{N}_i}) + p_{\mathcal{N}_i} g_{\mathcal{N}_i}^+(e_{\mathcal{N}_i})u_i \right. \\ \left. + |\mathcal{N}_i| u_i^T R u_i + \frac{1}{2} \sum_{j \in \mathcal{N}_i} Q(e_{(i,j)}) \right\}, \end{aligned} \quad (16)$$

which guarantees the global optimality of the differential graphical game, where

$$p_{\mathcal{N}_i} = [\dots, p_{(i,j)}, \dots] : j \in \mathcal{N}_i, \quad (17a)$$

$$g_{\mathcal{N}_i}^+(e_{\mathcal{N}_i}) = [\dots, g_e^+T(e_{(i,j)}), \dots]^T : j \in \mathcal{N}_i, \quad (17b)$$

$$f_{e_{\mathcal{N}}}(e_{\mathcal{N}_i}) = [\dots, f_e^T(e_{(i,j)}), \dots]^T : j \in \mathcal{N}_i. \quad (17c)$$

Proof: The Hamiltonian H in (15a) is formulated as shown in (18) at the top, where $p = [p_{(1,2)}, \dots, p_{(i,j)}, \dots, p_{(N-1,N)}] \in \mathbb{R}^{|\mathcal{A}|n}$ and

$$\mathcal{N}_i^\# = \{j \in \mathcal{N}_i \mid \theta_{ij} \geq 0\}, \quad (19a)$$

$$\mathcal{N}_i^\flat = \{j \in \mathcal{N}_i \mid \theta_{ij} < 0\}. \quad (19b)$$

The global Hamiltonian H is decomposed to the decoupled local

$$\begin{aligned}
H(e, p) &= \inf_{u \in [U_N]} \left\{ \frac{1}{2} \sum_{i=1}^N \left(\sum_{j \in \mathcal{N}_i^\#} p_{(i,j)} \left(g_e^\#(e_{(i,j)}) u_i + g_e^b(e_{(i,j)}) u_j \right) + \sum_{j \in \mathcal{N}_i^?} p_{(i,j)} \left(g_e^b(e_{(i,j)}) u_i + g_e^\#(e_{(i,j)}) u_j \right) \right. \right. \\
&\quad \left. \left. + \sum_{j \in \mathcal{N}_i} p_{(i,j)} f_e(e_{(i,j)}) + \sum_{j \in \mathcal{N}_i} L(e_{(i,j)}, u_{ij}) \right) \right\} \\
&= \frac{1}{2} \inf_{u \in [U_N]} \left\{ \sum_{i=1}^N \left(\sum_{j \in \mathcal{N}_i^\#} p_{(i,j)} g_e^\#(e_{(i,j)}) + \sum_{j \in \mathcal{N}_i^?} p_{(i,j)} g_e^b(e_{(i,j)}) \right) u_i + \sum_{j=1}^N \left(\sum_{i \in \mathcal{N}_j^\#} p_{(j,i)} g_e^\#(e_{(j,i)}) + \sum_{i \in \mathcal{N}_j^?} p_{(j,i)} g_e^b(e_{(j,i)}) \right) u_j \right. \\
&\quad \left. + \sum_{i=1}^N |\mathcal{N}_i| u_i^T R u_i + \sum_{j=1}^N \sum_{i \in \mathcal{N}_j} u_i^T R u_i + \sum_{i=1}^N \sum_{j \in \mathcal{N}_i} (p_{(i,j)} f_e(e_{(i,j)}) + Q(e_{(i,j)})) \right\} \\
&= \sum_{i=1}^N \inf_{u_i \in [U]} \left\{ \left(\sum_{j \in \mathcal{N}_i^\#} p_{(i,j)} g_e^\#(e_{(i,j)}) + \sum_{j \in \mathcal{N}_i^?} p_{(i,j)} g_e^b(e_{(i,j)}) \right) u_i + |\mathcal{N}_i| u_i^T R u_i + \frac{1}{2} \sum_{j \in \mathcal{N}_i} (p_{(i,j)} f_e(e_{(i,j)}) + Q(e_{(i,j)})) \right\}, \\
&= \sum_{i=1}^N \inf_{u_i \in [U]} \left\{ \frac{1}{2} p_{\mathcal{N}_i} f_{e_{\mathcal{N}_i}}(e_{\mathcal{N}_i}) + p_{\mathcal{N}_i} g_{\mathcal{N}_i}^+(e_{\mathcal{N}_i}) u_i + |\mathcal{N}_i| u_i^T R u_i + \frac{1}{2} \sum_{j \in \mathcal{N}_i} Q(e_{(i,j)}) \right\}, \tag{18}
\end{aligned}$$

Hamiltonian $H_{\mathcal{N}_i}$, and u_i only appears once in each $H_{\mathcal{N}_i}$, hence the solution of $H_{\mathcal{N}_i}$ guarantees the Pareto optimum. ■

B. Capture Basin

In the viability theory, the capture basin characterizes the set of states, in which a solution of a control policy exists such that the states can reach the target. In this paper, we determine the terminal time T in *Assumption 4* and estimate V_e by computing the capture basin. The links between the viability theory and finite horizon optimal control theory are specified in [32]. Based on that, we see the link between the differential game problem and the capture basin as follows:

Theorem 2: Considering the relative error system F_e for only two vehicles v_i and v_j , the value function V_e is related to the capture basin $\text{Capt}_{\hat{F}_e}(\varepsilon_p(\phi_L(\tau, e_{(i,j)})), \varepsilon_p(\phi_\Psi(\tau, e_{(i,j)})))$ by the formula

$$V_e(e_{(i,j)}(\tau)) = \inf_{(\tau, e_{(i,j)}, \zeta) \in \text{Capt}_{\hat{F}_e}(\varepsilon_p(\phi_L(\tau, e_{(i,j)})), \varepsilon_p(\phi_\Psi(\tau, e_{(i,j)})))} \zeta, \tag{20}$$

where $\varepsilon_p(\phi_L(\tau, e_{(i,j)}))$ and $\varepsilon_p(\phi_\Psi(\tau, e_{(i,j)}))$ are epigraphs of $\phi_L(\tau, e_{(i,j)})$ and $\phi_\Psi(\tau, e_{(i,j)})$, respectively. As specified in [32], an epigraph $\varepsilon_p(f(x))$ of a function $f(x)$ is expressed as $\varepsilon_p(f(x)) = \{(x, y) \mid y \geq f(x)\}$. Then, $\varepsilon_p(\phi_L(\tau, e_{(i,j)}))$ and $\varepsilon_p(\phi_\Psi(\tau, e_{(i,j)}))$ are expressed as follows:

$$\varepsilon_p(\phi_L(\tau, e_{(i,j)})) = \left\{ (\tau, e_{(i,j)}, \zeta) \mid \tau \in [0, T], \right. \\ \left. e_{(i,j)} \in K, \zeta \geq \int_\tau^T L(e_{(i,j)}(t), u_{ij}(t)) dt \right\}, \tag{21a}$$

$$\varepsilon_p(\phi_\Psi(\tau, e_{(i,j)})) = \left\{ (\tau, e_{(i,j)}, \zeta) \mid \tau = T, \right. \\ \left. e_{(i,j)} \in K, \zeta \geq \psi_C(e_{(i,j)}(T)) \right\}, \tag{21b}$$

and $\hat{F}_e$ is an auxiliary system

$$\begin{cases} \dot{e}_{(i,j)} = f_e(e_{(i,j)}) + g_e^+(e_{(i,j)}) u_i + g_e^-(e_{(i,j)}) u_j, & (22a) \\ \dot{\zeta} = -L(e_{(i,j)}, u_{ij}), & (22b) \\ u_{ij} \in [U_2], \quad e_{(i,j)} \in K, \quad t \in [0, T], & (22c) \end{cases}$$

Proof: Given the auxiliary system $\hat{F}_e$, $(\tau, e_{(i,j)}, \zeta) \in \text{Capt}_{\hat{F}_e}(\varepsilon_p(\phi_L(\tau, e_{(i,j)})), \varepsilon_p(\phi_\Psi(\tau, e_{(i,j)})))$ stands for the set of initial states $(\tau, e_{(i,j)}, \zeta)$ satisfying

$$\begin{cases} \zeta(\tau) \geq \int_\tau^T L(e_{(i,j)}(t), u_{ij}(t)) dt, & \tau \in [0, T], & (23a) \\ \zeta(\tau) \geq \psi_C(e_{(i,j)}(T)), & \tau = T, & (23b) \end{cases}$$

and (23) can be written as, for $\tau \in [0, T]$,

$$\zeta(T) \geq 0, \tag{24a}$$

$$\zeta(\tau) \geq \psi_C(e_{(i,j)}(T)) + \int_\tau^T L(e_{(i,j)}(t), u_{ij}(t)) dt, \tag{24b}$$

which implies

$$V_e(e_{(i,j)}(\tau)) \leq \psi_C(e_{(i,j)}(T)) + \int_\tau^T L(e_{(i,j)}(t), u_{ij}(t)) dt \\ \leq \inf_{(\tau, e_{(i,j)}, \zeta) \in \text{Capt}_{\hat{F}_e}(\varepsilon_p(\phi_L(\tau, e_{(i,j)})), \varepsilon_p(\phi_\Psi(\tau, e_{(i,j)})))} \zeta. \tag{25}$$

Given that $V_e(e_{(i,j)}(T)) = \psi_C(e_{(i,j)}(T)) \geq 0$, then

$$\begin{cases} V_e(e_{(i,j)}(T)) + \varepsilon \geq 0, & (26a) \\ V_e(e_{(i,j)}(\tau)) + \varepsilon \geq \psi_C(e_{(i,j)}(T)) \\ \quad + \int_\tau^T L(e_{(i,j)}(t), u_{ij}(t)) dt, & (26b) \end{cases}$$

and this implies $(\tau, e_{(i,j)}, V_e(e_{(i,j)}(\tau)) + \varepsilon) \in \text{Capt}_{\hat{F}_e}(\varepsilon_p(\phi_L(\tau, e_{(i,j)})), \varepsilon_p(\phi_\Psi(\tau, e_{(i,j)})))$ and

$$V_e(e_{(i,j)}(\tau)) + \varepsilon \geq \inf_{(\tau, e_{(i,j)}, \zeta) \in \text{Capt}_{\hat{F}_e}(\varepsilon_p(\phi_L(\tau, e_{(i,j)})), \varepsilon_p(\phi_\Psi(\tau, e_{(i,j)})))} \zeta. \tag{27}$$

Let $\varepsilon \rightarrow 0$, we summarize from (25) and (27) that

$$V_e(e_{(i,j)}(\tau)) = \inf_{(\tau, e_{(i,j)}, \zeta) \in \text{Capt}_{\hat{F}_e}(\varepsilon_p(\phi_L(\tau, e_{(i,j)})), \varepsilon_p(\phi_\Psi(\tau, e_{(i,j)})))} \zeta. \tag{28}$$

■

$$\begin{aligned}
\text{Capt}_{\tilde{F}_e}^*(\varepsilon_p(\varphi_L), \varepsilon_p(\varphi_\psi)) &= \left\{ (\tau, e_{(i,j)}, \zeta^*) \mid \forall \tau \in [0, T], \exists u_{ij}^*(\tau) \in [U_2], e_{(i,j)}(\tau) \in K, \right. \\
&\quad \left. \zeta^*(\tau) = \int_\tau^T L(e_{(i,j)}(t), u_{ij}^*(t)) dt, \zeta^*(T) = \psi_C(e_{(i,j)}(T)) \right\} \\
&= \left\{ (\tau, e_{(i,j)}, \zeta^*) \mid \exists \tau \in [0, T], \forall t \in [\tau, T], \exists u_{ij}^*(t) \in [U_2], \right. \\
&\quad \left. \varphi([0, T]; e_{(i,j)}(\tau), u_{ij}^*) \subseteq K, \varphi(T; e_{(i,j)}(\tau), u_{ij}^*) \in \partial C \right\}, \tag{32}
\end{aligned}$$

Denote $\text{Capt}_{\tilde{F}_e}^*(\varepsilon_p(\varphi_L), \varepsilon_p(\varphi_\psi))$ as the solution corresponding to $\inf_{u_{ij} \in [U_2]} \zeta(\tau) : \tau \in [0, T]$, the following theorem holds:

Theorem 3: Under Assumption 3, solving $\text{Capt}_{\tilde{F}_e}^*(\varepsilon_p(\varphi_L), \varepsilon_p(\varphi_\psi))$ is equivalent to solving $\text{Capt}_{\tilde{F}_e}(K, C)$ by Pontryagin Minimum Principle (PMP), where $\tilde{F}_e$ stands for the auxiliary system

$$\begin{cases} \dot{e}_{(i,j)} = \nabla_{p(i,j)} H_e(e_{(i,j)}, p(i,j)), \end{cases} \tag{29a}$$

$$\begin{cases} \dot{p}_{(i,j)} = -\nabla_{e_{(i,j)}} H_e(e_{(i,j)}, p(i,j)), \end{cases} \tag{29b}$$

$$\begin{cases} e_{(i,j)}(T) \in \partial C, \quad p_{(i,j)}(T) = \nabla \psi_C(e_{(i,j)}(T)), \end{cases} \tag{29c}$$

$$\begin{aligned}
H_e(e_{(i,j)}, p_{(i,j)}) &= \inf_{u_{ij} \in [U_2]} \{ p_{(i,j)} (f_e(e_{(i,j)}) + g_e^\pm(e_{(i,j)}) u_{ij}) \\
&\quad + L(e_{(i,j)}, u_{ij}) \}. \tag{29d}
\end{aligned}$$

Proof: Given that $C := \{e_{(i,j)} \in K \mid \psi_C(e_{(i,j)}) \leq 0\}$ and $e_{(i,j)}(T) \in \partial C$, for $\forall (\tau, e_{(i,j)}, \zeta) \in \text{Capt}_{\tilde{F}_e}(\varepsilon_p(\varphi_L(\tau, e_{(i,j)})), \varepsilon_p(\varphi_\psi(\tau, e_{(i,j)})))$, we have

$$\begin{cases} \inf_{u_{ij} \in [U_2]} \zeta(T) = \psi_C(e_{(i,j)}(T)) = 0, \end{cases} \tag{30a}$$

$$\begin{cases} \inf_{u_{ij} \in [U_2]} \zeta(\tau) = V_e(e_{(i,j)}(\tau)), \end{cases} \tag{30b}$$

and by PMP, (30) implies that $\forall e_{(i,j)}(T) \in \partial C$ and $\zeta(T) = 0$. This means that

$$\zeta(T) = \psi_C(e_{(i,j)}(T)) \wedge \zeta(\tau) = \int_\tau^T L(e_{(i,j)}(t), u_{ij}^*(t)) dt \tag{31a}$$

$$\begin{aligned}
\zeta(\tau) &= \psi_C(e_{(i,j)}(T)) + \int_\tau^T L(e_{(i,j)}(t), u_{ij}^*(t)) dt \\
&= V_e(e_{(i,j)}(\tau)). \tag{31b}
\end{aligned}$$

Then, let $\zeta^*(\tau) = \inf_{u_{ij} \in [U_2]} \zeta(\tau)$, we have (32) and it can be deduced that $(\tau, e_{(i,j)}, \zeta^*) \in \text{Capt}_{\tilde{F}_e}^*(\varepsilon_p(\varphi_L), \varepsilon_p(\varphi_\psi))$, therefore, $e_{(i,j)} \in \text{Capt}_{\tilde{F}_e}(K, C)$. ■

IV. DISTRIBUTED CONTROL POLICY

Our optimal distributed control law is designed based on the capture basin for the following reasons:

- The capture basin characterizes the set of states, in which solutions exist.
- The capture basin specifies the feasible learning region when adopting reinforcement learning algorithms.
- Approximation of capture basin provides a priori knowledge for the reinforcement learner.
- Solution to the differential game problems is related to the capture basin.

Fig. 1. Flow chart for the design of distributed control policy.

- Reachability of state is guaranteed for the dynamic system with constrained state and bounded control.

The distributed control policy is generated by two stages. The first stage is the approximation of the capture basin $\text{Capt}_{\tilde{F}_e}(K, C)$, which initializes the approximation of the value function V_e and provides a priori knowledge for the on-line cooperative reinforcement learner. The second stage is the on-line cooperative reinforcement learning, which performs an accurate approximation for the value function V_e , and optimizes the control policy. The corresponding flow chart is shown in Fig. 1.

A. Backward Propagation

The capture basin $\text{Capt}_{\tilde{F}_e}(K, C)$ is approximated by backward set propagation on C . Guaranteed approximations on the set propagation are performed using interval methods. Denote X as the state space of interest; we compose X by smaller non-overlapping interval vectors $\{[x]\}$ with identical side lengths $d([x]) = \varepsilon_X$, where $\varepsilon_X \subset \mathbb{R}^n$ is a set of resolutions of the state space. Furthermore, every box is numbered with index $h \in \mathbb{N}^n$, and sequentially any point $p \in K$ can be localized by the index through a function $\rho(q) : \mathbb{R}^n \rightarrow \mathbb{Z}^n$, where $q \in [x_h] : h = \rho(q)$. Subpavings are defined as the unions of non-overlapping interval vectors in X [29]. We introduce the following subpavings: $\mathbb{C}, \bar{\mathbb{C}}, \mathbb{K}, \bar{\mathbb{K}}, \text{Capt}_{\tilde{F}_e}(K, C)$ and $\overline{\text{Capt}_{\tilde{F}_e}(K, C)}$ as follows

Definition 5: $\mathbb{C}, \bar{\mathbb{C}}$ are inner and outer approximations of C ; $\mathbb{K}, \bar{\mathbb{K}}$ are inner and outer approximations of K ; $\text{Capt}_{\tilde{F}_e}(K, C)$, $\overline{\text{Capt}_{\tilde{F}_e}(K, C)}$ are inner and outer approximations of $\text{Capt}_{\tilde{F}_e}(K, C)$.

The relations among them are given by:

$$\mathbb{C} \subset C \subset \bar{\mathbb{C}} \subset \mathbb{K}, \quad \partial C \subset \partial \mathbb{C} = \bar{\mathbb{C}} \setminus \mathbb{C}, \quad (33a)$$

$$\mathbb{K} \subset K \subset \bar{\mathbb{K}} \subset X, \quad \partial K \subset \partial \mathbb{K} = \bar{\mathbb{K}} \setminus \mathbb{K}, \quad (33b)$$

$$\text{Capt}_{\bar{F}_e}(K, C) \subset \text{Capt}_{\bar{F}_e}(K, C) \subset \overline{\text{Capt}_{\bar{F}_e}(K, C)} \subseteq \mathbb{K}. \quad (33c)$$

A discrete hypersurface $\mathcal{D}_0 \in \partial \mathbb{D}_0 = \partial \mathbb{C}$ is generated to approximate ∂C , and all its vertices $\mathbf{Q}_0 = \{q_{0,i}\} \subset \partial C$. Its propagated sets are denoted as $\mathcal{D}_k$, where k is the retrieve of the iterations. The vertices $\mathbf{Q}_k = \{q_{k,i}\}$ of the k th discrete hypersurface $\mathcal{D}_k$ satisfy $q_{k,i} \in [\mathbf{x}_{k,i}] \setminus \partial[\mathbf{x}_{k,i}] : i \in [1, I_k]$ and $\partial \mathbb{D}_k = \bigcup \{[\mathbf{x}_{k,i}]\}$. Only certain parts of the vertices are chosen to be propagated when they meet the propagation condition. Denote $\mathbb{I}_k$ as all of the indexes of the vertices satisfying propagation condition in the k th iteration, $q_{k,i} : i \in \mathbb{I}_k$ are chosen to be propagated as their corresponding box $[\mathbf{x}_{k,i}] : i \in \mathbb{I}_k$ satisfies the following condition:

Condition 1:

$$[f_e](\mathbf{x}_{k,i}) + [g_e^\pm](\mathbf{x}_{k,i})u_\pm \subset \mathcal{T}_{\mathcal{D}_k}(q_{k,i}) \wedge [\mathbf{x}_{k,i}] \subset \mathbb{K}, \quad (34)$$

where $u_\pm = [u_+^T, u_-^T]^T$, $[f](\cdot)$ represents an inclusion function which is expressed in [29]; $\mathcal{T}_{\mathcal{D}_k}(q_{k,i})$ stands for Bouligand's tangent cone [28], and

$$u_\pm = \arg \min_{u_\pm \in [U_2]} \{p_{k,i}(f_e(q_{k,i}) + g_e^\pm(q_{k,i})u_\pm) + u_\pm^T R_2 u_\pm\}, \quad (35)$$

and the control policy $u_\pm$ always follows (35) during the propagation. If $\nexists [\mathbf{x}_{k,i}] : i \in [1, I_k]$ satisfies (34), the propagation stops, and the propagation time t determines the terminal time T . We denote the retrieve of the terminal iteration as K_b .

As specified in [32], the backward propagation is denoted as $\overleftarrow{(\cdot)}$, and $\overleftarrow{(\cdot)}(t) = (\cdot)(T - t)$. Given the auxiliary system $\hat{F}_e$ and $\bar{F}_e$, by denoting $\phi_{k,i} = [\overleftarrow{q}_{k,i}^T, \overleftarrow{p}_{k,i}^T, \overleftarrow{\zeta}_{k,i}^T]^T$, the backward dynamic system $\overleftarrow{F}_\phi$ relying on PMP is defined as

$$\overleftarrow{q}_{k,i}(t) = -f_e(\overleftarrow{q}_{k,i}(t)) - g_e^\pm(\overleftarrow{q}_{k,i}(t))u_\pm, \quad (36a)$$

$$\overleftarrow{p}_{k,i}(t) = \overleftarrow{p}_{k,i}(t) (\nabla f_e(\overleftarrow{q}_{k,i}(t)) + \nabla g_e^\pm(\overleftarrow{q}_{k,i}(t))u_\pm) + \nabla_{q_{k,i}} L(\overleftarrow{q}_{k,i}(t), u_\pm), \quad (36b)$$

$$\overleftarrow{\zeta}_{k,i}(t) = L(\overleftarrow{q}_{k,i}(t), u_\pm), \quad (36c)$$

$$\overleftarrow{q}_{0,i}(0) \in \mathbf{Q}_0, \quad \overleftarrow{p}_{0,i}(0) = \nabla \psi_C(\overleftarrow{q}_{0,i}(0)), \quad \overleftarrow{\zeta}_{0,i}(0) = 0.$$

Then, the backward propagation for $\mathcal{D}_k$ is performed by

$$\phi_{k+1,i} = \phi_{\bar{F}_\phi}(\Delta t_k; \phi_{k,i}, u_\pm), \quad i \in \mathbb{I}_k, \quad (37a)$$

$$\Delta t_k : \sup_{i \in [1, I_k]} \|\phi_{\bar{F}_\phi}(\Delta t_k; \overleftarrow{q}_{k,i}, u_\pm) - \overleftarrow{q}_{k,i}\| < \min\{\varepsilon_X\}, \quad (37b)$$

and the propagation time Δt_k of the k th iteration ensures that the propagated boxes in each loop are connected (i.e. $\partial \mathbb{D}_{k+1} \cap \partial \mathbb{D}_k \subset \partial \mathbb{D}_k, k \in [0, K_b - 1]$), and the proof is similar to [28].

Assume $V^e(\overleftarrow{q}_{k,i}(t))$ as the true value function for a single relative error system, and $\hat{V}_e$ as its approximated value function. As the state space is composed of grids, the approximation is performed piecewise. For $k \in [0, K_b]$ and $i \in [1, I_k]$, assume

$[\mathbf{x}_h] \subset \overline{\text{Capt}_{\bar{F}_e}(K, C)} \setminus \mathbb{C}$, we have

$$\hat{V}_e(\overleftarrow{q}_{k,i}) = \hat{V}_e^{[\mathbf{x}_h]}(\overleftarrow{q}_{k,i}), \quad (38a)$$

$$\hat{V}_e^{[\mathbf{x}_h]}(\overleftarrow{q}_{k,i}, \hat{W}_e^{[\mathbf{x}_h]}) = \hat{W}_e^{[\mathbf{x}_h]T} \sigma(\overleftarrow{q}_{k,i}), \quad (38b)$$

$$V^e(\overleftarrow{q}_{k,i}) = W_e^{[\mathbf{x}_h]T} \sigma(\overleftarrow{q}_{k,i}) + \varepsilon_e^{[\mathbf{x}_h]}(\overleftarrow{q}_{k,i}), \quad (38c)$$

where $\sigma : \mathbb{R}^n \rightarrow \mathbb{R}^l$ is the activation function of the neural network; $\hat{W}_e^{[\mathbf{x}_h]}(t_e^{[\mathbf{x}_h]}) \in \mathbb{R}^l$ is the estimated weight of their corresponding activation functions; $t_e^{[\mathbf{x}_h]}$ stands for the local time for the parameters assigned to the box $[\mathbf{x}_h]$ and $h = \rho(\overleftarrow{q}_{k,i}(t))$; $\overleftarrow{q}_{k,i}(t) = \overleftarrow{q}_{k,i}(t) - \text{mid}([\mathbf{x}_h])$; $\text{mid}([\mathbf{x}_h])$ represents the middle point of $[\mathbf{x}_h]$; $\varepsilon_e^{[\mathbf{x}_h]}$ is the function reconstruction error satisfying $\sup_{\overleftarrow{q}_{k,i} \in [\mathbf{x}_h]} \|\varepsilon_e^{[\mathbf{x}_h]}(\overleftarrow{q}_{k,i})\| \leq \varsigma_\varepsilon^{[\mathbf{x}_h]}$ and $\sup_{\overleftarrow{q}_{k,i} \in [\mathbf{x}_h]} \|\nabla \varepsilon_e^{[\mathbf{x}_h]}(\overleftarrow{q}_{k,i})\| \leq \varsigma_{\nabla \varepsilon}^{[\mathbf{x}_h]}$.

Our reinforcement learning algorithm is based on actor-critic methods, but only the critic is applied during propagation, where one box corresponds to one critic. Once the first propagated point $\overleftarrow{q}_{k,i}(t)$ enters the box $[\mathbf{x}_h] : h = \rho(\overleftarrow{q}_{k,i}(t))$, we initialize the parameters $\Gamma_e^{[\mathbf{x}_h]}$, $\hat{W}_e^{[\mathbf{x}_h]}$ and $\omega_e(t)$ for the critic of the local box, where $\Gamma_e^{[\mathbf{x}_h]} \in \mathbb{R}^{l \times l}$ is initialized to a positive definite matrix, and $\hat{W}_e^{[\mathbf{x}_h]}$ and $\omega_e(t)$ will be initialized such that

$$\overleftarrow{\zeta}_{k,i}(t) = \hat{W}_e^{[\mathbf{x}_h]T}(0) \sigma(\overleftarrow{q}_{k,i}(t)), \quad (39a)$$

$$\overleftarrow{q}_{k,i}^T(t) L(\overleftarrow{q}_{k,i}(t), u_\pm) = \hat{W}_e^{[\mathbf{x}_h]T}(0) \nabla \sigma(\overleftarrow{q}_{k,i}(t)) \|\overleftarrow{q}_{k,i}(t)\|^2. \quad (39b)$$

The parameters for a box are all denoted as $(\cdot)_e^{[\mathbf{x}_h]}$, and for each box they have their own local time $t_e^{[\mathbf{x}_h]}$. When $\overleftarrow{q}_{k,i}(t) \in [\mathbf{x}_h]$, the parameters for the box $[\mathbf{x}_h]$ are updated, and meanwhile the local timer $t_e^{[\mathbf{x}_h]}$ is running. Hence, although global time t is always running, if $\overleftarrow{q}_{k,i}(t) \notin [\mathbf{x}_h]$, the local time $t_e^{[\mathbf{x}_h]}$ freezes until any point enters the box $[\mathbf{x}_h]$. Let

$$\omega_e(t) = -\nabla \sigma(\overleftarrow{q}_{k,i}(t)) \overleftarrow{q}_{k,i}(t), \quad (40a)$$

$$\delta_e^{[\mathbf{x}_h]}(t, t_e^{[\mathbf{x}_h]}) = \hat{W}_e^{[\mathbf{x}_h]T}(t_e^{[\mathbf{x}_h]}) \omega_e(t) + L(\overleftarrow{q}_{k,i}(t), u_\pm(t)), \quad (40b)$$

$$\lambda_e^{[\mathbf{x}_h]}(t, t_e^{[\mathbf{x}_h]}) = k_\lambda \left(1 - e^{-|\delta_e^{[\mathbf{x}_h]}(t, t_e^{[\mathbf{x}_h]})|}\right), \quad (40c)$$

where $k_\lambda \in \mathbb{R}^+$ is a positive tuning parameter, $\delta_e^{[\mathbf{x}_h]}$ is the Bellman error, and $\lambda_e^{[\mathbf{x}_h]}$ is the forgetting factor [37]. When $\overleftarrow{q}_{k,i}(t) \in [\mathbf{x}_h]$, the critic is designed as follows:

$$\begin{aligned} \dot{\hat{W}}_e^{[\mathbf{x}_h]}(t_e^{[\mathbf{x}_h]}) &= -k_c \Gamma_e^{[\mathbf{x}_h]}(t_e^{[\mathbf{x}_h]}) \\ &\times \frac{\omega_e(t)}{1 + \omega_e(t)^T \Gamma_e^{[\mathbf{x}_h]}(t_e^{[\mathbf{x}_h]}) \omega_e(t)} \delta_e^{[\mathbf{x}_h]}(t, t_e^{[\mathbf{x}_h]}), \end{aligned} \quad (41a)$$

$$\begin{aligned} \Gamma_e^{[\mathbf{x}_h]}(t_e^{[\mathbf{x}_h]}) &= -k_c \left(-\lambda_e^{[\mathbf{x}_h]}(t, t_e^{[\mathbf{x}_h]}) \Gamma_e^{[\mathbf{x}_h]}(t_e^{[\mathbf{x}_h]}) + \Gamma_e^{[\mathbf{x}_h]}(t_e^{[\mathbf{x}_h]}) \right. \\ &\times \left. \frac{\omega_e(t) \omega_e(t)^T}{1 + \omega_e(t)^T \Gamma_e^{[\mathbf{x}_h]}(t_e^{[\mathbf{x}_h]}) \omega_e(t)} \Gamma_e^{[\mathbf{x}_h]}(t_e^{[\mathbf{x}_h]}) \right), \end{aligned} \quad (41b)$$

where $k_c \in \mathbb{R}^+$ is a positive tuning parameter. We denote the dynamic systems in (41) as F_c . Following the above derivations, we can propose the theorem infra:

Theorem 4: Assume $\frac{\omega_e}{\sqrt{1+\omega_e^T \Gamma_e^{[x_h]} \omega_e}}$ is persistently exciting. By

adopting the critic shown in (41), the estimation error $\|\tilde{W}_e^{[x_h]}\| = \|W_e^{[x_h]} - \hat{W}_e^{[x_h]}\|$ is uniformly ultimately bounded (UUB). If the neural network (NN) in (38b) is ideal and $u_\pm$ is generated by backward propagation using PMP, $\hat{W}_e^{[x_h]}$ exponentially and asymptotically converges to the HJB solution to the game described in (8) and (10).

Proof: For simplicity, we denote $u_\pm = u_\pm(e_{(i,j)}, W_e^{[x_h]})$; $u_\pm^* = u_\pm^*(e_{(i,j)}, W_e^{[x_h]})$, where $u_\pm^*$ is the Pareto optimum to the game described in (8) and (10); $\tilde{u}_\pm = u_\pm^* - u_\pm$. The local timer $t_e^{[x_h]}$ is omitted in the subsequent context for simplicity. Given (38c) and (40a), we can derive

$$0 = W_e^{[x_h]T} \omega_e (\bar{q}_{k,i}, u_\pm^*) + L(\bar{q}_{k,i}, u_\pm^*) + \nabla \mathcal{E}_e^{[x_h]}(\bar{q}_{k,i}) (f_e(\bar{q}_{k,i}) + g_e^\pm(\bar{q}_{k,i}) u_\pm^*). \quad (42)$$

Then, denoting $\delta_e^{[x_h]} = \delta_e^{[x_h]}(\hat{W}_e^{[x_h]}, W_e^{[x_h]}, \bar{q}_{k,i}, u_\pm, u_\pm^*)$ and $\omega_e = \omega_e(\bar{q}_{k,i}, u_\pm)$, (40b) can be rewritten as

$$\begin{aligned} \delta_e^{[x_h]} &= -\tilde{W}_e^{[x_h]T} \omega_e (\bar{q}_{k,i}, u_\pm) - \tilde{u}_\pm^T R_2 (u_\pm^* + u_\pm) \\ &\quad + \nabla \mathcal{E}_e^{[x_h]}(\bar{q}_{k,i}) (f_e(\bar{q}_{k,i}) + g_e^\pm(\bar{q}_{k,i}) u_\pm^*) \\ &\quad - W_e^{[x_h]T} \nabla \sigma(\bar{q}_{k,i}) g_e^\pm(\bar{q}_{k,i}) \tilde{u}_\pm. \end{aligned} \quad (43)$$

Given (43) and a Lyapunov function $\mathcal{V}_e^{[x_h]}(\tilde{W}_e^{[x_h]}) = \tilde{W}_e^{[x_h]T} \Gamma_e^{[x_h]-1} \tilde{W}_e^{[x_h]}$, we have

$$\begin{aligned} \dot{\mathcal{V}}_e^{[x_h]} &= \dot{\tilde{W}}_e^{[x_h]T} \Gamma_e^{[x_h]-1} \tilde{W}_e^{[x_h]} + \tilde{W}_e^{[x_h]T} \frac{d}{dt} \left(\Gamma_e^{[x_h]-1} \tilde{W}_e^{[x_h]} \right) \\ &= 2\tilde{W}_e^{[x_h]T} \Gamma_e^{[x_h]-1} \dot{\tilde{W}}_e^{[x_h]} + \tilde{W}_e^{[x_h]T} \frac{d\Gamma_e^{[x_h]-1}}{dt} \tilde{W}_e^{[x_h]} \\ &= -k_c \frac{\tilde{W}_e^{[x_h]T} \omega_e \omega_e^T \tilde{W}_e^{[x_h]}}{1 + \omega_e^T \Gamma_e^{[x_h]} \omega_e} \\ &\quad - k_c k_\lambda \left(1 - e^{-|\delta_e^{[x_h]}|} \right) \tilde{W}_e^{[x_h]T} \Gamma_e^{[x_h]-1} \tilde{W}_e^{[x_h]} \\ &\quad + 2k_c \frac{\tilde{W}_e^{[x_h]T} \omega_e}{1 + \omega_e^T \Gamma_e^{[x_h]} \omega_e} \left(-\tilde{u}_\pm^T R_2 (u_\pm^* + u_\pm) \right. \\ &\quad \left. + \nabla \mathcal{E}_e^{[x_h]}(\bar{q}_{k,i}) (f_e(\bar{q}_{k,i}) + g_e^\pm(\bar{q}_{k,i}) u_\pm^*) \right. \\ &\quad \left. - W_e^{[x_h]T} \nabla \sigma(\bar{q}_{k,i}) g_e^\pm(\bar{q}_{k,i}) \tilde{u}_\pm \right). \end{aligned} \quad (44)$$

As $\sup_{u_\pm^* \in [U_2]} \|u_\pm^*\| \leq \varsigma_u$, $\sup_{u_\pm \in [U_2]} \|u_\pm\| \leq \varsigma_u$ and *Assumption 3* hold, $\|\tilde{u}_\pm\|$

and $\|f_e(\bar{q}_{k,i}) + g_e^\pm(\bar{q}_{k,i}) u_\pm^*\|$ are bounded. Moreover, given that $\sup_{\bar{q}_{k,i} \in [\bar{q}]} \|\nabla \mathcal{E}_e^{[x_h]}(\bar{q}_{k,i})\| \leq \varsigma_{\nabla \mathcal{E}}^{[x_h]}$, then $\nabla \mathcal{E}_e^{[x_h]}(\bar{q}_{k,i}) \bar{q}_{k,i}$ is bounded.

Provided that $\frac{\omega_e}{\sqrt{1+\omega_e^T \Gamma_e^{[x_h]} \omega_e}}$ is persistently exciting, therefore,

$\|\Gamma_e^{[x_h]}\|$ is bounded [37]. Thus, there exists a bound $\varsigma_{\tilde{W}_e^{[x_h]}}$, when

$$\varsigma_{\tilde{W}_e^{[x_h]}} \leq \|\tilde{W}_e^{[x_h]}\|, \quad \dot{\mathcal{V}}_e^{[x_h]}(\tilde{W}_e^{[x_h]}) < 0.$$

If the NN structure is ideal, then $\sup_{\bar{q}_{k,i} \in [\bar{q}]} \|\nabla \mathcal{E}_e^{[x_h]}(\bar{q}_{k,i})\| = 0$. Provided that both $[U_2]$ and $J(e_{(i,j)})$ are convex and $u_\pm$ is computed from (35), the sufficient condition is satisfied for ensuring the optimality by PMP. Thus, when $u_\pm = u_\pm^*$, then $\dot{\mathcal{V}}_e^{[x_h]} < 0$ always holds. ■

When the backward propagation for one iteration has been completed, the subpavings $\mathbb{D}$ and $\partial \mathbb{D}$ are updated as follows:

$$\partial \mathbb{D}_k = \cup \{ [\mathbf{x}] \in \tilde{\mathbb{K}} \mid [\mathbf{x}] \setminus \partial [\mathbf{x}] \cap \partial \mathcal{D}_k \neq \emptyset \}, \quad (45a)$$

$$\mathbb{D}_k = \cup \{ [\mathbf{x}] \in \tilde{\mathbb{K}} \mid [\mathbf{x}] \subset \mathcal{D}_k \}. \quad (45b)$$

Moreover, we remove and interpolate points to ensure that every box $[\mathbf{x}_h] \subset \partial \mathbb{D}_k$ contains one point. The interpolated states $\hat{\phi}_{k,i}$ can be estimated as follows:

$$\hat{q}_{k,i} : [\mathbf{x}_h] \subset \partial \mathbb{D}_k, \quad \hat{q}_{k,i} \in [\mathbf{x}_h] \setminus \partial [\mathbf{x}_h] \cap \partial \mathcal{D}_k, \quad (46a)$$

$$\hat{p}_{k,i} = \frac{\sum_{j \in \mathbb{N}_i} \frac{p_{k,j}}{\|\hat{q}_{k,i} - q_{k,j}\|}}{\sum_{j \in \mathbb{N}_i} \frac{1}{\|\hat{q}_{k,i} - q_{k,j}\|}}, \quad \hat{\zeta}_{k,i} = \frac{\sum_{j \in \mathbb{N}_i} \frac{\zeta_{k,j}}{\|\hat{q}_{k,i} - q_{k,j}\|}}{\sum_{j \in \mathbb{N}_i} \frac{1}{\|\hat{q}_{k,i} - q_{k,j}\|}}, \quad (46b)$$

where $q_{k,j} : j \in \mathbb{N}_i$ stands for the vertices that share the common segments of $\mathcal{D}_k$ with $q_{k,i}$, and $\mathbb{N}_i$ is the set of their indexes.

The pseudo code in **Algorithm 1** shows the detailed steps to perform the off-line backward propagation.

B. Cooperative Learning

The on-line reinforcement learning process is taken under the following assumption:

Assumption 5: $\forall t \geq 0$, the communication graph $\mathcal{G}_N$ is always connected, and

$$\begin{aligned} \forall i, j \in [1, N], \quad i \neq j, \quad e_{(i,j)}(t) \in \text{Capt}_{\tilde{F}_e}(X, C) \\ \wedge \forall e_{(i,j)}(t) \notin \text{Capt}_{\tilde{F}_e}(K, C), \quad \alpha_{ij} = 0, \end{aligned} \quad (47)$$

such that there always exists a distributed control strategy for the network system to reach consensus.

We assign a critic and an actor for each local network system, where the critic learns the local value function and the actor learns the optimal control policy. Regarding the i th vehicle v_i , the parameters of its critic and actor are partially shared by the critics and actors of its neighbouring vehicles $v_{\mathbb{N}_i}$, respectively. Denoting $(\cdot)_{\mathbb{N}_i} = \{(\cdot)_{(i,j)}\}$, $(\cdot)_{(i,j)}$ is shared by the i th and the j th (or actor).

The designed critic in cooperative learning aims to update $\hat{W}_{\mathcal{N}_i}^c = \left[\dots, \hat{W}_{(i,j)}^{[x_h]T}, \dots \right]^T : j \in \mathbb{N}_i$, when $\alpha_{ij} > 0 \wedge h = \rho(e_{(i,j)}(t)) \wedge [\mathbf{x}_h] \subset \text{Capt}_{\tilde{F}_e}(K, C) \setminus \mathbb{C}$. The critic is composed of two independent parts, and they are updated simultaneously. The first part aims to approximate $W_e^{[x_h]}$ as shown in (38c). The parameters $(\cdot)_{\mathbb{N}_i}^{[x_h]}$ in Section IV-A are shared throughout the whole graph. These parameters, together with their timer $t_e^{[x_h]}$, are updated via (40) and (41) with slight modifications:

$$\omega_{(i,j)} = \nabla \sigma(\tilde{e}_{(i,j)}) (f_e(e_{(i,j)}) + g_e^\pm(e_{(i,j)}) u_\pm), \quad (48a)$$

$$u_\pm = \arg \min_{u_{ij} \in [U_2]} \left\{ \hat{W}_e^{[x_h]T} \nabla \sigma(\tilde{e}_{(i,j)}) g_e^\pm(e_{(i,j)}) u_{ij} + u_{ij}^T R_2 u_{ij} \right\}, \quad (48b)$$

$$\tilde{e}_{(i,j)} = e_{(i,j)} - \text{mid}([\mathbf{x}_h]). \quad (48c)$$

Algorithm 1 Computation of an inner approximation of $\text{Capt}_{\bar{F}_e}(K, C)$.

Generate grids and subpavings $\mathbb{C}, \bar{\mathbb{C}}, \mathbb{K}, \bar{\mathbb{K}}$;
 $\mathbb{D}_0 = \mathbb{C}; \partial \mathbb{D}_0 = \partial \mathbb{C}; k = 0; t = 0$;
 Allocate vertices $\mathbf{Q}_0$ such that $\bar{q}_{0,i}(0) \in [\mathbf{x}_{0,i}] \setminus \partial[\mathbf{x}_{0,i}] \cap \partial C$;
 $\bar{p}_{0,i}(0) = \nabla \psi_C(\bar{q}_{0,i}(0)); \bar{z}_{0,i}(0) = 0$;
while $\partial \mathbb{D}_{k+1} \neq \partial \mathbb{D}_k$ **do**
 $I_k = \text{size}(\partial \mathbb{D}_k); \Delta t_k = 0$;
 while $\Delta t_k = 0$ satisfies (37b) **do**
 for $i = 1; i \leq I_k; i++$ **do**
 $u_{\pm}$ is calculated by (35);
 if Condition 1 holds true **then**
 Set $\Delta \tau_k \ll \Delta t_k$;
 $\phi_{k,i} = \varphi_{F_\phi}(\Delta t_k; \phi_{k,i}, u_{\pm})$; $h = \rho(\bar{q}_{k,i})$;
 if $\bar{q}_{k,i}$ is the first point that enters $[\mathbf{x}_h]$ **then**
 $\hat{W}_e^{[\mathbf{x}_h]}$ and ω_e are initialized by (39);
 else
 $\hat{W}_e^{[\mathbf{x}_h]} = \varphi_{F_c}(\Delta t_k; \hat{W}_e^{[\mathbf{x}_h]}, u_{\pm})$;
 $\Gamma_e^{[\mathbf{x}_h]} = \varphi_{F_c}(\Delta t_k; \Gamma_e^{[\mathbf{x}_h]}, u_{\pm})$;
 end if
 end if
 end for
 $\Delta t_k = \Delta t_k + \Delta \tau_k$;
 end while
 $k = k + 1; t = t + \Delta t_k$; $\partial \mathbb{D}_k$ and $\mathbb{D}_k$ are updated by (45);
 Adjust $\mathbf{Q}_k$ by removing and interpolating states such that
 $\forall [\mathbf{x}] \subset \partial \mathbb{D}_k$ contains only one point q ; the interpolated states
 are estimated as shown in (46)
end while
 $T = t$; $\text{Capt}_{\bar{F}_e}(K, C) = \mathbb{D}_k$;

The second part aims to approximate the instantaneous weight $\beta_{(i,j)} \in \mathbb{R}^+$ on $W_e^{[\mathbf{x}_h]}$. The value functions for local network systems can be constructed by NN as follows:

$$V_{\mathcal{N}_i}(e_{\mathcal{N}_i}) = \sum_{j \in \mathcal{N}_i} \mathcal{W}_{(i,j)}^{[\mathbf{x}_h]} \sigma(\tilde{e}_{(i,j)}) + \sum_{j \in \mathcal{N}_i} \varepsilon_{(i,j)}^{[\mathbf{x}_h]}(\tilde{e}_{(i,j)}), \quad (49)$$

where $h = \rho(e_{(i,j)}) : e_{(i,j)} \in [\mathbf{x}_h]$; $\mathcal{W}_{(i,j)}^{[\mathbf{x}_h]}$ stands for the true NN weight and $\mathcal{W}_{(i,j)}^{[\mathbf{x}_h]}(t) = \beta_{(i,j)}(t) W_e^{[\mathbf{x}_h]}$; $\varepsilon_{(i,j)}^{[\mathbf{x}_h]}(t, \tilde{e}_{(i,j)}(t)) = \beta_{(i,j)}(t) \varepsilon_e^{[\mathbf{x}_h]}(\tilde{e}_{(i,j)}(t))$; $\sup_{e_{(i,j)} \in [\mathbf{x}_h]} \|\varepsilon_e^{[\mathbf{x}_h]}(\tilde{e}_{(i,j)})\| \leq \zeta_e^{[\mathbf{x}_h]}$; and $\sup_{e_{(i,j)} \in [\mathbf{x}_h]} \|\nabla \varepsilon_e^{[\mathbf{x}_h]}(\tilde{e}_{(i,j)})\| \leq \zeta_{\nabla \varepsilon}^{[\mathbf{x}_h]}$. The estimated value functions $\hat{V}_{\mathcal{N}_i}$ can be written as

$$\hat{V}_{\mathcal{N}_i}(e_{\mathcal{N}_i}) = \sum_{j \in \mathcal{N}_i} \hat{\mathcal{W}}_{(i,j)}^{[\mathbf{x}_h]} \sigma(\tilde{e}_{(i,j)}), \quad (50)$$

where $\hat{\mathcal{W}}_{(i,j)}^{[\mathbf{x}_h]}(t) = \hat{\beta}_{(i,j)}(t) \hat{W}_e^{[\mathbf{x}_h]}$ and $\hat{\beta}_{(i,j)}$ is the estimated dynamic weight. Before the on-line reinforcement training, the parameters of critics for all the relative error systems are initialized as follows: for $\forall \alpha_{ij} > 0$, $\hat{\mathcal{W}}_{(i,j)}^{[\mathbf{x}_h]}(0) = \hat{\beta}_{(i,j)}(0) \hat{W}_e^{[\mathbf{x}_h]}$ when $[\mathbf{x}_h] \subset \text{Capt}_{\bar{F}_e}(K, C) \setminus \mathbb{C}$, and $\hat{\mathcal{W}}_{(i,j)}^{[\mathbf{x}_h]}(0) = \mathbf{0}$ when $[\mathbf{x}_h] \subset \mathbb{C}$.

Let $\tilde{e}_{\mathcal{N}_i} = [\cdots, \tilde{e}_{(i,j)}^T, \cdots]^T : j \in \mathcal{N}_i, \nabla \sigma_{\mathcal{N}_i}(\tilde{e}_{\mathcal{N}_i}) =$

$\text{diag}([\cdots, \nabla \sigma(\tilde{e}_{(i,j)}), \cdots]) : j \in \mathcal{N}_i$, and for the other variables, let $(\cdot)_{\mathcal{N}_i} = [\cdots, (\cdot)_{(i,j)}^T, \cdots]^T : j \in \mathcal{N}_i$, for $i \in [1, N], j \in \mathcal{N}_i, i \neq j$ and $[\mathbf{x}_h] \subset \text{Capt}_{\bar{F}_e}(K, C) \setminus \mathbb{C}$, the estimated dynamic weight $\hat{\beta}_{(i,j)}(t)$ is updated as shown below:

$$\dot{\hat{\beta}}_{(i,j)} = -k_\beta \hat{W}_e^{[\mathbf{x}_h]} \nabla \sigma(\tilde{e}_{(i,j)}) \left(\delta_{\mathcal{N}_i} \left(\frac{1}{2} f_e(e_{(i,j)}) + g_e^+(e_{(i,j)}) u_i \right) + \delta_{\mathcal{N}_j} \left(\frac{1}{2} f_e(e_{(j,i)}) + g_e^+(e_{(j,i)}) u_j \right) \right), \quad (51)$$

where k_β is a positive tuning parameter, $\hat{\beta}_{(i,j)}(t)$ is constrained to $[0, \infty)$ and $\hat{\beta}_{(i,j)}(0) = 1$; $\delta_{\mathcal{N}_i}$ is the Bellman error corresponding to the local Hamiltonian $H_{\mathcal{N}_i}$ in (19), which is defined as

$$\delta_{\mathcal{N}_i} = \hat{W}_{\mathcal{N}_i}^c \nabla \sigma_{\mathcal{N}_i}(\tilde{e}_{\mathcal{N}_i}) g_{\mathcal{N}_i}^+(e_{\mathcal{N}_i}) u_i + \frac{1}{2} \hat{W}_{\mathcal{N}_i}^c \nabla \sigma_{\mathcal{N}_i}(\tilde{e}_{\mathcal{N}_i}) f_{e_{\mathcal{N}_i}}(e_{\mathcal{N}_i}) + |\mathcal{N}_i| u_i^T R u_i + \frac{1}{2} \sum_{j \in \mathcal{N}_i} Q(e_{(i,j)}), \quad (52)$$

and in this way, the i th and the j th critics update their common parameter $\hat{\beta}_{(i,j)}$ to minimize both $|\delta_{\mathcal{N}_i}|$ and $|\delta_{\mathcal{N}_j}|$.

The updating law for the actor is designed below, for $j \in \mathcal{N}_i$ and $i \in [1, N]$:

$$\dot{\hat{S}}_{(i,j)}^a = \frac{k_c k_a \|\delta_e^{[\mathbf{x}_h]}\| \|\Gamma_e^{[\mathbf{x}_h]} \omega_{(i,j)}\| (\hat{W}_e^{[\mathbf{x}_h]} - \hat{S}_{(i,j)}^a)}{(\lambda_e^{[\mathbf{x}_h]} + \varepsilon_a) |1 + \omega_{(i,j)}^T \Gamma_e^{[\mathbf{x}_h]} \omega_{(i,j)}|}, \quad (53a)$$

$$\hat{W}_{(i,j)}^a = \hat{\beta}_{(i,j)} \hat{S}_{(i,j)}^a, \quad (53b)$$

where $S \in \mathbb{R}^l$; $\hat{S}_{(i,j)}^a(0) = \hat{W}_e^{[\mathbf{x}_h]}$; $\varepsilon_a \in \mathbb{R}^+$ is a small positive constant and $k_a \in \mathbb{R}^+$ is a positive tuning parameter.

The distributed control policy for the i th vehicle v_i is designed as below

$$u_i(e_{\mathcal{N}_i}, \hat{W}_{\mathcal{N}_i}^a) = \arg \min_{u_i \in [U]} \left\{ |\mathcal{N}_i| u_i^T R u_i + \hat{W}_{\mathcal{N}_i}^a \nabla \sigma_{\mathcal{N}_i}(\tilde{e}_{\mathcal{N}_i}) g_{\mathcal{N}_i}^+(e_{\mathcal{N}_i}) u_i \right\}, \quad (54)$$

and if $\forall i, j \in [1, N], e_{(i,j)} \in \partial C$, then $\hat{W}_e^{[\mathbf{x}_h]} = \nabla \psi_C(e_{(i,j)})$. Denote $\mathcal{W}_{\mathcal{N}_i} = [\cdots, \mathcal{W}_{(i,j)}^{[\mathbf{x}_h]}]^T : j \in \mathcal{N}_i$, given (18) and (49), the Pareto optimal distributed control policy can be written as

$$u_i^*(e_{\mathcal{N}_i}, \mathcal{W}_{\mathcal{N}_i}) = \arg \min_{u_i \in [U]} \left\{ \mathcal{W}_{\mathcal{N}_i}^T \nabla \sigma_{\mathcal{N}_i}(\tilde{e}_{\mathcal{N}_i}) g_{\mathcal{N}_i}^+(e_{\mathcal{N}_i}) u_i + |\mathcal{N}_i| u_i^T R u_i + \nabla \varepsilon_{\mathcal{N}_i}(\tilde{e}_{\mathcal{N}_i}) g_{\mathcal{N}_i}^+(e_{\mathcal{N}_i}) u_i \right\}. \quad (55)$$

C. Convergence Analysis

The estimation errors on the NN weights of the i th critic and the i th actor are derived by $\tilde{\mathcal{W}}_{\mathcal{N}_i}^c = \mathcal{W}_{\mathcal{N}_i} - \hat{\mathcal{W}}_{\mathcal{N}_i}^c$ and $\tilde{\mathcal{W}}_{\mathcal{N}_i}^a = \mathcal{W}_{\mathcal{N}_i} - \hat{\mathcal{W}}_{\mathcal{N}_i}^a$, respectively. Their elements are expressed as $\tilde{\mathcal{W}}_{(i,j)}^{[\mathbf{x}_h]}(t) = \beta_{(i,j)}(t) W_e^{[\mathbf{x}_h]} - \hat{\beta}_{(i,j)}(t) \hat{W}_e^{[\mathbf{x}_h]}$ and $\tilde{\mathcal{W}}_{(i,j)}^a(t) = \beta_{(i,j)}(t) W_e^{[\mathbf{x}_h]} - \hat{\beta}_{(i,j)}(t) \hat{S}_{(i,j)}^a$, respectively. Moreover, the control error is given by $\tilde{u}_i = u_i^* - u_i$ where $\|\tilde{u}_i\| \leq \zeta_{\tilde{u}_i}$ when $t \rightarrow \infty$.

Theorem 5: Assume $\frac{\omega_{(i,j)}}{\sqrt{1 + \omega_{(i,j)}^T \Gamma_e^{[\mathbf{x}_h]} \omega_{(i,j)}}}$ is persistently exciting. From the local critic given (41) and (51) as well as the local actor

$$\begin{aligned} \dot{\mathcal{V}}_\beta = & - \sum_{i,j=1}^N \alpha_{ij} \tilde{\beta}_{(i,j)} \hat{W}_e^{[x_h]^T} \nabla \sigma(\tilde{e}_{(i,j)}) \\ & \times \left(\left(\frac{1}{2} f_e(e_{(i,j)}) + g_e^+(e_{(i,j)}) u_i \right) \left(\left(\frac{1}{2} f_{e_N}(e_{N_i}) + g_{N_i}^+(e_{N_i}) u_i \right)^T \nabla \sigma_{N_i}^T(\tilde{e}_{N_i}) \tilde{\beta}_{N_i} \otimes I_l \hat{W}_{N_i}^e + \mathfrak{K}_{\beta_i} + \mathfrak{K}_{W_i} + \mathfrak{K}_{\varepsilon_i} + \mathfrak{K}_{R_i} \right) \right. \\ & \left. + \left(\frac{1}{2} f_e(e_{(j,i)}) + g_e^+(e_{(j,i)}) u_j \right) \left(\left(\frac{1}{2} f_{e_N}(e_{N_j}) + g_{N_j}^+(e_{N_j}) u_j \right)^T \nabla \sigma_{N_j}^T(\tilde{e}_{N_j}) \tilde{\beta}_{N_j} \otimes I_l \hat{W}_{N_j}^e + \mathfrak{K}_{\beta_j} + \mathfrak{K}_{W_j} + \mathfrak{K}_{\varepsilon_j} + \mathfrak{K}_{R_j} \right) \right). \quad (62) \end{aligned}$$

formulated in (53), $\forall i \in [1, N], j \in \mathcal{N}_i$, $\|\tilde{W}_{N_i}^c\|$ and $\|\tilde{W}_{N_i}^a\|$ are UUB. If the NN structure and the tuning parameters are ideal, and the state resolution ε_X in backward propagation is small enough, then $\varsigma_{\tilde{u}_i}$ can be arbitrarily small such that u_i can converge to the Pareto optimal strategy u_i^* of the game described in (13) and (14).

Proof: For simplicity, denote $u_i^* = u_i^*(e_{N_i}, \mathcal{W}_{N_i})$, $u_i = u_i(e_{N_i}, \hat{W}_{N_i}^e)$. For the true NN weight $\mathcal{W}_{N_i}$, the following property always holds:

$$\begin{aligned} 0 = & \frac{1}{2} \mathcal{W}_{N_i}^T \nabla \sigma_{N_i}(\tilde{e}_{N_i}) f_{e_N}(e_{N_i}) + \mathcal{W}_{N_i}^T \nabla \sigma_{N_i}(\tilde{e}_{N_i}) g_{N_i}^+(e_{N_i}) u_i^* \\ & + \frac{1}{2} \nabla \varepsilon_{N_i}(\tilde{e}_{N_i}) f_{e_N}(e_{N_i}) + \nabla \varepsilon_{N_i}(\tilde{e}_{N_i}) g_{N_i}^+(e_{N_i}) u_i^* \\ & + |\mathcal{N}_i| u_i^{*T} R u_i^* + \frac{1}{2} \sum_{j \in \mathcal{N}_i} \mathcal{Q}(e_{(i,j)}), \quad (56) \end{aligned}$$

and given (56), by denoting $\delta_{N_i} = \delta_{N_i}(\hat{W}_{N_i}^c, \mathcal{W}_{N_i}, e_{N_i}, u_i, u_i^*)$, (52) can be recast to

$$\begin{aligned} \delta_{N_i} = & \hat{W}_{N_i}^{c,T} \nabla \sigma_{N_i}(\tilde{e}_{N_i}) g_{N_i}^+(e_{N_i}) u_i + \frac{1}{2} \hat{W}_{N_i}^{c,T} \nabla \sigma_{N_i}(\tilde{e}_{N_i}) f_{e_N}(e_{N_i}) \\ & + |\mathcal{N}_i| u_i^T R u_i + \frac{1}{2} \sum_{j \in \mathcal{N}_i} \mathcal{Q}(e_{(i,j)}) \\ & - \frac{1}{2} \mathcal{W}_{N_i}^T \nabla \sigma_{N_i}(\tilde{e}_{N_i}) f_{e_N}(e_{N_i}) \\ & - \mathcal{W}_{N_i}^T \nabla \sigma_{N_i}(\tilde{e}_{N_i}) g_{N_i}^+(e_{N_i}) u_i^* \\ & - \frac{1}{2} \nabla \varepsilon_{N_i}(\tilde{e}_{N_i}) f_{e_N}(e_{N_i}) - \nabla \varepsilon_{N_i}(\tilde{e}_{N_i}) g_{N_i}^+(e_{N_i}) u_i^* \\ & - |\mathcal{N}_i| u_i^{*T} R u_i^* - \frac{1}{2} \sum_{j \in \mathcal{N}_i} \mathcal{Q}(e_{(i,j)}) \\ = & \hat{W}_{N_i}^{c,T} \nabla \sigma_{N_i}(\tilde{e}_{N_i}) \left(\frac{1}{2} f_{e_N}(e_{N_i}) + g_{N_i}^+(e_{N_i}) u_i \right) \\ & - \mathcal{W}_{N_i}^T \nabla \sigma_{N_i}(\tilde{e}_{N_i}) \left(\frac{1}{2} f_{e_N}(e_{N_i}) + g_{N_i}^+(e_{N_i}) u_i^* \right. \\ & \left. - g_{N_i}^+(e_{N_i}) u_i + g_{N_i}^+(e_{N_i}) u_i \right) - \frac{1}{2} \nabla \varepsilon_{N_i}(\tilde{e}_{N_i}) f_{e_N}(e_{N_i}) \\ & - \nabla \varepsilon_{N_i}(\tilde{e}_{N_i}) g_{N_i}^+(e_{N_i}) u_i^* + |\mathcal{N}_i| u_i^T R u_i - |\mathcal{N}_i| u_i^{*T} R u_i^* \\ = & \tilde{W}_{N_i}^{c,T} \nabla \sigma_{N_i}(\tilde{e}_{N_i}) \left(\frac{1}{2} f_{e_N}(e_{N_i}) + g_{N_i}^+(e_{N_i}) u_i \right) \\ & - \mathcal{W}_{N_i}^T \nabla \sigma_{N_i}(\tilde{e}_{N_i}) g_{N_i}^+(e_{N_i}) \tilde{u}_i - \frac{1}{2} \nabla \varepsilon_{N_i}(\tilde{e}_{N_i}) f_{e_N}(e_{N_i}) \\ & - \nabla \varepsilon_{N_i}(\tilde{e}_{N_i}) g_{N_i}^+(e_{N_i}) u_i^* - |\mathcal{N}_i| \tilde{u}_i^T R (u_i^* + u_i), \quad (57) \end{aligned}$$

where

$$\begin{aligned} \tilde{W}_{N_i}^c = & \left[\cdots, \left(\beta_{(i,j)} \tilde{W}_e^{[x_h]} + \tilde{\beta}_{(i,j)} \hat{W}_e^{[x_h]} \right)^T, \cdots \right]^T : j \in \mathcal{N}_i, \\ = & \beta_{N_i} \otimes I_l \tilde{W}_{N_i}^e + \tilde{\beta}_{N_i} \otimes I_l \hat{W}_{N_i}^e, \quad \tilde{W}_{N_i}^e, \hat{W}_{N_i}^e \in \mathbb{R}^{l|\mathcal{N}_i|}, \quad (58) \end{aligned}$$

where $I_l \in \mathbb{R}^{l \times l}$, $\tilde{W}_{N_i}^e = \left[\cdots, \tilde{W}_e^{[x_h]^T}, \cdots \right]^T : j \in \mathcal{N}_i$, $\hat{W}_{N_i}^e = \left[\cdots, \hat{W}_e^{[x_h]^T}, \cdots \right]^T : j \in \mathcal{N}_i$, and we recast (57) to

$$\begin{aligned} \delta_{N_i} = & - \hat{W}_{N_i}^{e,T} \tilde{\beta}_{N_i}^T \otimes I_l \nabla \sigma_{N_i}(\tilde{e}_{N_i}) \left(\frac{1}{2} f_{e_N}(e_{N_i}) + g_{N_i}^+(e_{N_i}) u_i \right) \\ & - \mathfrak{K}_{\beta_i} - \mathfrak{K}_{W_i} - \mathfrak{K}_{\varepsilon_i} - \mathfrak{K}_{R_i}, \quad (59) \end{aligned}$$

where $\mathfrak{K}_{\varepsilon_i}$, $\mathfrak{K}_{W_i}$, $\mathfrak{K}_{R_i}$, and $\mathfrak{K}_{\beta_i}$ are bounded terms expressed as

$$\mathfrak{K}_{\varepsilon_i} = \frac{1}{2} \nabla \varepsilon_{N_i}(\tilde{e}_{N_i}) f_{e_N}(e_{N_i}) + \nabla \varepsilon_{N_i}(\tilde{e}_{N_i}) g_{N_i}^+(e_{N_i}) u_i^*, \quad (60a)$$

$$\mathfrak{K}_{W_i} = \mathcal{W}_{N_i}^T \nabla \sigma_{N_i}(\tilde{e}_{N_i}) g_{N_i}^+(e_{N_i}) \tilde{u}_i, \quad (60b)$$

$$\mathfrak{K}_{R_i} = |\mathcal{N}_i| \tilde{u}_i^T R (u_i^* + u_i), \quad (60c)$$

$$\mathfrak{K}_{\beta_i} = \tilde{W}_{N_i}^{e,T} \beta_{N_i}^T \otimes I_l \nabla \sigma_{N_i}(\tilde{e}_{N_i}) \left(\frac{1}{2} f_{e_N}(e_{N_i}) + g_{N_i}^+(e_{N_i}) u_i \right). \quad (60d)$$

Let $\tilde{\beta}_{(i,j)} = \beta_{(i,j)} - \hat{\beta}_{(i,j)}$, we construct Lyapunov function $\mathcal{V}_\beta$ as follows

$$\mathcal{V}_\beta = \frac{1}{2k_\beta} \sum_{i,j=1}^N \alpha_{ij} \left(\tilde{\beta}_{(i,j)} \right)^2, \quad (61)$$

and the time derivative of $\mathcal{V}_\beta$ is given by (62). Let

$$\mathfrak{R}_{N_i} = \hat{W}_{N_i}^{e,T} \tilde{\beta}_{N_i}^T \otimes I_l \nabla \sigma_{N_i}(\tilde{e}_{N_i}) \left(\frac{1}{2} f_{e_N}(e_{N_i}) + g_{N_i}^+(e_{N_i}) u_i \right), \quad (63)$$

then, (62) can be recast to

$$\dot{\mathcal{V}}_\beta = - \sum_{i=1}^N (\mathfrak{R}_{N_i})^2 - \sum_{i=1}^N \mathfrak{R}_{N_i} (\mathfrak{K}_{\beta_i} + \mathfrak{K}_{W_i} + \mathfrak{K}_{\varepsilon_i} + \mathfrak{K}_{R_i}), \quad (64)$$

and given that $\hat{W}_e^{[x_h]}$ is initialized through backward propagation in IV-A, therefore, $\forall e_{(i,j)} \in [x_h]$, $\|\hat{W}_e^{[x_h]}\| \geq \varsigma_{\hat{W}_e^{[x_h]}}$. Then, inferred from (64), there exists a bound $\varsigma_{\tilde{\beta}_{(i,j)}}$ such that $\dot{\mathcal{V}}_\beta < 0$ when $|\tilde{\beta}_{(i,j)}| > \varsigma_{\tilde{\beta}_{(i,j)}}$: $j \in \mathcal{N}_i$, and $|\tilde{\beta}_{(i,j)}|$ is thereby UUB. As $\|\tilde{W}_e^{[x_h]}\|$ is UUB proved in *Theorem 4*, $\|\tilde{W}_{(i,j)}^{[x_h]}\| = \|\beta_{(i,j)} \tilde{W}_e^{[x_h]} + \tilde{\beta}_{(i,j)} \hat{W}_e^{[x_h]}\|$ is also UUB.

Let $\tilde{S}_{(i,j)}^a = \hat{W}_e^{[x_h]} - \hat{S}_{(i,j)}^a$, the time derivative of the Lyapunov function $\mathcal{V}_{(i,j)}^a = \frac{1}{2} \tilde{S}_{(i,j)}^a T \tilde{S}_{(i,j)}^a$ can be written as shown below:

$$\begin{aligned} \dot{\mathcal{V}}_{(i,j)}^a &= - \frac{k_c k_a \left\| \delta_e^{[x_h]} \right\| \left\| \Gamma_e^{[x_h]} \omega_{(i,j)} \right\|}{\left(\lambda_e^{[x_h]} + \varepsilon_a \right) \left| 1 + \omega_{(i,j)}^T \Gamma_e^{[x_h]} \omega_{(i,j)} \right|} \tilde{S}_{(i,j)}^a T \tilde{S}_{(i,j)}^a \\ &\quad - k_c \delta_e^{[x_h]} \tilde{S}_{(i,j)}^a T \frac{\Gamma_e^{[x_h]} \omega_{(i,j)}}{1 + \omega_{(i,j)}^T \Gamma_e^{[x_h]} \omega_{(i,j)}} \\ &\leq - \frac{k_c k_a \left\| \delta_e^{[x_h]} \right\| \left\| \Gamma_e^{[x_h]} \omega_{(i,j)} \right\|}{\left(\lambda_e^{[x_h]} + \varepsilon_a \right) \left| 1 + \omega_{(i,j)}^T \Gamma_e^{[x_h]} \omega_{(i,j)} \right|} \left\| \tilde{S}_{(i,j)}^a \right\|^2 \\ &\quad + \frac{k_c \left\| \delta_e^{[x_h]} \right\| \left\| \Gamma_e^{[x_h]} \omega_{(i,j)} \right\|}{\left| 1 + \omega_{(i,j)}^T \Gamma_e^{[x_h]} \omega_{(i,j)} \right|} \left\| \tilde{S}_{(i,j)}^a \right\|, \end{aligned} \quad (65)$$

and $\dot{\mathcal{V}}_{(i,j)}^a < 0$ when $\left\| \tilde{S}_{(i,j)}^a \right\| \geq \frac{1+\varepsilon_a}{k_a}$. Thus, $\left\| \tilde{W}_{(i,j)}^a \right\| = \left\| \beta_{(i,j)} \left(\tilde{W}_e^{[x_h]} + \tilde{S}_{(i,j)}^a \right) + \tilde{\beta}_{(i,j)} \tilde{S}_{(i,j)}^a \right\|$ is UUB.

If the NN is ideal, then for $\forall \alpha_{ij} > 0$ and $[x_h] \subset \text{Capt}_{\tilde{F}_e}(K, C) \setminus \mathbb{C}$, it is obvious that $\left\| \nabla \mathcal{E}_{(i,j)}^{[x_h]}(\tilde{e}_{(i,j)}) \right\| = 0$ and $\mathfrak{K}_{\varepsilon_i} = 0$. The control error $\tilde{u}_{\pm}$ is given below:

$$\begin{aligned} \tilde{u}_{\pm} &= \arg \min_{u_{ij} \in [U_2]} \left\{ W_e^{[x_h]T} \nabla \sigma(\tilde{e}_{(i,j)}) g_e^{\pm}(e_{(i,j)}) u_{ij} + u_{ij}^T R_2 u_{ij} \right\} \\ &\quad - \arg \min_{u_{ij} \in [U_2]} \left\{ \hat{W}_e^{[x_h]T} \nabla \sigma(\tilde{e}_{(i,j)}) g_e^{\pm}(e_{(i,j)}) u_{ij} + u_{ij}^T R_2 u_{ij} \right\}. \end{aligned} \quad (66)$$

According to (44), we can deduce that

$$\begin{aligned} \dot{\mathcal{V}}_e^{[x_h]} &< -k_c \frac{\left\| \omega_{(i,j)} \right\|^2 \left\| \tilde{W}_e^{[x_h]} \right\|^2}{1 + \omega_{(i,j)}^T \Gamma_e^{[x_h]} \omega_{(i,j)}} + 2k_c \frac{\left\| \omega_{(i,j)} \right\| \left\| \tilde{W}_e^{[x_h]} \right\|}{1 + \omega_{(i,j)}^T \Gamma_e^{[x_h]} \omega_{(i,j)}} \\ &\quad \times \left(\left\| W_e^{[x_h]T} \nabla \sigma(\tilde{e}_{(i,j)}) g_e^{\pm}(e_{(i,j)}) \right\| \left\| \tilde{u}_{\pm} \right\| \right. \\ &\quad \left. + \sigma_{\max}(R_2) \left\| u_{\pm}^* + u_{ij} \right\| \left\| \tilde{u}_{\pm} \right\| \right). \end{aligned} \quad (67)$$

Then, let

$$\begin{aligned} \varsigma_{\tilde{W}_e^{[x_h]}} &= \frac{2 \left\| W_e^{[x_h]T} \nabla \sigma(\tilde{e}_{(i,j)}) g_e^{\pm}(e_{(i,j)}) \right\| \left\| \tilde{u}_{\pm} \right\|}{\left\| \omega_{(i,j)} \right\|} \\ &\quad + \frac{2\sigma_{\max}(R_2) \left\| u_{\pm}^* + u_{ij} \right\| \left\| \tilde{u}_{\pm} \right\|}{\left\| \omega_{(i,j)} \right\|}, \end{aligned} \quad (68)$$

and when $\left\| \tilde{W}_e^{[x_h]} \right\| > \varsigma_{\tilde{W}_e^{[x_h]}}$, $\dot{\mathcal{V}}_e^{[x_h]} < 0$. Thus, $\left\| \tilde{W}_e^{[x_h]} \right\|$ is ultimately bounded by $\varsigma_{\tilde{W}_e^{[x_h]}}$ weighed by $\left\| \tilde{u}_{\pm} \right\|$. By tuning k_c and k_λ , $\varsigma_{\tilde{u}_{\pm}}^{[x_h]} = \max_{e_{(i,j)} \in [x_h]} \left\| \tilde{u}_{\pm}(e_{(i,j)}) \right\|$ can be arbitrarily small after backward propagation, if the state resolution ε_X in the propagation is small enough, as the trajectories of the estimated states $\hat{\phi}$ in box $[x_h]$ discussed in (46) will be close to the optimal trajectories generated by backward propagation using PMP. Apart from that, $\hat{W}_e^{[x_h]}$ is updated by on-line reinforcement learning without increasing $\varsigma_{\tilde{u}_{\pm}}^{[x_h]}$, as $\left\| \tilde{W}_e^{[x_h]} \right\|$ asymptotically converges to its bound $\varsigma_{\tilde{W}_e^{[x_h]}}$. To summarize, a smaller set of resolutions ε_X and larger parameters

k_c and k_λ will lead to a smaller ultimate bound $\varsigma_{\tilde{W}_e^{[x_h]}}$ of $\left\| \tilde{W}_e^{[x_h]} \right\|$.

By adjusting the tuning parameter k_a , $\varsigma_{\tilde{S}_a}$ can be arbitrarily small when $t \rightarrow \infty$, such that $\hat{S}_{(i,j)}^a$ asymptotically converges to $\hat{W}_e^{[x_h]}$ when $e_{(i,j)} \in [x_h]$. Assuming that ε_X is small enough, we have

$$\begin{aligned} \forall [x_o] \subset \text{Capt}_{\tilde{F}_e}(K, C) \setminus \mathbb{C}, \quad [x_o] \cap [x_h] &\neq \emptyset, \\ \max_o \left\{ \left\| \hat{W}_e^{[x_h]} - \hat{W}_e^{[x_o]} \right\| \right\} &\rightarrow \varepsilon_{[x]}, \end{aligned} \quad (69)$$

where $\varepsilon_{[x]}$ is a positive constant which can be arbitrarily small. If k_a is sufficiently large, $\left\| \tilde{S}_{(i,j)}^a \right\|$ can be arbitrarily small for $\forall e_{(i,j)} \in \text{Capt}_{\tilde{F}_e}(K, C) \setminus \mathbb{C}$. We assume that ε_X is sufficiently small, there exist k_c , k_λ and k_a such that the control error $\tilde{u}_i$ can be arbitrarily small. Then, based on the result of (64), the convergence of $\left\| \tilde{\beta}_{\mathcal{N}_i} \right\|$ reduces $\left\| \tilde{u}_i \right\|$, and meanwhile the bounded terms $\mathfrak{K}_{\beta_i}$, $\mathfrak{K}_{W_i}$, $\mathfrak{K}_{\varepsilon_i}$, and $\mathfrak{K}_{R_i}$ decrease due to the decay of $\left\| \tilde{u}_i \right\|$. When these bounded terms are eliminated, $\dot{\mathcal{V}}_\beta < 0$ always holds, and $\left\| \tilde{\beta}_{\mathcal{N}_i} \right\| \rightarrow 0$. Given that

$$\begin{aligned} \left\| \tilde{W}_{\mathcal{N}_i}^c \right\| &\leq \left\| \beta_{\mathcal{N}_i} \right\| \left\| \tilde{W}_{\mathcal{N}_i}^e \right\| + \left\| \tilde{\beta}_{\mathcal{N}_i} \right\| \left\| \hat{W}_{\mathcal{N}_i}^e \right\|, \\ \left\| \tilde{W}_{\mathcal{N}_i}^a \right\| &\leq \left\| \beta_{\mathcal{N}_i} \right\| \left\| \tilde{S}_{\mathcal{N}_i}^a \right\| + \left\| \tilde{\beta}_{\mathcal{N}_i} \right\| \left\| \hat{S}_{\mathcal{N}_i}^a \right\|, \end{aligned} \quad (70)$$

where $\tilde{S}_{\mathcal{N}_i}^a = [\dots, W_e^{[x_h]} - \hat{S}_{(i,j)}^a, \dots]^T : j \in \mathcal{N}_i$. Thus $\left\| \tilde{W}_{\mathcal{N}_i}^c \right\|$ and $\left\| \tilde{W}_{\mathcal{N}_i}^a \right\|$ can be arbitrarily small, and u_i ultimately converge to u_i^* . ■

V. SIMULATION RESULTS

In this section, the simulations of autonomous vehicle networks are demonstrated. Consider the kinematic model of the i th vehicle system in the global coordinate frame:

$$\dot{x}_i = v_i \cos(\theta_i), \quad (71a)$$

$$\dot{y}_i = v_i \sin(\theta_i), \quad (71b)$$

$$\dot{\theta}_i = w_i, \quad (71c)$$

where $u_i = [v_i, w_i]^T \in [U]$ and $x_i, y_i, \theta_i, v_i, w_i \in \mathbb{R}$. Then, let $e_{(i,j)} = [x_{(i,j)}, y_{(i,j)}, \theta_{(i,j)}]^T$, and the relative error system for two connected vehicles is expressed as

$$g_e^\#(e_{(i,j)}) = \begin{bmatrix} -1 & y_{(i,j)} \\ 0 & -x_{(i,j)} \\ 0 & -1 \end{bmatrix}, \quad g_e^b(e_{(i,j)}) = \begin{bmatrix} \cos(\theta_{(i,j)}) & 0 \\ \sin(\theta_{(i,j)}) & 0 \\ 1 & 0 \end{bmatrix}, \quad (72a)$$

$$\dot{e}_{(i,j)} = g_e^+(e_{(i,j)}) u_i + g_e^-(e_{(i,j)}) u_j. \quad (72b)$$

According to (10), $\psi_C(e_{(i,j)})$, $Q(e_{(i,j)})$ and R_2 are selected as shown below:

$$\begin{aligned} \psi_C(e_{(i,j)}) &= 125 \text{sign}(x_{(i,j)}) (x_{(i,j)})^3 + 625 (y_{(i,j)})^4 \\ &\quad + 6.32 \text{sign}(\theta_{(i,j)}) (\theta_{(i,j)}) - 1, \end{aligned} \quad (73a)$$

$$Q(e_{(i,j)}) = e_{(i,j)}^T \begin{bmatrix} 0.25 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.1621 \end{bmatrix} e_{(i,j)}, \quad (73b)$$

$$R = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}, \quad (73c)$$

Fig. 2. Inner approximation of the capture basin $\text{Capt}_{\tilde{F}_e}(K, C)$.

Fig. 3. Switching communication graph $A(t)$.

and the control constraint $[U]$ and the communication zone K are determined as

$$[U] = \begin{bmatrix} 0.02 & 0.2 & -\frac{\pi}{10} & \frac{\pi}{10} \end{bmatrix}^T, \quad (74a)$$

$$K = \left\{ e_{(i,j)} \in X \mid \frac{1}{25} (x_{(i,j)})^2 + \frac{1}{25} (y_{(i,j)})^2 < 1 \right\}. \quad (74b)$$

and the resolution in backward propagation is set to $\varepsilon_X = [0.05, 0.05, \frac{\pi}{60}]^T$. The approximation result of the capture basin is depicted by its vertices shown in Fig. 2.

The switching graph is depicted in Fig. 3, in which $\alpha_{(i,j)}$ represents the undirected communication link between v_i and v_j . Fig. 4 shows the 3-D plot of the states of five vehicles by the distributed control with the cooperative learning, and they converge at around $t = 39s$. The convergences of $\beta_{(i,j)}$ are shown in Fig. 5(a), and all the $\beta_{(i,j)}$ converge at $t = 300s$. The convergences of bellman errors δ_{N_i} is shown in Fig. 5(b).

Given the case in Fig. 4, Fig. 6(a) illustrates the comparison of the global accumulative costs $J(T, e(0), u)$ and $\mathbf{J}(T, e(0), \mathbf{u})$, specified in (14), corresponding to the algorithms with and without the

Fig. 4. The state convergences of five vehicles by the distributed control with the cooperative learning.

(a)

(b)

Fig. 5. Convergences of $\beta_{(i,j)}$ and bellman errors δ_{N_i} : (a) Convergences of $\beta_{(i,j)}$; (b) Convergences of bellman errors δ_{N_i} .

cooperative learning, respectively. Fig. 6(b) illustrates the global accumulative cost comparisons of four groups of simulations for different numbers of agents, and each group contains ten simulation cases. Defining $\mathbf{T} = \inf \{ T \in (0, \infty) \mid \forall \alpha_{ij} = 1, e_{(i,j)}(T) \in \underline{\mathbb{C}} \}$ for a pair of cases to compare, the symbols represent $\frac{J(\mathbf{T}, e(0), \mathbf{u})}{\mathbf{J}(\mathbf{T}, e(0), \mathbf{u})}$.

VI. CONCLUSION AND FUTURE WORK

Based on the viability theory, we have proposed a novel distributed control protocol for nonlinear systems with bounded control by formulating distributed control problems to differential graphical games. Our algorithm involves the game theory, optimal theory, viability theory, interval methods, reinforcement learning technique, etc. We adopted the reinforcement learning technique

Fig. 6. Global cost comparison between the algorithms with and without the cooperative learning: (a) Cost comparison of a pair of cases; (b) Cost comparisons of four groups of simulations for different numbers of agents.

when approximating the capture basin. The off-line computation of the capture basin generates a priori knowledge and characterizes the feasible learning scope for the on-line reinforcement learner. By using on-line cooperative reinforcement learning, the dimensionality issue of implementing the tools of the viability theory has been overcome. We approximated the Pareto optimality of the graphical game rather than the Nash equilibrium, thus achieving an optimal convergence cost of states in a global sense. Our future work will focus on differential graphical games among groups of vehicles with uncertainties, deep distributed reinforcement learning, and the viability property of complex network systems.

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