Asymptotic path-independent integrals for the evaluation of crack-tip parameters in a neo-Hookean material

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Abstract

In this paper, we develop new asymptotic path-independent integrals for the evaluation of the crack tip parameters in a 2D neo-Hookean material. The new integrals are of both J -integral and interaction energy integral type and rely on the separation of the asymptotic boundary value problem into independent problems for each of the deformed coordinates. Both the plane stress and plane strain cases are considered. The integrals developed are used to compute the amplitude parameters of the asymptotic crack tip fields, which allows for direct extraction of these parameters from numerical results. A long strip with an edge crack under mixed loading modes is considered for both homogeneous and biomaterial cases. It is found that the asymptotic J -integrals produce good results for the first-order parameters while the interactions integrals produce good results for both the first and second-order parameters.

Keywords: Asymptotic path-independent integrals; Crack tip fields; neo-Hookean materials; Interface crack

1 Introduction

It has become increasingly important to fabricate soft materials with high strength and toughness to fulfill the requirements of the extraordinary mechanical performance in the design of soft robotics

and artificial soft tissues (Gong 2010; Sun et al. 2012). To understand fracture behavior, a key problem is to quantitatively reveal the crack tip fields for given material microstructure, so as to determine crack initiation and propagation criteria and develop suitable strategies to enhance the toughness.

As a subclass of fracture in soft materials, the crack problem in hyperelastic solids has been extensively studied in the past five decades, mostly focused on asymptotic analysis of the crack tip fields within the framework of finite elasticity. In such analyses, the deformed crack tip coordinates are usually assumed to take the form of a product of two single-variable functions of the radius and polar angle in a polar coordinate system with the crack tip at the origin of a reference coordinate system. Then, the governing equations of the crack tip are reduced to an eigenvalue problem for the angular functions. Wong and Shield (1969) investigated a crack problem in a biaxially stretched neo-Hookean sheet with large stretch throughout the entire domain. The crack tip fields for a hyperelastic solid based on the asymptotic analysis were derived by Knowles and Sternberg (1973, 1974) where the first- and high-order asymptotic analyses were conducted under the condition of plane strain and a closed form solution for the deformed coordinate was found by solving a nonlinear eigenvalue problem. A similar approach was pursued in the analysis of cracks in an incompressible generalized neo-Hookean sheet (Geubelle and Knauss 1994a; Long et al. 2011), where the effects of strain hardening behavior at a large strain on the crack fields was considered. Stephenson (1982) studied the crack tip fields of an incompressible Mooney-Rivlin material under the condition of plane strain, and recently his solution was extended to a variety of compressible elastomers (Begley et al. 2015). The bimaterial interface crack problem in hyperelastic solids was considered by Knowles and Sternberg (1983) for the case of neo-Hookean materials under plane stress, Herrmann (1989) for the plane strain interface crack, Ravichandran and Knauss (1989) for a finite element study of the neo-Hookean crack problem, Geubelle and Knauss (1994b) for Generalized neo-Hookean materials in plane stress, and Lengyel et al. (2016) for an interface crack between a compressible elastomer and a rigid substrate with finite slippage. Other related work on crack problems within the framework of finite elasticity includes the anti-plane shear field near the tip of a crack (Knowles 1977; Wu 2017), and 3D crack tip field formulations for a neo-Hookean incompressible material composed of a plane deformation and an anti-plane one (Arfaoui et al. 2018). A review of crack tip fields in hyperelastic materials subjected to large quasi-static deformation can be found in (Long and Hui 2015). In these works, the mathematical structure of the crack tip fields is revealed, which plays an essential role in deriving the energy release rate at the crack tip (or the J -integral) and understanding the main features for the

crack tip fields in different materials and crack modes. For example, the Cauchy stress shows r^{-1} singularity near the crack tip, rather than the $r^{-1/2}$ singularity for linear elastic materials.

Based on the asymptotic analysis, the solutions for the deformed coordinates are expressed as linear combinations of the eigenvectors, whose amplitude coefficients are determined by the far-field boundary conditions. In several previous works, these amplitude parameters were related to the path-independent J -integral through the asymptotic solutions (Knowles and Sternberg 1983; Ravichandran and Knauss 1989; Long et al. 2011). Usually, there are more than one undetermined parameters in the crack tip fields, thus the J -integral is not sufficient to determine all the parameters. Chang and Li (2004) introduced a J_2 -integral in addition to the original J -integral, and defined two coordinate-based P -integrals based on the asymptotic displacement fields for the crack in neo-Hookean sheet. In Ravichandran and Knauss (1989), a numerical fitting method was used to identify the amplitude parameters for the neo-Hookean interface crack field. These methods can produce good estimates of the amplitude parameters if the finite element mesh near the crack tip is well designed so that the path chosen for numerical fitting suitably passes through the nodes or Gauss points of the finite element mesh. This is not usually the case for unstructured meshes, where a suitable path for numerical fitting or calculating the P -integral are not available. Other methods employed in linear elasticity either require development of special element techniques (Su and Feng 2005), or consist of numerical fitting to determine the higher-degree coefficients (Ayatollahi and Nejati 2011; Mirzaei et al. 2019). Their applications in finite elasticity remain to be studied. Thus, we seek a robust method to evaluate these amplitude parameters.

In this paper, we take the neo-Hookean material as an example to explore efficient path-independent integrals for evaluation of the crack tip parameters. As mentioned earlier, the asymptotic solution for the crack tip fields of an incompressible neo-Hookean material in the plane stress condition was obtained by Knowles and Sternberg (1983), and the solution in the plane strain case was reported in Stephenson (1982). Based on the results in Knowles and Sternberg (1983), we developed several new asymptotic path-independent integrals for the evaluation of crack tip parameters. We consider both plane stress and plane strain conditions. For the plane stress case, we consider both cracks in a homogeneous material and interface cracks and for the plane strain case, homogeneous materials only. Specifically, we develop new asymptotic integrals of J -integral type and also of in-

interaction energy integral type. These integrals are derived based on the observation that the asymptotic governing equations decompose into independent equations for each of the deformed coordinates y_1 and y_2 . The integrals of J -integral type follow usual procedures, but for each degree of freedom, separately and thus give rise to individual expressions for two of the four parameters that characterize the first and second order terms in the asymptotic expressions. The interaction energy integrals follow the approach in (Nakamura and Parks 1992) for evaluating the elastic T-stress term along three-dimensional crack fronts and permit the four crack tip parameters to be evaluated. Domain integrals formulations (Li et al. 1985; Shih et al. 1986; Moran and Shih 1987) are presented and discussed. Both homogenous and biomaterial interface crack problems for the plane stress conditions are considered for a long strip under mixed loading modes. We consider also the homogeneous crack problem for the plane strain case. The deformation coordinates and stress evaluated by the predicted crack tip parameters agree well with the finite element results, showing the usefulness of the proposed approach.

This paper is organized as follows. In Section 2, we review the asymptotic crack tip fields of a neo-Hookean material obtained by Knowles and Sternberg (1983) and Stephenson (1982). In Section 3, we introduce several asymptotic path-independent integrals to evaluate the crack tip parameters for both the plane stress and plane strain. Domain integration procedures for the new integrals are then discussed. In Section 4, a long strip with an edge crack is introduced as an example to verify the usefulness of integrals for computing the crack tip parameters. Discussions for cases with different loading modes and material constants are given.

2 Asymptotic crack tip fields for neo-Hookean sheets

2.1 The governing equations of plane-stress finite elasticity

Following Knowles and Sternberg (1983) and using similar notation, we consider a thin sheet of material occupying the three-dimensional undeformed region, $\mathfrak{R}$, i.e.,

$$\mathfrak{R} = \{\mathbf{x} \mid (x_1, x_2) \in \Pi, -\frac{1}{2}t \leq x_3 \leq \frac{1}{2}t\} \quad (1)$$

where Π is the cross-section of $\mathfrak{R}$ in the plane $x_3 = 0$, and t is the constant thickness of the sheet in the reference configuration. Then, the deformation gradient is written as $\mathbf{F} = (\nabla \mathbf{y})^T$, where

$\nabla = \mathbf{e}_i \partial / \partial x_i$ is the gradient operator with respect to the undeformed coordinates, and $\mathbf{y} = \mathbf{y}(\mathbf{x})$ are the deformed coordinates. For incompressible materials, we have the volume-preserving constraint, $\det \mathbf{F} = 1$. The equilibrium equation is written as

$$\nabla \cdot \mathbf{P}^T = 0 \quad (2)$$

where $\mathbf{P}$ is the first Piola-Kirchhoff stress which is related to the Cauchy (true) stress $\boldsymbol{\sigma}$ by $\boldsymbol{\sigma} = \mathbf{P} \cdot \mathbf{F}^T$.

We will confine our attention to the neo-Hookean material, for which the strain energy function is given by

$$W = \frac{1}{2} \mu (I_1 - 3) \quad (3)$$

where μ is the shear modulus, I_1 is the first invariable of the right Cauchy-Green deformation tensor $\mathbf{C}$, i.e., $I_1 = \text{tr} \mathbf{C} = \text{tr}(\mathbf{F}^T \mathbf{F})$. The first Piola-Kirchhoff stress can be obtained by the constitutive equation

$$\mathbf{P} = \frac{\partial W}{\partial \mathbf{F}} - p \mathbf{F}^{-T} = \mu \mathbf{F} - p \mathbf{F}^{-T} \quad (4)$$

where the second term is a result of the incompressibility constraint and p is a Lagrange multiplier.

In a state of plane stress, we assume that the deformation is symmetric about the plane $x_3 = 0$. Let $F_{33} = \lambda$ and $J = \det [F_{\alpha\beta}]$, $\alpha, \beta = 1, 2$, then, from the incompressibility constraint,

$$J = \lambda^{-1}, \quad C_{33}^{-1} = \lambda^{-2}, \quad F_{\alpha 3} = F_{3\alpha} = 0 \quad (5)$$

From the condition $P_{33} = 0$ and Eq. (4), it follows that $p = \mu \lambda^2$, and the stresses are given by

$$P_{\alpha\beta} = \mu F_{\alpha\beta} - \mu \lambda^2 F_{\beta\alpha}^{-1} = \mu y_{\alpha,\beta} - \mu \lambda^3 \varepsilon_{\alpha u} \varepsilon_{\beta v} y_{u,v} \quad (6)$$

where $\varepsilon_{\alpha\beta}$ is the two-dimensional alternating symbol ($\varepsilon_{11} = \varepsilon_{22} = 0, \varepsilon_{12} = -\varepsilon_{21} = 1$). Substituting Eq. (6) into the equilibrium equation $P_{\alpha\beta,\beta} = 0$ and dividing by μ , we get

$$y_{\alpha,\beta\beta} - \lambda^3 \varepsilon_{\beta u} \varepsilon_{\alpha v} y_{u,v\beta} - 3\lambda^2 \lambda_{,\beta} \varepsilon_{\beta u} \varepsilon_{\alpha v} y_{u,v\beta} = 0 \quad (7)$$

which is a nonlinear partial differential equation for y_α . The nominal tractions t_α on a surface with outward normal n_β are given by

$$t_\alpha = P_{\alpha\beta} n_\beta = \mu [y_{\alpha,\beta} n_\beta - \lambda^3 \varepsilon_{\beta u} \varepsilon_{\alpha v} y_{u,v} n_\beta] \quad (8)$$

2.2 Asymptotic solutions for the plane stress case

Referring to Figure 1, the origin of the coordinate system is attached to the crack tip during deformation process but it does not rotate as the material deforms. A polar coordinate system (r, θ) is centered on crack tip, which can be related to the Cartesian coordinates by $x_1 = r \cos \theta$ and $x_2 = r \sin \theta$.

Figure 1 Sketch of the crack coordinate system

The out of plane stretch exhibits the asymptotic form $\lambda = J^{-1} = O(r^q)$ as $r \rightarrow 0$ (Wong and Shield 1969; Knowles and Sternberg 1983). This implies that some of the components of the deformation gradient will be unbounded which will simplify the asymptotic forms of the governing equations as we retain leading order terms only. In particular, the out of plane stretch $\lambda \rightarrow 0$ as $r \rightarrow 0$. Because of this, for the lowest order analysis, the stress, the governing equation and the traction free boundary conditions on the crack surfaces can be written as

$$P_{\alpha\beta} = \mu y_{\alpha,\beta} \quad (9)$$

$$y_{\alpha,\beta\beta} = 0 \quad (10)$$

$$y_{\alpha,\beta} n_\beta = 0, \text{ at } x_1 \leq 0, x_2 = 0 \quad (11)$$

The Laplace equation (10) together with the boundary condition (11) will be transformed into an eigenvalue problem whose solution is given by $y_\alpha = r^{m/2} v(\theta)$, where $v(\theta) = \sin(\frac{m}{2}\theta)$ for $m=1,3,5,\dots$ and $v(\theta) = \cos(\frac{m}{2}\theta)$ for $m=2,4,6,\dots$ (Knowles and Sternberg 1983). Taking the two lowest order solutions for $m=1$ and 2 gives rise to deformed coordinates near the crack tip

$$y_\alpha \sim p_\alpha r^{1/2} \sin \frac{\theta}{2} + q_\alpha r \cos \theta, \quad \alpha = 1, 2 \quad (12)$$

where $p_\alpha, q_\alpha, \alpha = 1, 2$ are the crack tip parameters determined by far-field boundary conditions. In the front of the crack tip with $\theta = 0$, the deformed coordinate y_1 satisfies

$$y_1|_{\theta=0} \sim q_1 r > 0 \Rightarrow q_1 > 0, \quad (13)$$

For a biomaterial interface crack problem, we assume the shear modulus for the materials above and below the crack line as $\mu_1 = \mu$ and $\mu_2 = \mu/s$, respectively, where s is a modulus ratio satisfying $s = \mu_1/\mu_2$. Then, the expression for y_α should be modified as (Knowles and Sternberg 1983)

$$y_\alpha \sim p_\alpha h(\theta) r^{1/2} \sin \frac{\theta}{2} + q_\alpha r \cos \theta, \quad \alpha = 1, 2 \quad (14)$$

where $h(\theta)$ is jump function defined as $h(\theta) = 1$ for $\theta \in (0, \pi]$ and $h(\theta) = s$ for $\theta \in [-\pi, 0)$.

Using Eq. (12), the asymptotic forms for the components of $\mathbf{F}$ can be derived as

$$\begin{aligned} F_{11} &\sim -\frac{1}{2} p_1 h(\theta) r^{-1/2} \sin \frac{\theta}{2} + q_1, & F_{12} &\sim \frac{1}{2} p_1 h(\theta) r^{-1/2} \cos \frac{\theta}{2} \\ F_{21} &\sim -\frac{1}{2} p_2 h(\theta) r^{-1/2} \sin \frac{\theta}{2} + q_2, & F_{22} &\sim \frac{1}{2} p_2 h(\theta) r^{-1/2} \cos \frac{\theta}{2} \end{aligned} \quad (15)$$

and using Eqs. (3), (5) and (9), the components of the first Piola-Kirchhoff stress, the Jacobian, the Cauchy stress and strain energy function are given by

$$P_{\alpha\beta}/\mu \sim F_{\alpha\beta} \quad (16)$$

$$J \sim \frac{1}{2} h(\theta) r^{-1/2} (p_2 q_1 - p_1 q_2) \cos \frac{\theta}{2} \quad (17)$$

$$\begin{aligned} \sigma_{11}/\mu &\sim \frac{1}{4} p_1^2 h(\theta) r^{-1} - p_1 q_1 r^{-1/2} \sin \frac{\theta}{2} + q_1^2/h(\theta) \\ \sigma_{22}/\mu &\sim \frac{1}{4} p_2^2 h(\theta) r^{-1} - p_2 q_2 r^{-1/2} \sin \frac{\theta}{2} + q_2^2/h(\theta) \\ \sigma_{12}/\mu &\sim \frac{1}{4} p_1 p_2 h(\theta) r^{-1} - \frac{1}{2} r^{-1/2} (p_2 q_1 + p_1 q_2) \sin \frac{\theta}{2} + q_1 q_2/h(\theta) \end{aligned} \quad (18)$$

$$W/\mu \sim \frac{1}{8} h(\theta) r^{-1} (p_1^2 + p_2^2) - \frac{1}{2} (p_1 q_1 + p_2 q_2) r^{-1/2} \sin \frac{\theta}{2} + \frac{1}{2h(\theta)} (q_1^2 + q_2^2) - \frac{3}{2} \quad (19)$$

By setting s to 1 in Eqs. (15)~(19), we can obtain the corresponding solutions for the homogenous neo-Hookean material.

2.3 Asymptotic solutions for the plane strain case

For the plane strain case, $F_{33} = 1$, and the strain energy function and constitutive equations for the neo-Hookean material are given by (e.g., Begley et al. 2015)

$$W = \frac{1}{2} \mu (I_1 - 2) - p(J - 1) \text{ and } P_{\alpha\beta} = \mu F_{\alpha\beta} - p(F_{\eta\eta} \delta_{\alpha\beta} - F_{\alpha\beta}) \quad (20)$$

where $I_1 = F_{\alpha\beta} F_{\alpha\beta}$, $J = \det[F_{\alpha\beta}]$, $\alpha, \beta = 1, 2$ and p is a Lagrange multiplier. The corresponding asymptotic fields at the crack tip follows (Stephenson 1982)

$$\begin{cases} y_1 \sim a_1 r^{1/2} \sin \frac{\theta}{2} + \frac{r}{a^2} \left[a_1 b_1 \cos \theta - a_2 b_2 \sin^2 \frac{\theta}{2} \right] \\ y_2 \sim a_2 r^{1/2} \sin \frac{\theta}{2} + \frac{r}{a^2} \left[a_1 b_2 \sin^2 \frac{\theta}{2} + a_2 b_1 \cos \theta \right] \\ p \sim -\frac{2\mu b_2}{a^2} r^{1/2} \cos \frac{\theta}{2} \end{cases} \quad (21)$$

where $a_\alpha, b_\alpha, \alpha = 1, 2$ are the parameters to be determined, and $a^2 = a_1^2 + a_2^2$. It is convenient, for our subsequent discussion, to also write the asymptotic solution for y_1 and y_2 as

$$\begin{cases} y_1 \sim p_1 r^{1/2} \sin \frac{\theta}{2} + q_1 r \cos \theta - \frac{p_2}{a^2} (p_2 q_1 - p_1 q_2) r \\ y_2 \sim p_2 r^{1/2} \sin \frac{\theta}{2} + q_2 r \cos \theta + \frac{p_1}{a^2} (p_2 q_1 - p_1 q_2) r \end{cases} \quad (22)$$

where

$$p_\alpha = a_\alpha, \quad q_1 = \frac{1}{2a^2} (2a_1 b_1 + a_2 b_2), \quad q_2 = \frac{1}{2a^2} (2a_2 b_1 - a_1 b_2) \quad (23)$$

It is observed that Eq. (22) has the same form as Eq. (12) for the solution of the plane stress case, except for an additional r term in the plane strain case. We note also that for $y_\alpha, \alpha = 1, 2$ in Eq. (22),

$$y_{\alpha, \beta\beta} = B_\alpha r^{-1} \quad (24)$$

where $B_1 = -(a_2 b_2)/(2a^2)$, and $B_2 = (a_1 b_2)/(2a^2)$. Thus, the fields (21) or (22) do not satisfy the Laplace equation, but rather, include a term of order r^{-1} . We will refer this result later in Section 3.3.

3 Crack tip integrals

3.1 Global path-independent integrals

For a homogeneous hyperelastic material with traction free crack surface, zero body forces and quasistatic loading, the path-independent J -integral is expressed as (Rice 1968; Knowles and Sternberg 1973; Shih et al. 1986)

$$J_1 = \int_{\Gamma_1} (W \delta_{1j} - P_{ij} y_{i,1}) n_j d\Gamma, \quad i, j = 1, 2 \quad (25)$$

where Γ_1 is a contour with arbitrary shape and location surrounding the crack tip (Figure 2), n_j is its unit outer normal, and the integrand $W \delta_{1j} - P_{ij} y_{i,1}$ is divergence free. For purposes of description in the paper, we will refer to this type of path-independence as global path-independence. The integral

(25) is path independent and its value does not depend on a limiting process in which the crack tip contour Γ_1 is shrunk onto the crack tip (Shih et al. 1986).

Figure 2 Integral contour $\Gamma = \Gamma_1 - \Gamma_2 + \Gamma^+ + \Gamma^-$ encloses a simply connected region surrounding the crack tip

Using the asymptotic crack tip fields in Eqs. (15)~(19), it can be shown that the path-independent J_1 integral on a circular contour with radius r (as $r \rightarrow 0$) gives the results

$$J_1 = \int_{-\pi}^{\pi} (W \delta_{1j} - P_{ij} y_{i,1}) n_j r d\theta = \frac{\mu\pi}{4} (p_1^2 + p_2^2) \quad (26)$$

where the component of the unit outer normal n_j are given as $n_1 = \cos \theta$ and $n_2 = \sin \theta$. By choosing a specific far-field contour as Γ_1 , the two parameters p_1 and p_2 in the J -integral can be related to the far-field boundary conditions (see the example in Section 4). For the symmetric case, the crack tip fields are symmetric about $x_2 = 0$ and the coefficient p_1 of the first leading order terms is zero. Eq. (26) thus reduces to $J_1 = \frac{\mu\pi}{4} p_2^2$. In this case, p_2 can be directly computed with J_1 . In general, we cannot use the standard J -integral in Eq. (26) to extract the parameters p_1 and p_2 separately since they come as a combined quantity. For mixed crack modes, at a minimum, four such integrals are required to evaluate all the four parameters p_α and q_α , $\alpha = 1, 2$ in Eq. (12).

The above holds for plane stress or plane strain cases. In the latter case, the notation $J_1 = \frac{\mu\pi}{4}(a_1^2 + a_2^2)$ is used primarily, see Chang and Li (2004) for example.

Replacing the derivatives with respect to x_1 in J_1 with those with respect to x_2 , the J_2 integral is given by

$$J_2 = \int_{\Gamma_1} (W\delta_{2j} - P_{ij}y_{i,2})n_j d\Gamma, \quad i, j = 1, 2 \quad (27)$$

Though the integrand is divergence free, this integral is not path-independent in the global sense describe above, because, unlike the case for J_1 , the integrand does not vanish on the crack surfaces Γ^+ and Γ^- , e.g., $(W\delta_{2j} - P_{ij}y_{i,2})n_j = -W$ on Γ^+ . When the strain energy W is the same on the top and bottom crack faces (not true in general case), and the contour starts and ends on opposite points on the crack faces, global path independence is obtained.

In the general case, by specifying a circular contour with radius r_0 as Γ_1 within the zone of dominance of the asymptotic crack tip fields (14) and using the relations in Eqs. (15)~(19), we can still calculate J_2 as

$$J_2 = -2\mu(p_1q_1 + p_2q_2)r_0^{1/2} \quad (28)$$

Thus the J_2 integral can be used, but without full power to choose an arbitrary integration path. It can be easily seen that $J_2 \rightarrow 0$ as $r_0 \rightarrow 0$ so J_2 vanishes as the contour is shrunk to the crack tip. In the plane strain case, using Eqs. (20) and (22) in Eq. (27), we can obtain the same expression for J_2 as that in Eq. (28) (also shown in Chang and Li (2004)), i.e., $J_2 = -2\mu(p_1q_1 + p_2q_2)r_0^{1/2} = -2\mu b_1 r_0^{1/2}$, where the relations in Eq. (23) are used.

At this point, the expressions $\frac{\mu\pi}{4}(p_1^2 + p_2^2)$ for J_1 and $-2\mu(p_1q_1 + p_2q_2)r_0^{1/2}$ for J_2 are insufficient to compute the four parameters p_α and $q_\alpha, \alpha = 1, 2$. Additional relations are needed. For that purpose, we develop a set of asymptotic path-independent integrals that permit each of p_α and $q_\alpha, \alpha = 1, 2$ to be computed separately.

3.2 Asymptotic path-independent integrals for the plane-stress case

The J-integral method. Referring to the asymptotic boundary value problem (10) and (11), the governing equations separate into two independent Laplace equations for each degree of freedom y_1 and y_2 . Based on this observation, we construct asymptotic J_1 integrals for y_1 and y_2 separately as $r \rightarrow 0$, i.e.,

$$J_1^{y_\alpha} = \lim_{\Gamma_1 \rightarrow 0} \int_{\Gamma_1} \left(W^{y_\alpha} \delta_{1j} - P_j^{y_\alpha} y_{\alpha,1} \right) n_j d\Gamma, \quad \alpha = 1, 2, \text{ no summation on } \alpha \quad (29)$$

where W^{y_α} and $P_j^{y_\alpha}$ satisfy

$$W^{y_\alpha} = \frac{1}{2} \mu (y_{\alpha,1}^2 + y_{\alpha,2}^2) \text{ and } P_j^{y_\alpha} = P_{\alpha j} \quad (30)$$

which we refer to as the asymptotic strain energy function and Piola-Kirchhoff stresses, respectively, and which hold in the region dominated by the asymptotic solution. The divergence of the integrand in Eq. (29) satisfies

$$\begin{aligned} & \left(W^{y_\alpha} \delta_{1j} - P_j^{y_\alpha} y_{\alpha,1} \right)_{,j} \\ &= \mu y_{\alpha,1j} y_{\alpha,j} - P_{j,j}^{y_\alpha} y_{\alpha,1} - P_j^{y_\alpha} y_{\alpha,1j} \quad (\text{Eq. (6) and } P_{j,j}^{y_\alpha} = P_{\alpha j,j} = 0) \\ &= \mu \lambda^3 \varepsilon_{\alpha u} \varepsilon_{\beta v} y_{u,v} y_{\alpha,\beta 1} \end{aligned} \quad (31)$$

Using (5), (17) and (14), it can be shown that $\lambda \sim r^{1/2}$, $y_{u,v} \sim r^{-1/2}$, $y_{\alpha,\beta 1} \sim r^{-3/2}$ and thus $\mu \lambda^3 \varepsilon_{\alpha u} \varepsilon_{\beta v} y_{u,v} y_{\alpha,\beta 1} \sim r^{-1/2}$, which means that Eq. (31) is not sufficiently singular to contribute to an area integral as $r \rightarrow 0$, i.e.,

$$\begin{aligned} & \lim_{r \rightarrow 0} \int_A \left(W^{y_\alpha} \delta_{1j} - P_j^{y_\alpha} y_{\alpha,1} \right)_{,j} dA \\ &= \lim_{r \rightarrow 0} \int_A \mu \lambda^3 \varepsilon_{\alpha u} \varepsilon_{\beta v} y_{u,v} y_{\alpha,\beta 1} dA \sim \lim_{r \rightarrow 0} \int_A r^{-1/2} f(\theta) dA \sim r^{3/2} \sim 0, \quad r \rightarrow 0 \end{aligned} \quad (32)$$

where the particular form of $f(\theta)$ is not required for the argument. Thus, the integrals (29) are asymptotically path-independent, i.e., the integrands are asymptotically divergence free, vanishes on

the crack faces and the integrals are independent of the shape and location of the contour Γ_1 so long as it is taken in the region of dominance of the singular fields.

By evaluating the integrals $J_1^{y_\alpha}$ over circular contours as $r \rightarrow 0$ and using Eq. (12), we obtain the expressions

$$J_1^{y_\alpha} = \frac{\mu\pi}{4} p_\alpha^2, \quad \alpha = 1, 2 \quad (33)$$

Thus, the integrals $J_1^{y_\alpha}$ can be used to extract the crack tip parameters p_1 and p_2 , separately. It is noted that, recalling Eq. (26), $J_1 = J_1^{y_1} + J_1^{y_2}$.

In a similar way, the J_2 integral can be used, separately, for each deformed coordinate, i.e.,

$$J_2^{y_\alpha} = \lim_{r \rightarrow r_0} \int_{\Gamma_1} \left(W^{y_\alpha} \delta_{2j} - P_j^{y_\alpha} y_{\alpha,2} \right) n_j d\Gamma, \quad \alpha = 1, 2, \text{ no summation on } \alpha \quad (34)$$

By a similar argument to that made about $J_1^{y_\alpha}$, the integral $J_2^{y_\alpha}$ is shown to be asymptotically divergence free. Similarly also to the case of the J_2 integral, the integral $J_2^{y_\alpha}$ is not fully path-independent (even in the asymptotic sense) due to the contribution of strain energy W^{y_α} along the crack faces. And similarly again, $J_2^{y_\alpha}$ is useful if a circular contour is used to generate an expression in terms of the parameters q_α , i.e.,

$$J_2^{y_\alpha} = -2\mu p_\alpha q_\alpha r_0^{1/2}, \quad \alpha = 1, 2, \text{ no summation on } \alpha \quad (35)$$

Considering Eq. (28), we have $J_2 = J_2^{y_1} + J_2^{y_2}$. Thus, using Eqs. (33) and (35), the parameters p_α and q_α , $\alpha = 1, 2$ can be computed based on, for example, finite element results in the crack-tip region governed by the asymptotic fields.

The interaction energy integral method. For the case $p_1 = 0$, which occurs for a Mode I crack, for example, the expression (35) is identically zero for $\alpha = 1$ and thus fails to yield an expression for the parameter q_1 and so alternatives must be found. In this case, we seek a suitable interaction energy integral to resolve the issue. In the work by Nakamura and Parks (1992), in order to extract the T

stress distribution along three dimensional crack front, a local conservation interaction energy integral was defined, where the crack tip solution was superimposed with an auxiliary field due to a line load at the crack tip. Such an integral technique was also found useful to compute the mixed-mode stress intensity factors in elastic crack fields (Nakamura and Parks 1989). By again considering the separable nature of the asymptotic solution (12) (e.g. a Laplace equation for each y_1 and y_2), we can develop interaction energy integrals, which are path-independent asymptotically and which can be used to find expressions for the crack tip parameters, i.e.,

$$I^{y_\alpha} = \lim_{\Gamma_1 \rightarrow 0} \int_{\Gamma_1} \left(P_j^{y_\alpha} y_{,j}^{\text{aux}} n_1 - P_j^{y_\alpha} y_{,1}^{\text{aux}} n_j - P_j^{\text{aux}} y_{\alpha,1} n_j \right) d\Gamma \quad \text{for } \alpha = 1, 2 \quad (36)$$

where y^{aux} is an auxiliary field satisfying the asymptotic governing equations (10) and (11), the auxiliary stress field is given by $P_j^{\text{aux}} = \mu y_{,j}^{\text{aux}}$, and $P_j^{y_\alpha} = P_{\alpha j}$ is shown in Eq. (6). It can be shown that the integrand in Eq. (36) is divergence free within the region dominated by the asymptotic solution in the plane stress case and the integrand vanishes on the crack surface, thus the interaction energy integral is asymptotically path-independent, i.e.,

$$\left(P_k^{y_\alpha} y_{,k}^{\text{aux}} \delta_{1j} - P_j^{y_\alpha} y_{,1}^{\text{aux}} - P_j^{\text{aux}} y_{\alpha,1} \right)_{,j} = \mu (\lambda^3 \varepsilon_{\alpha u} \varepsilon_{\beta v} y_{u,v})_{,1} y_{\alpha,\beta}^{\text{aux}} \quad (37)$$

which again, using Eqs. (5), (17) and (14), is not sufficiently singular to contribute to an area integral as $r \rightarrow 0$ for the auxiliary fields used here, which are $y^{\text{aux}} = r^{1/2} \sin \frac{\theta}{2}$ and $y^{\text{aux}} = \ln r$ as will be seen below. Thus, the integral (36) is asymptotically path-independent.

Choosing the auxiliary field as $y^{\text{aux}} = r^{1/2} \sin \frac{\theta}{2}$, and calculating I^{y_α} on a circular contour as its radius approaches zeros, it can be shown that

$$I_1^{y_\alpha} = \frac{\pi \mu p_\alpha}{2}, \quad \alpha = 1, 2, \quad \text{for } y^{\text{aux}} = r^{1/2} \sin \frac{\theta}{2} \quad (38)$$

from which, the crack tip parameters, p_α , of the first order term in Eq. (12) can be evaluated.

If y^{aux} is set to $y^{\text{aux}} = \ln r$, we obtain

$$I_2^{y_\alpha} = -2\pi \mu q_\alpha, \quad \alpha = 1, 2, \quad \text{for } y^{\text{aux}} = \ln r \quad (39)$$

This allows the calculation of the parameters, q_α , of the second order term in Eq. (12), separately.

For the interface crack problem, following similar procedures as in the homogenous case and using Eqs. (15)~(19), $J_1^{y_\alpha}$, $I_1^{y_\alpha}$, and $I_2^{y_\alpha}$ on a circular contour with radius r_0 are given by

$$J_1^{y_\alpha} = \frac{\mu\pi(1+s)}{8} p_\alpha^2, \quad \alpha = 1, 2 \quad (40)$$

$$I_1^{y_\alpha} = \frac{\pi\mu p_\alpha(1+s)}{4}, \quad \alpha = 1, 2, \quad \text{for } y^{\text{aux}} = h(\theta)r^{1/2} \sin \frac{\theta}{2} \quad (41)$$

$$I_2^{y_\alpha} = -(1+1/s)\pi\mu q_\alpha, \quad \alpha = 1, 2, \quad \text{for } y^{\text{aux}} = \ln r \quad (42)$$

3.3 Asymptotic path-independent integrals for the plane-strain case

The methodology described above for the plane stress case can also be used for the plane strain case. For example, using Eq. (21) in Eq. (29) or Eq. (34) and computing the integral on a circular contour, we can obtain

$$J_1^{y_\alpha} = \frac{\mu\pi}{4} a_\alpha^2, \quad \alpha = 1, 2 \quad (43)$$

$$J_2^{y_\alpha} = \begin{cases} -\frac{2}{3a^2} a_1 r_0^{1/2} (3a_1 b_1 + 2a_2 b_2), & \alpha = 1 \\ -\frac{2}{3a^2} a_2 r_0^{1/2} (3a_2 b_1 - 2a_1 b_2), & \alpha = 2 \end{cases} \quad (44)$$

For the plane stress case, the auxiliary fields are harmonic and satisfy the asymptotic governing equations (10), a Laplace equation. We choose the same auxiliary fields here, which again satisfy the Laplace equation, but not Eq. (24) which has an additional term of order r^{-1} . By computing the interaction energy integrals on circular contours, we obtain the following results: Taking the auxiliary fields as $y^{\text{aux}} = r^{1/2} \sin \frac{\theta}{2}$ or $y^{\text{aux}} = \ln r$, and defining the auxiliary stress field as $P_j^{\text{aux}} = \mu y_{,j}^{\text{aux}}$, one can show that, in the plane strain case, the interaction energy integral from Eq. (36) evaluated on a circular contour gives,

$$I_1^{y_\alpha} = \frac{\pi\mu a_\alpha}{2}, \quad \alpha = 1, 2, \quad \text{for } y^{\text{aux}} = r^{1/2} \sin \frac{\theta}{2} \quad (45)$$

$$\begin{cases} I_2^{y_1} = -\pi\mu(2a_1b_1 + a_2b_2)/a^2 \\ I_2^{y_2} = -\pi\mu(2a_2b_1 - a_1b_2)/a^2 \end{cases}, \quad \text{for } y^{\text{aux}} = \ln r \quad (46)$$

i.e., the integrals can be used to compute all parameters a_α and b_α , $\alpha = 1, 2$.

In this case, the integral $I_1^{y_\alpha}$ is asymptotically path-independent, because, using Eq. (21) and $y^{\text{aux}} = r^{1/2} \sin \frac{\theta}{2}$, similar to the plane stress case,

$$\left(P_k^{y_\alpha} y_{,k}^{\text{aux}} \delta_{1j} - P_j^{y_\alpha} y_{,1}^{\text{aux}} - P_j^{\text{aux}} y_{\alpha,1} \right)_{,j} = [-p(y_{k,k} \delta_{\alpha\beta} - y_{\alpha,\beta})]_{,1} y_{,\beta}^{\text{aux}} \sim r^{-3/2} f(\theta) \quad (47)$$

Thus, Eq. (47) is not sufficiently singular to contribute to an area integral as $r \rightarrow 0$ (the integrand is asymptotically divergence free).

For the $I_2^{y_\alpha}$ integral, however, the situation is different, since $[-p(y_{k,k} \delta_{\alpha\beta} - y_{\alpha,\beta})]_{,1} y_{,\beta}^{\text{aux}} \sim r^{-2} f(\theta)$ and it does not vanish as $r \rightarrow 0$ for $y^{\text{aux}} = \ln r$ in an area integral, i.e.,

$$\lim_{r \rightarrow 0} \int_A [-p(y_{k,k} \delta_{\alpha\beta} - y_{\alpha,\beta})]_{,1} y_{,\beta}^{\text{aux}} dA \sim \int_A r^{-2} f(\theta) dA \sim O(1) \quad (48)$$

However, the result (46) indicates that, the $I_2^{y_\alpha}$ integral evaluated on a circular contour yields an expression of the same form as the plane stress case. The underlying reason for this is that the contribution of the $B_\alpha r^{-1}$ term (Eq. (24)) to the contour integrals (36) is zero as $r \rightarrow 0$. Thus, although not asymptotically path-independent, the $I_2^{y_\alpha}$ integral yields the expression (46) for a circular contour as $r \rightarrow 0$. This feature will also come into play in the domain integral representation of $I_2^{y_\alpha}$ in Section 4. We can think of the $I_2^{y_\alpha}$ integral as asymptotically path-independent with the stipulation that the contour be circular.

4 Parameter evaluation based on the domain integration method

In a typical finite element simulation, domain integral methods with a suitably specified weight function g , are efficient ways of calculating singular crack tip integrals (Li et al. 1985; Shih et al. 1986; Moran and Shih 1987). An example for J_1 in Eq. (25) can be given as

$$J_1 = \int_A \left(P_{ij} y_{i,1} - W \delta_{1j} \right) \frac{\partial g}{\partial x_j} dA \quad (49)$$

where A is a simply connected region enclosed by contours $\Gamma_1, \Gamma_2, \Gamma^+$ and Γ^- (Figure 2), the mechanical quantities P_{ij} , $y_{i,j}$ and W are the finite element results available at the Gauss integration points of each element in the area A , and g is a sufficiently smooth function defined in A that is unity on Γ_2 and vanishes on Γ_1 . Note that we have used the divergence free property of the integrand in this expression. In the case that Γ_1 is a circular contour with radius r_0 and Γ_2 shrinks to the crack tip, g can be specified as

$$g = \frac{1}{r_0} (r_0 - r), \quad r \in [0, r_0] \quad (50)$$

Similarly, the $J_1^{y_\alpha}$ integral can be written in a domain form, i.e.,

$$\begin{aligned} J_1^{y_\alpha} &= \lim_{r \rightarrow 0} \oint_{\Gamma} \left(P_j^{y_\alpha} y_{\alpha,1} - W^{y_\alpha} \delta_{1j} \right) g n_j d\Gamma \\ &= \lim_{r \rightarrow 0} \int_A \left(P_j^{y_\alpha} y_{\alpha,1} - W^{y_\alpha} \delta_{1j} \right) \frac{\partial g}{\partial x_j} dA + \underbrace{\lim_{r \rightarrow 0} \int_A \left(P_j^{y_\alpha} y_{\alpha,1} - W^{y_\alpha} \delta_{1j} \right)_{,j} g dA}_0 \end{aligned} \quad (51)$$

where in the region dominated by the asymptotic solution, the second integral on the right hand side vanishes because $P_j^{y_\alpha} y_{\alpha,1} - W^{y_\alpha} \delta_{1j}$ is asymptotically divergence free (see Section 3.2). Thus, for a small enough external contour Γ_1 inside of which the asymptotic solution dominates, the $J_1^{y_\alpha}$ integral is given by

$$J_1^{y_\alpha} = \lim_{r \rightarrow 0} \int_A \left(P_j^{y_\alpha} y_{\alpha,1} - W^{y_\alpha} \delta_{1j} \right) \frac{\partial g}{\partial x_j} dA = \frac{\mu\pi}{4} p_\alpha^2, \quad \alpha = 1, 2 \quad (52)$$

For the $J_2^{y_\alpha}$ integral, similar to Eq. (51), we obtain

$$\begin{aligned} J_2^{y_\alpha} &= -2\mu p_\alpha q_\alpha r_0^{1/2} \\ &= \int_A \left(P_j^{y_\alpha} y_{\alpha,2} - W^{y_\alpha} \delta_{2j} \right) \frac{\partial g}{\partial x_j} dA - \int_{\Gamma^+ + \Gamma^-} W^{y_\alpha} n_2 g d\Gamma \end{aligned} \quad (53)$$

where $g=0$ on the outer contour Γ_1 (see Figure 2) and $g=1$ on the inner contour, W^{y_α} are the strain energy satisfying, for example, $W^{y_\alpha}/\mu \sim \frac{1}{8} r^{-1} p_1^2 - \frac{1}{2} p_1 q_1 r^{-1/2} + \frac{1}{2} q_1^2$ on the top crack surface Γ^+ . The term $-2\mu p_\alpha q_\alpha r_0^{1/2}$ is the integral on the inner contour Γ_2 . We note here that for small r_0 (which will be the case in numerical simulation in which the area A resides within the zone of dominance of the asymptotic fields), both left and right hand side of Eq. (53) will be correspondingly small (independently of the value of p_α and q_α) and thus the evaluation of the parameter q_α depends on the ratio of two small numbers and is prone to inaccuracy.

For the interaction energy integrals, similar to the J integrals, I^{y_α} can be re-expressed as

$$I^{y_\alpha} = \int_A \left(P_j^{y_\alpha} y_{,j}^{\text{aux}} \frac{\partial g}{\partial x_1} - \left(P_j^{y_\alpha} y_{,1}^{\text{aux}} n_j + P_j^{\text{aux}} y_{\alpha,1} n_j \right) \frac{\partial g}{\partial x_j} \right) dA, \alpha = 1, 2 \quad (54)$$

Note that to obtain the above equation, we make use of the condition that $P_j^{y_\alpha} y_{,j}^{\text{aux}} \delta_{1j} - P_j^{y_\alpha} y_{,1}^{\text{aux}} - P_j^{\text{aux}} y_{\alpha,1}$ is asymptotically divergence free in the region dominated by the asymptotic solution for the plane stress case for $I_1^{y_\alpha}$ and $I_2^{y_\alpha}$ and for the plane strain case for $I_1^{y_\alpha}$ (see discussions following Eqs. (37) and (47)). For the plane strain case for $I_2^{y_\alpha}$ and using a circular domain, the area integral term with integrand $(P_j^{y_\alpha} y_{,j}^{\text{aux}} \delta_{1j} - P_j^{y_\alpha} y_{,1}^{\text{aux}} - P_j^{\text{aux}} y_{\alpha,1})_{,j} g$ does not contribute, because the contribution of the term $B_\alpha r^{-1}$ (Eq. (24)) to the contour integrals (36) is zero as $r \rightarrow 0$ (see discussion following Eq. (48)). Thus, the domain integrals $I_1^{y_\alpha}$ and $I_2^{y_\alpha}$ can be used to extract the crack tip parameters q_α and p_α , respectively.

5 Results and analysis

5.1 The finite element model

We employ a total Lagrange finite element scheme to solve the nonlinear governing equations in Eqs, (7) and (8). The discretized finite element equilibrium equation can be generally expressed as

$$\mathbf{f}^{\text{int}} - \mathbf{f}^{\text{ext}} = \sum_e \int_{\Omega^e} \mathbf{B}^T \mathbf{P}^d d\Omega - \mathbf{f}^{\text{ext}} = \mathbf{0} \quad (55)$$

where $\mathbf{f}^{\text{int}}$ and $\mathbf{f}^{\text{ext}}$ are the internal and external forces, respectively, $\mathbf{P}^d$ is the vector form of the first Piola-Kirchhoff stress, and $\mathbf{B}$ is the strain matrix satisfying $\mathbf{F}^d = \mathbf{B}\mathbf{y}^d$ with $\mathbf{F}^d$ and $\mathbf{y}^d$ the vector forms of the deformation gradient and the coordinate vector of element Ω^e . In the 2D plane stress case, $\mathbf{P}^d = \{P_{11}, P_{22}, P_{12}, P_{21}\}^T$, $\mathbf{F}^d = \{F_{11}, F_{22}, F_{12}, F_{21}\}^T$, $\mathbf{B} = [B_{ij}]_{4 \times 2n}$, $\mathbf{y}^d = \{y_1^1, y_2^1, \dots, y_1^n, y_2^n\}^T$ with n the total nodes of element Ω^e . Equation (55) is solved by the Newton-Raphson scheme with fixed step size. The increment algebraic equation at the m th iteration follows

$$\mathbf{K}_m \Delta \mathbf{y}_m = \mathbf{f}^{\text{ext}} - \mathbf{f}_m^{\text{int}}, \quad \mathbf{K}_m = \sum_e \int_{\Omega^e} \mathbf{B}^T \mathbf{D} \mathbf{B} d\Omega \quad (56)$$

where $\mathbf{D}$ is the tangent modulus matrix composed by the element of the fourth-order tensor $\mathbb{C}_{ijkl} = \partial P_{ij} / \partial F_{kl}$. Using Eq. (6), $\mathbb{C}$ can be obtained as

$$\mathbb{C}_{ijkl} = \mu \delta_{ik} \delta_{jl} + \mu \lambda^2 (2F_{ji}^{-1} F_{lk}^{-1} + F_{jk}^{-1} F_{li}^{-1}) \quad (57)$$

It can be verified that $\mathbb{C}_{ijkl} = \mathbb{C}_{klij}$ and thus both $\mathbf{D}$ and $\mathbf{K}_m$ are symmetric. An eight-node quadrilateral element is developed to discretize 2D domains in the plane stress case. More details about the finite element implementation are referred to (Wriggers 2008; Belytschko et al. 2013).

For the plane strain case, a hybrid eight-node serendipity element is used, in which, the displacement is interpolated by quadratic functions, and three internal DOFs for approximating the Lagrange multiplier p in Eq. (4) are introduced to deal with the incompressibility issue of the material, similar to the one reported in (Liu et al. 2015).

5.2 Verification for the case of a crack in a long strip

The plane stress case. We consider a long and narrow strip with an initial edge crack on its middle line as shown in Figure 3. The size of the strip is $L_0 \times H_0$. A uniaxial stretch λ_y and a shear deformation κ are applied on the top and bottom surfaces to trigger the mixed mode deformation at the crack tip. The smallest mesh size near the crack tip is about $10^{-6} L_0$ (Figure 3b) and is tested to produce converged results.

For a homogeneous material, the domain can be roughly divided into four regions $Q_1 \sim Q_4$ (Rivlin and Thomas 1953). As shown in Figure 3, the strain fields in Q_2 and Q_4 are non-uniform due to the influence of the crack tip fields and the boundary conditions, while it is nearly uniform in Q_3 because of the large length of the strip compared to its height. In region Q_1 , there is almost no deformation. These observations are demonstrated by a representative distribution of the Von Mises stress in Figure 3c. The J -integral in Eq. (25) on a boundary contour (highlighted by green lines in Figure 3) survives only on the path Γ_{Q_3} inside the region Q_3 , which gives the result

$$J_1 = W(\mathbf{F}_s)H_0 = \frac{H_0}{2} \left(\kappa^2 + \lambda_y^2 + \frac{1}{\lambda_y^2} - 2 \right) \quad (58)$$

where the deformation gradient in Q_3 satisfies $\mathbf{F}_s = [1 \ \kappa \ 0; 0 \ \lambda_y \ 0; 0 \ 0 \ \frac{1}{\lambda_y}]$ and H_0 is the height of the strip. We first compare the J -integrals calculated by Eqs. (49) and (58) to verify the computational scheme.

Figure 4a gives the J -integrals obtained from Eq. (58) (i.e., the analytical solutions) and Eq. (49) (i.e., the finite element results) for the homogeneous strip in the plane stress condition for different stretch levels and loading modes. The results agree with each other well for both the mode I crack with $\kappa = 0$ and the mixed crack modes with $\kappa = \lambda_y - 1$ for different stretch levels. The finite element results are computed using Eq. (49) in the regions bounded by several different contours $\Gamma_{11} \sim \Gamma_{15}$, where $\Gamma_{11} \sim \Gamma_{14}$ are circular contours with radii $r/L_0 = 2.5 \times 10^{-5}$, 2.5×10^{-4} , 5×10^{-3} , and 5×10^{-2} , respectively, and Γ_{15} is a square contour with an edge length $5 \times 10^{-5} L_0$ and the crack tip as its center point. For the circular contours $\Gamma_{11} \sim \Gamma_{14}$, the weight function g defined in (50) is used,

while for the square contour Γ_{15} , g is assigned to be zero on Γ_{15} , and to be one for the interior nodes. From Figure 4a, the finite element results are almost identical for different contours $\Gamma_{11} \sim \Gamma_{15}$ and also agree well with the analytical solution (58), which verifies the path-independence of the J -integral.

The plane strain case. For the plane strain case, we still consider the long strip shown in Figure 3, but use the hybrid eight-node serendipity element to discretize the model. The applied deformations are $\kappa = 0.4$, $\lambda_y = 1.4$. Similarly to the plane stress case, we compare the global path-independent J -integrals for the finite element results and the analytical analysis. Unlike in the plane stress case, the deformation in region Q_3 is not now uniform in the x_2 direction because of the combination of plane strain, incompressibility and clamped boundary conditions, which highly constrain the deformation near the upper and lower boundaries (Figure 3d), thus the deformation gradient at the top or bottom surfaces is $\mathbf{F}_s = \mathbf{I}$, while that at the mid-line is given by $\mathbf{F} = \text{diag}(\frac{1}{\lambda_y}, \lambda_y, 1)$.

To verify the J -integral for the plain strain, we first consider the case where the horizontal displacement on the upper and lower boundaries is unconstrained (free to slide in the x_1 -direction). The uniformity of deformation and stress in region Q_3 can be seen in Figure 3e. Thus, the deformation gradient on Γ_{Q_3} satisfies $\mathbf{F} = \text{diag}(\frac{1}{\lambda_y}, \lambda_y, 1)$ and the term that survives in the J -integral is that on Γ_{Q_3} which gives

$$J_1 = W(\mathbf{F}_s)H_0 = \frac{H_0}{2} \left(\lambda_y^2 + \frac{1}{\lambda_y^2} - 2 \right) \quad (59)$$

A comparison of J_1 between the above analytical solution and the finite element results on different contours is shown in Figure 4b. The results obtained from different methods agree well with each other, indicating the accuracy of the domain integration method for computing the J -integral.

Figure 3 (a) Sketch of a long stretched strip and (b) the mesh scheme near the crack tip for circular contours. The distributions of Von Mises stress on the deformed configuration for (c) the plane stress case with $\kappa = 0$, $\lambda_y = 2.0$, (d) and (e) the plane strain cases with clamped boundary condition ($\kappa = 0$, $\lambda_y = 1.4$) and sliding boundary condition ($\lambda_y = 1.4$), respectively.

Figure 4 Comparison of the J -integral between the finite element results and the analytical results for (a) the plane stress case and (b) the plane strain case.

5.3 The parameters in the plane stress case

In the plane stress case, the evaluated parameters for different modulus ratios $s = 1, 1/2$ and $1/5$ are given in Table 1~3, respectively. Both the results from the J -integrals J^{y_a} , Eqs. (52) and (53), and the interaction energy integrals $I_1^{y_a}$ and $I_2^{y_a}$, Eqs. (41) and (42), are shown. For all the results, the x_2 -directional stretch is fixed as $\lambda_y = 2$, and the x_1 directional shear takes the values $\kappa = -1, 0$ and 1 . To estimate the accuracy of the results, an error indicator e_J is introduced as

$$e_J = \frac{|J_1^a - \tilde{J}_1|}{\tilde{J}_1} \quad (60)$$

where $J_1^a = \frac{\pi\mu}{4}(p_1^2 + p_2^2)$ is obtained by the computed p_α in Tables 1~3, and $\tilde{J}_1$ is the average value of J_1 (Eq. (49)) on contours $\Gamma_{11} \sim \Gamma_{14}$. Unlike the path-independent integral J_1 , showing almost identical value on the contours $\Gamma_{11} \sim \Gamma_{15}$ (Figure 4) for fixed stretches, J_1^a shows increasing difference compared with $\tilde{J}_1$ when the radius of the contour increases. This can be easily understood because the asymptotic crack tip fields hold only in the region close to the crack tip and the asymptotic path independent integrals should be evaluated on domain close to the crack tip.

For a small enough contour (e.g., Γ_{11} , Γ_{12} and Γ_{15}), the results for all s and κ are very close to the reference solution in terms of e_J as shown in Table 1~3. Comparing the results obtained by the two kinds of integrals, we find that the J -integrals J^{y_α} produce slightly better predictions of p_1 and p_2 on different contours than do the interaction energy integrals. In the case of uniaxial stretch ($\lambda_y = 2.0$, $\kappa = 0$), $J_1^{y_1}$ is found to be negative for large contours (e.g., Γ_{13} and Γ_{14}) and thus fails to predict p_1 (Tables 2 and 3). The interaction energy integrals $I_1^{y_\alpha}$ and $I_2^{y_\alpha}$ can provide good estimates of the crack tip parameters in all cases. Over all, the errors e_J for the contours Γ_{11} , Γ_{12} and Γ_{13} , are small and the precision of the corresponding parameters are acceptable. Thus, the interaction energy integrals show better performance in evaluating these coefficients than the J -integrals in terms of full power to evaluate all the parameters. In practical plane stress crack problems, it is suggested that one first calculate p_1 and p_2 using the J -integrals J^{y_α} in Eq. (40), then compute q_1 and q_2 using the interaction energy integrals in Eq. (41). One may need to use Eq. (42) to compute p_1 and p_2 if Eq. (40) fails to do so.

It is found that in all of these cases, though q_2 is not zero, it is usually several times smaller than p_2 . In the asymptotic analysis, it can be seen that $q_2 r \ll p_2 r^{1/2}$, and thus the effect of $q_2 r \cos \theta$ on the distribution of y_2 can be safely ignored. This observation is consistent with the findings reported in (Ravichandran and Knauss 1989). In the next section, we show the deformed coordinate and stress distribution to further demonstrate the efficacy of the method for calculating the crack tip fields.

5.4 Crack tip fields in the plane stress case

To further verify the results in the plane stress cases, we plot the deformed coordinates y_1 , y_2 and stress components in Figure 5~7, respectively. The finite element results for y_1 and y_2 are extracted from the nodal values, while the Cauchy stresses are from the Gauss points. The analytical solution is given by Eq. (12) or (14) with the crack tip parameters obtained using the contour Γ_{11} using the interaction integrals (Table 1~3). It can be found from Figure 5 that the circumferential distributions of y_1 and y_2 show good agreement between the finite element results and the analytical results for different modulus ratios s and the x_1 -directional shear deformation κ . As s decreases or the modulus of the lower half of the strip increases, the deformation amplitude increases in the region $x_2 > 0$ (or $\theta \in [0, \pi]$), and it decreases in the region $x_2 < 0$ (or $\theta \in [0, \pi]$), and slope discontinuity in the front of crack tip $\theta = 0$ becomes more pronounced. The deformed coordinate shows significant difference for different shear deformations. The amplitude of y_1 is in the order of 10^{-4} for $\kappa = 0$, and exhibits a reverse of its value when κ changes from 1 to -1. It is interesting that κ has almost no influence on the distribution of y_2 since the x_2 -directional stretch is kept at a constant $\lambda_y = 2$. Distributions of y_1 and y_2 in the radial direction at $\theta = \pi/4$ and $-\pi/4$ in Figure 6 also show good agreement between the finite element and the analytical results. For the uniaxial stretch of a homogeneous strip, i.e., $s = 1$, $\lambda_y = 2$ and $\kappa = 0$, we have $p_1 = 0$ and $y_1 \sim q_1 \cos \theta$, thus y_1 at $\theta = \pi/4$ and $-\pi/4$ merges into one line as shown in Figure 6a, while for the case $s \neq 0$, the symmetry of the deformation is broken and y_1 at $\theta = \pi/4$ is larger than that at $-\pi/4$. It can be seen that y_2 exhibits an almost perfect linear relation with r in log-log plot (Figure 6b) for different modulus ratios s , since, as mentioned earlier, $q_2 r \ll p_2 r^{1/2}$ and thus $y_2 \sim p_2 r^{1/2} \sin \frac{\theta}{2}$.

The distributions of Cauchy stress components are shown in Figure 7 and Figure 8 and good agreement between the finite element results and analytical results can be found, which indicates the accuracy of the determined parameters. In the case of uniaxial stretch (Figure 7), σ_{22} is much larger than σ_{11} and σ_{12} for all the modulus ratios s . The major stress component σ_{22} shows a discontinuous point at $\theta = 0$ when $s \neq 1$, while it is almost a constant in the regions $\theta < 0$ and $\theta > 0$. This is because the second order term $q_2 r \cos \theta$ plays an insignificant role in y_2 , and thus

$y_2 \sim p_2 r^{1/2} \sin \frac{\theta}{2}$ and $\sigma_{22} r / \mu \sim \frac{1}{4} p_2^2 h(\theta)$. The tensile stress σ_{22} is larger in the top half material for the cases $s = 1/2$ and $1/5$, because it has larger deformation due to a lower shear modulus compared to the lower half. It is observed that changing the x_1 -directional shear does not significantly change the distribution of σ_{22} (Figure 8c), because from Table 2, p_2 only varies a little for different κ . The stress σ_{12} and σ_{11} increase to the same order as σ_{22} when κ was changed to -1 and 1 (Figure 8a and b).

Figure 5 Distribution of y_1 and y_2 on a circular contour at $r/L_0 = 2.5 \times 10^{-5}$ for different modulus ratios $s = 1, 1/2$ and $1/5$. (a)-(c) show y_1 for $\kappa = 0, 1$ and -1 , respectively, and (d) shows all results for $\kappa = 0, 1$ and -1 .

Figure 6 Distribution of (a) y_1 and (b) y_2 in the radial direction at $\theta = \pi/4$ and $-\pi/4$ for different modulus ratios $s = 1, 1/2$ and $1/5$. The strip is in uniaxial stretch with $\lambda_y = 2.0$ and $\kappa = 0$.

Figure 7 Distribution of stress components (a) σ_{11} , (b) σ_{12} and (c) σ_{22} in the circumferential direction at $r/L_0 = 2.5 \times 10^{-5}$ for different modulus ratios $s = 1, 1/2$ and $1/5$. The uniaxial stretch is applied for all the cases with $\lambda_y = 2.0$ and $\kappa = 0$.

Figure 8 Distribution of stress components (a) σ_{11} , (b) σ_{12} and (c) σ_{22} in the circumferential direction at $r/L_0 = 2.5 \times 10^{-5}$ for different shear deformation $\kappa = -1, 0$ and 1 . The modulus ratio is fixed as $s = 1/2$.

Table 1 Evaluated parameters p_α , q_α , $\alpha = 1, 2$ and the error indicator e_J for different deformation levels and external contours. J^{y_α} denotes the J -integral method by Eqs. (52) and (53) and I^{y_α} the interaction energy integrals by Eqs. (38), (39) and (54).

	Case 1: $\lambda_y = 1.0, \kappa = -1.0$			Case 2: $\lambda_y = 1.0, \kappa = 0$			Case 3: $\lambda_y = 1.0, \kappa = 1.0$		
	p_α	q_α	e_J	p_α	q_α	e_J	p_α	q_α	e_J
Γ_{11}	J^{y_α}	$p_1:-0.3871$ $p_2:0.5991$	$q_1:1.0731$ $q_2:0.0597$	0.30%					
	I^{y_α}	-0.3887 0.6016	1.1002 0.0538	0.53%					
Γ_{12}	J^{y_α}	-0.3890 0.6020	1.0847 0.0653	0.67%					
	I^{y_α}	-0.3886 0.6014	1.0995 0.0492	0.47%					
Γ_{13}	J^{y_α}	-0.3873 0.5994	1.0559 0.0472	0.20%					
	I^{y_α}	-0.3870 0.5989	1.0607 0.0189	0.37%					
Γ_{14}	J^{y_α}	-0.3905 0.6079	0.7261 -0.0724	2.30%					
	I^{y_α}	-0.3895 0.6246	0.9651 0.0008	6.18%					
Γ_{15}	J^{y_α}	-0.3856 0.5968	----	1.07%					
	I^{y_α}	-0.3856 0.5958	1.1015 0.0512	1.30%					

Table 2 Evaluated parameters p_α , q_α , $\alpha = 1, 2$ and the error indicator e_J for the interface crack problem in plane stress. The modulus ratio is $s = 1/2$. J^{y_α} denotes the J -integral method by Eqs. (40) and (53) and I^{y_α} the interaction energy integrals by Eqs. (41), (42) and (54).

Case 1: $\lambda_y = 1.0, \kappa = -1.0$					Case 2: $\lambda_y = 1.0, \kappa = 0$			Case 3: $\lambda_y = 1.0, \kappa = 1.0$		
p_α q_α e_J					p_α	q_α	e_J	p_α	q_α	e_J
Γ_{11}	J^{y_α}	$p_1 : -0.5048$	$q_1 : ----$	0.43%	0.0104	----	0.38%	0.5253	----	0.41%
		$p_2 : 0.7914$	$q_2 : ----$		0.7814	----		0.7794	----	
	I^{y_α}	-0.5026	1.1860	1.29%	0.0103	1.0875	1.25%	0.5231	1.0192	0.67%
		0.7880	-0.1207		0.7780	-0.2196		0.7794	-0.2333	
Γ_{12}	J^{y_α}	-0.5076	----	0.66%	0.0104	----	0.69%	0.5282	----	0.68%
		0.7957	----		0.7856	----		0.7836	----	
	I^{y_α}	-0.5066	1.1910	0.27%	0.0104	1.0894	0.31%	0.5281	1.0233	0.66%
		0.7942	-0.1286		0.7841	-0.2215		0.7836	-0.2292	
Γ_{13}	J^{y_α}	-0.5067	----	0.16%	----	----	1.10%	0.5248	----	0.17%
		0.7917	----		0.7645	----		0.7811	----	
	I^{y_α}	-0.5064	1.1299	0.25%	0.0093	1.0272	0.10%	0.5245	0.9836	0.34%
		0.7914	-0.1531		0.7825	-0.2002		0.7803	-0.1826	
Γ_{14}	J^{y_α}	-0.5067	----	0.16%	----	----	3.38%	0.5224	----	2.11%
		0.7917	----		0.6851	----		0.7955	----	
	I^{y_α}	-0.5065	1.0018	6.16%	0.0116	0.9531	8.32%	0.5292	0.9328	5.85%
		0.8264	-0.0756		0.8148	-0.0988		0.8117	-0.0903	
Γ_{15}	J^{y_α}	-0.5031	----	1.11%	0.0103	----	1.07%	0.5236	----	1.07%
		0.7887	----		0.7787	----		0.7768	----	
	I^{y_α}	-0.5024	1.1940	1.40%	0.0103	1.0932	1.37%	0.5227	1.0259	1.40%
		0.7875	-0.1261		0.7775	-0.2218		0.7755	-0.2320	

Table 3 Evaluated parameters p_α , q_α , $\alpha = 1, 2$ and the error indicator e_J for the interface crack problem. The modulus ratio is $s = 1/5$. J^{y_α} denotes the J -integral method by Eqs. (40) and (53) and I^{y_α} the interaction energy integrals by Eqs. (41), (42) and (54).

		Case 1: $\lambda_y = 1.0, \kappa = -1.0$			Case 2: $\lambda_y = 1.0, \kappa = 0$			Case 3: $\lambda_y = 1.0, \kappa = 1.0$		
		p_α	q_α	e_J	p_α	q_α	e_J	p_α	q_α	e_J
Γ_{11}	J^{y_α}	$p_1: -0.6211$ $p_2: 0.9513$	$q_1: ----$ $q_2: ----$	0.71%	0.0280 0.9338	----	0.42%	0.6704 0.9234	----	0.41%
	I^{y_α}	-0.6185 0.9490	1.3062 -0.2384	1.29%	0.0279 0.9298	1.1073 -0.3866	1.27%	0.6675 0.9194	0.9522 -0.3686	1.27%
Γ_{12}	J^{y_α}	-0.6247 0.9582	----	0.65%	0.0276 0.9389	----	0.67%	0.6738 0.9285	----	0.67%
	I^{y_α}	-0.6235 0.9564	1.2968 -0.2537	0.27%	0.0279 0.9372	1.0920 -0.3846	0.31%	0.6726 0.9268	0.9440 -0.3534	0.30%
Γ_{13}	J^{y_α}	-0.6269 0.9528	----	0.07%	---- 0.9374	----	0.90%	0.6666 0.9281	----	0.13%
	I^{y_α}	-0.6262 0.9573	1.1492 -0.2174	0.66%	0.0252 0.9387	0.9968 -0.2769	0.61%	0.6677 0.9274	0.9010 -0.2494	0.12%
Γ_{14}	J^{y_α}	-0.6407 0.9709	----	4.09%	---- 0.9624	----	4.18%	0.6558 0.9504	----	1.98%
	I^{y_α}	-0.6216 0.9817	1.0317 -0.0744	3.86%	0.0275 0.9683	0.9677 -0.1071	7.07%	0.6729 0.9594	0.9220 -0.1085	5.04%
Γ_{15}	J^{y_α}	-0.6191 0.9499	----	1.11%	0.0278 0.9307	----	1.08%	0.6680 0.9203	----	1.09%
	I^{y_α}	-0.6182 0.9484	1.3061 -0.2491	1.41%	0.0278 0.9293	1.1023 -0.3880	1.37%	0.6670 0.9189	0.9505 -0.3012	1.39%

5.5 Crack tip parameters and fields in plane strain case

In the plane strain case (clamped boundary conditions), the asymptotic J -integrals J^{y_α} in Eqs. (43) and (44) and the interaction energy integrals $I_1^{y_\alpha}$ and $I_2^{y_\alpha}$ in Eqs. (45) and (46) are used to extract the crack tip parameters. The results obtained from several contours (the same as those in the plane stress case) are shown in Table 4 for mixed crack modes, where λ_y is fixed at 1.4 and the shear deformation κ takes 0.0, 0.2 and 0.4. The error indicator e_j defined in Eq. (60) is used to measure the error of the computed parameters. Note that the reference J -integral value, $\tilde{J}_1$ (using a domain form of integration (49)), corresponds to the case of the clamped boundary condition, rather than the value shown in Figure 4b for the sliding boundary condition.

It can be seen that e_j is smallest for the parameters calculated by the interaction energy integrals on Γ_{12} for different shear deformation κ . The error increases significantly when the radius of the contour increases from Γ_{12} to Γ_{14} . These results are consistent with the plane stress case, that is, the larger the integration area, the larger the error in the computed parameters, due to the failure of the asymptotic solution to describe the crack tip fields far from the crack tip. On the rectangular contour Γ_{15} , a_α also has good precision and differs very little from the values on Γ_{12} . In the region inside the smallest contour Γ_{11} , the mesh exhibits the most severe deformation and thus may contribute to errors in evaluating the parameters. This is in part why the parameters obtained using contour Γ_{12} are better than those obtained using Γ_{11} . Using the parameters predicted by the interaction energy integrals on contour Γ_{12} , the distribution of deformed coordinates can be obtained (Figure 9). Good agreement between the finite element results and the analytical solution can be seen in both circumferential and radial directions for different κ . This indicates the efficacy of the method for estimating the crack tip parameters.

For the small circular and rectangular contours, Γ_{12} and Γ_{15} , $a_\alpha, \alpha = 1, 2$ obtained by either $J_1^{y_\alpha}$ or $I_1^{y_\alpha}$ agree with each other, while on the rectangular contour Γ_{15} , $I_2^{y_\alpha}$ fails to extract accurate value of b_1 and b_2 (and are not list in Table 5). This justifies the conclusion in Section 3.3 that the integrands of $J_1^{y_\alpha}$, $I_1^{y_\alpha}$ and $I_2^{y_\alpha}$ are asymptotically path-independent near the crack only ($I_2^{y_\alpha}$ also

requires a circular domain). Thus the use of $I_2^{y_\alpha}$ leads to large error in computing b_α using a non-circular contour as expected.

The distributions of y_1 and y_2 in Eq. (21) with the crack tip parameters given by I^{y_α} on Γ_{12} are given in Figure 9. Good agreement between the analytical solution and the finite element results can be seen in both circumferential and radial directions for different κ . For the mixed loading modes with $\kappa = 0.2$ and 0.4 , y_1 and y_2 has the same distribution features as those shown in Figure 5b-d, since $r^{1/2} \sin \frac{\theta}{2}$ becomes the leading order term in both y_1 and y_2 . Increasing κ leads to increase of y_1 but does not affect the amplitude of y_2 , which indicates the two deformation components should be independent with other when large stretch is applied.

Figure 9 Distributions of y_1 and y_2 in (a-b) the circumferential and (c-d) radial directions. The crack tip parameters are obtained by the interaction energy integral $I_1^{y_\alpha}$ and $I_2^{y_\alpha}$ on Γ_{12} given in Table 4.

Table 4 Evaluated parameters a_α , b_α , $\alpha = 1, 2$ for the plane strain case. J^{y_α} denotes the J -integrals in Eqs. (43) and (44) and I^{y_α} the interaction energy integrals in Eqs. (45) and (46).

	Case 1: $\lambda_y = 1.0, \kappa = 0$			Case 2: $\lambda_y = 1.0, \kappa = 0.2$			Case 3: $\lambda_y = 1.0, \kappa = 0.4$		
	a_α	b_α	e_J	a_α	b_α	e_J	a_α	b_α	e_J
Γ_{11}	J^{y_α}	$a_1: 0.0149$ $a_2: 0.6587$	$b_1: 0.0422$ $b_2: 2.0919$	0.1379 0.6485	0.0615 1.4384	2.73% 3.08%	0.2637 0.6216	0.2131 1.6169	3.06%
	I^{y_α}	-0.0001 0.6646	0.0314 1.7756	0.1366 0.6548	0.0907 1.7759	1.03% 1.35%	0.2647 0.6272	0.2218 1.7052	1.46%
Γ_{12}	J^{y_α}	0.0450 0.6707	0.0002 -0.0095	0.1463 0.6604	0.1140 1.7330	0.88%	0.2712 0.6333	0.2255 1.7369	0.91%
	I^{y_α}	-0.0001 0.6693	0.0039 1.9073	0.1390 0.6590	0.1162 1.8828	0.02%	0.2672 0.6319	0.2339 1.8189	0.08%
Γ_{13}	J^{y_α}	0.1890 0.6461	0.0010 -0.0003	0.2273 0.6381	0.1155 1.0034	1.17%	0.3094 0.6165	0.2272 1.3897	1.16%
	I^{y_α}	0.0000 0.6556	0.0002 1.8821	0.1332 0.6468	0.1172 1.8615	3.84%	0.2573 0.6230	0.2315 1.8066	3.40%
Γ_{14}	J^{y_α}	0.4616 0.5489	0.0000 0.0001	0.4688 0.5471	0.1014 0.2474	14.46%	0.4895 0.5420	0.2002 0.4680	13.40%
	I^{y_α}	0.0000 0.6276	0.0000 1.6220	0.1059 0.6242	0.1018 1.6156	11.62%	0.2094 0.6164	0.2021 1.5975	9.89%
Γ_{15}	J^{y_α}	0.0305 0.6717	---- ----	0.1427 0.6595	---- ----	0.39%	0.2730 0.6340	---- ----	1.31%
	I^{y_α}	0.0007 0.6638	---- ----	0.1380 0.6517	---- ----	2.15%	0.2656 0.6264	---- ----	1.58%

5.6 A summary and comparison of the integrals

The basic features of the integrals used and developed here are summarized in Table 5, and we note the following:

- (a) In term of path independence, J_1 is globally path-independent, while $J_1^{y_\alpha}$, $I_1^{y_\alpha}$ and $I_2^{y_\alpha}$ are asymptotically path-independent, except in the plane strain case, where $I_2^{y_\alpha}$ is path independent only if the integration contour is circular.
- (b) In terms of the ability to extract the crack tip parameters, all asymptotic path-independent integrals, i.e., $J_1^{y_\alpha}$, $I_1^{y_\alpha}$ and $I_2^{y_\alpha}$, are able to extract either the first-order or the second-order parameters. The integral $J_2^{y_\alpha}$ in this paper can be used to extract $q_\alpha, \alpha = 1, 2$ in the homogeneous crack problem if the contour is circular. The expression for $J_2^{y_\alpha}$ shows that it vanishes as $r_0 \rightarrow 0$ and thus for a small but finite contour, the evaluation of the parameter q_α depends on the ratio of two small numbers and is prone to inaccuracy, and we have not used it for the interface crack problem in the paper.

Table 5 A summary of the basic properties of the integrals

	Path independence		Direct extracted parameters	
	Plane stress	Plane strain	Plane stress	Plane strain
J_1 (Eq. (26))	Global	Global	No*	No*
$J_1^{y_\alpha}$ (Eqs. (33), (40), (43))	Asymptotic	Asymptotic	$p_\alpha, \alpha = 1, 2$	$a_\alpha, \alpha = 1, 2$
$J_2^{y_\alpha}$ (Eqs. (53))	No	No	$q_\alpha, \alpha = 1, 2$ (circular)	$b_\alpha, \alpha = 1, 2$ (circular)
$I_1^{y_\alpha}$ (Eqs (38), (41), (45))	Asymptotic	Asymptotic	$p_\alpha, \alpha = 1, 2$	$a_\alpha, \alpha = 1, 2$
$I_2^{y_\alpha}$ (Eqs. (39), (42), (46))	Asymptotic	Asymptotic (circular)	$q_\alpha, \alpha = 1, 2$	$b_\alpha, \alpha = 1, 2$, (circular)

*This is because p_1 and p_2 in J_1 come into play as a sum of their squares. The exception is the mode I crack problem with $p_1 = 0$, where J_1 can be used to calculate p_2 , directly.

6 Conclusions

In this paper, we construct several asymptotic path-independent integrals to evaluate the amplitude parameters of the crack tip fields in a neo-Hookean material for plane stress and plane strain and for both homogeneous and interface cracks. In the finite element simulation, the integrals are calculated by a domain integration method. Numerical results for a long strip with a horizontal edge crack indicate that the present approach is suitable for computing the amplitude parameters, and thus the distributions of the deformed coordinates and stresses in the asymptotic region.

By observing that the asymptotic governing equations hold for each deformed coordinate separately, new asymptotic path-independent $J_1^{y_\alpha}$ integrals and interaction energy integrals $I_1^{y_\alpha}$ and $I_2^{y_\alpha}$ that permit the computation of all crack tip parameters up to the first two terms of the asymptotic solution are derived. Arfaoui et al. (2018) showed how crack tip fields in three-dimensional hyperelastic neo-Hookean crack problem reduce asymptotically to a combined plain strain and anti-plane form. The approach in this paper can be suitably extended for the computation of parameters in 3D and is the focus of ongoing work. Such information may prove useful in understanding crack growth, tearing, and other failure process in soft materials.

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