Analysis of a finite matrix with an inhomogeneous circular inclusion subjected to a non-uniform eigenstrain

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Abstract

The mechanical model of eigenstrains could not be always taken as uniform distributions in engineering applications when performing micromechanics analysis of the inclusion-matrix system. In the framework of plane strain, this paper presents the analytical solution to an inhomogeneous circular inclusion with a non-uniform eigenstrain concentrically embedded in a finite matrix. First, the equivalent eigenstrain equation is extended to satisfy the condition of the finite matrix through the equivalent eigenstrain principle. The modified equation is used to transform the inhomogeneous inclusion in a finite matrix into the corresponding homogeneous inclusion. Then, the model of the inhomogeneous circular inclusion is accordingly formulated and the stress and strain distributions are found. Finally, the stresses for the case of the polynomial series distribution of eigenstrains are obtained. The effects of non-uniformity of eigenstrains, the material mismatch and the inclusion size on stress distributions are shown graphically. The results indicate the stiffer inclusion induce the larger stress under the specific eigenstrain distribution. The analytical solutions obtained here also help to predict failure and optimize designs of composite structure.

Keywords: Inhomogeneous inclusion, Non-uniform eigenstrain, Finite matrix, Circular inclusion, Polynomial series distribution of eigenstrains.

1. Introduction

Inclusion problems have been widely studied due to their fundamental role and practical applications in micromechanics and advanced materials. Along with the intrinsic physical and engineering significance of understating material properties and designing new materials, numerous efforts have been dedicated to investigating inclusions in an infinite matrix. These studies, which can be mainly categorized into homogenous and inhomogeneous inclusion problems, have largely released the assumptions existed in the pioneering works of Eshelby [1-2]. Besides elliptical and ellipsoid inclusions and uniform eigenstrains distributed in inclusions, the studies have considered line [3-5], polygonal (or polyhedral) [6-11] and even arbitrary-shaped [12-18] inclusions and also extended the eigenstrains to non-uniformity [19-23]. A comprehensive review work can be found in literature [24].

In many situations encountered in practical engineering, assuming that the size of matrix is infinite cannot be always sufficiently accurate since the size of inclusions is not always relatively small. For instance, the volume fraction of carbon nanotubes in composites has been raised up to above 20% in order to improve their relatively low stiffness and strength [25-27]. In the particle-reinforced composite cases, Al_2O_3 took up a volume fraction of 50% in metal matrix composite [28]. In such cases, the boundary effect of matrix has to be considered. Thus, inclusions in a finite matrix draw much interests. For example, Li et.al. and Wang et.al. [29-32] investigated a homogenous circular concentrically surrounded by a circular domain and a spherical inclusion concentrically embedded in a spherical

matrix. The considered inclusion is prescribed with a uniform eigenstrain and the exterior boundary of the matrix is subjected to the displacement and traction boundary conditions. They obtained the closed-form expressions of Dirichlet-Eshelby and Neumann-Eshelby tensors related to different boundary conditions. Subsequently, based on the elasticity theory of a simplified strain gradient, Gao et.al [33] and Ma et.al [34] tested the similar problems and extended the solutions in the framework of high-order elasticity theory. Mejak [35] studied axisymmetrically placed inclusion in a sphere and found the Dirichlet-Eshelby tensor of an eccentric spherical inclusion through the power series expansion. Research on arbitrarily-shaped inclusion problems studied by Zou [36] is quite significant. In his paper, the authors proposed a general procedure based on the superposition principle, which overcame the unavailability of the Green's function. He also derived the explicit results for a concentric square inclusion and an eccentric circular inclusion. After that, Zou and He [37] used the superposition principle [36] to further analyze an eccentric spherical inclusion with a uniform eigenstrain located in a finite sphere. Chen [38] conducted a numerical investigation of an elliptical inclusion in a finite elliptic plate, considering a uniform eigenstrain in the inclusion and static loading on the outer boundary of matrix. Pan and Yu [39] considered the shape effect of matrix for circular inclusion problem using complex potential method along with conformal mapping. Subsequently, they [40] further extended their work to an arbitrarily-shaped inclusion by assuming that the shape of an inclusion can be replaced with an arbitrarily-shaped polygon. Both works were also verified by simulations.

However, all the researches that we investigated only considered the *homogeneous* inclusions with the *uniform* eigenstrain. In this paper, we aim to derive an analytical solution for the *inhomogeneous* circular inclusion problem with a *non-uniform* symmetrical eigenstrain concentrically embedded in a circular matrix. Symmetrical eigenstrains exist in problems of point heat source, diffusion of materials and compositional changes [41]. It is an extension to our previous work [42-43] in which the authors dealt with an inhomogeneous inclusion with non-uniform eigenstrains in an infinite matrix. Toward this, a procedure of solving an inhomogeneous arbitrarily-shaped inclusion with non-uniform eigenstrains in a finite matrix is presented. We enrich the equivalent eigenstrain principle and the modified equivalent eigenstrain equation is also derived.

This paper is organized as follows. Section 2 presents an modified equation of equivalent eigenstrain for solving inhomogeneous inclusion problems in a finite matrix. And in Section 3, analytical solutions to the circular *inhomogeneous* inclusion problem with a non-uniform eigenstrain is provided. We illustrate a typical example in Section 4 to study the *inhomogeneous* inclusion problem with a polynomial-distributed eigenstrain and its effect of eigenstrain distributions, the material mismatch and the inclusion size on elastic field is also analyzed. Finally, the main conclusions are summarized in Section 5.

2. An extended equivalent eigenstrain equation for a finite matrix

Consider an inhomogeneous arbitrarily-shaped inclusion Ω prescribed with a non-uniform eigenstrain ε_{ij}^* is embedded into a finite matrix, as shown in Fig. 1(a). The elastic moduli of the matrix and the inhomogeneous inclusion are C_{ijkl} and C_{ijkl}^* , respectively. The boundary ∂D is subjected to boundary loads $\mathbf{T}$.

Fig. 1. (a) The inhomogeneous arbitrarily-shaped inclusion in a finite elastic body subjected to a non-uniform eigenstrain and boundary loads, (b) The corresponding homogeneous inclusion problem. The inhomogeneous inclusion problem in Fig. 1(a) is equivalent to the corresponding homogeneous inclusion problem in Fig. 1(b) via the extended equivalent eigenstrain equation.

Based on the equivalent eigenstrain principle [29], the inhomogeneous inclusion problem in Fig. 1(a) can be transformed into the corresponding homogeneous inclusion problem (Fig. 1(b)). But the equivalent eigenstrain ε_{ij}^0 equation has to be extended as following,

$$\varepsilon_{ij}^0 = S_{ijkl} (C_{klpq} - C_{klpq}^*) \varepsilon_{pq} + S_{ijkl} C_{klpq}^* \varepsilon_{pq}^* \quad (1)$$

Where $S_{ijkl} = C_{ijkl}^{-1}$ is the elastic compliance of the matrix, the correction term ε_{ij} represents the total strain caused by both ε_{ij}^0 and $\mathbf{T}$ in the homogenous inclusion Ω . In the following step, we will introduce how to find the relation between ε_{ij} , ε_{ij}^0 and $\mathbf{T}$. The actual stress σ_{ij} in Fig. 1(b) can be evaluated by the sum of σ_{ij}^0 and σ_{ij}^1 , where σ_{ij}^0 is caused by ε_{ij}^0 in an infinite matrix and σ_{ij}^1 satisfies the boundary conditions on ∂D :

$$\sigma_{ij}^1|_{\partial D} = -\sigma_{ij}^0|_{\partial D} + \mathbf{T} \quad (2)$$

Once the stress σ_{ij} is obtained, then, Hooke's law yields

$$\varepsilon_{ij} = \varepsilon_{ij}^0 + S_{ijkl} \sigma_{kl}, \quad \text{in } \Omega \quad (3)$$

$$\varepsilon_{ij} = S_{ijkl} \sigma_{kl}, \quad \text{in } D \quad (4)$$

By combining Eqs. (1-4), the equivalent eigenstrain ε_{ij}^0 , the total stress σ_{ij} and the total strain ε_{ij} can be calculated. Since the original inhomogeneous problem is completely equivalent to the corresponding homogeneous inclusion problem, the elastic field in the origin problem is found.

One should note that the boundary loads $\mathbf{T}$ are remote loads and $\sigma_{ij}^1|_{\partial D} = \mathbf{T}$ when the domain of the matrix becomes infinite, which indicates that the problem in a finite domain is directly degenerated into the corresponding problem in an infinite domain. In such case, Eqs. (1) to (4) are consistent with the solutions in the previous work [42].

We here in theory solve the inhomogeneous inclusion with a non-uniform eigenstrain in a finite matrix. It provides a framework to obtain analytical as well as numerical solutions for specific inhomogeneous inclusion problems. In the following, we will use the procedure to derive the analytical solution for an inhomogeneous circular inclusion in a finite matrix.

3. Analytical solution for an inhomogeneous circular inclusion with a non-uniform symmetrical eigenstrain

In the framework of plane strain, the inhomogeneous circular inclusion subjected to a non-uniform symmetrical eigenstrain will be analyzed. As shown in Fig. 2(a), a finite circular domain ($\rho_1 < \rho < \rho_2$) with the elastic moduli C_{ijkl} concentrically contains an inhomogeneous inclusion ($\rho < \rho_1$) whose the elastic moduli are C_{ijkl}^* . Eigenstrain given in the inhomogeneous circular inclusion is assumed to be in the form of $\varepsilon_{ij}^*(\rho) = \delta_{ij} \varepsilon^*(\rho)$ and zero in the matrix. Fig. 2(b) shows the corresponding homogenous inclusion with the equivalent eigenstrain ε_{ij}^0 . For plain strain condition and isotropic materials, the components of C_{ijkl} , C_{ijkl}^* , $\varepsilon_{ij}^*(\rho)$, ε_{ij}^0 and the total strain ε_{ij} in question are expressed as follows,

$$\begin{aligned} [C_{ijkl}] &= \frac{2\mu}{1-2\nu} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & 1-2\nu \end{bmatrix}, \quad [C_{ijkl}^*] = \frac{2\mu^*}{1-2\nu^*} \begin{bmatrix} 1-\nu^* & \nu^* & 0 \\ \nu^* & 1-\nu^* & 0 \\ 0 & 0 & 1-2\nu^* \end{bmatrix} \\ [\varepsilon_{ij}^*] &= \begin{bmatrix} \varepsilon_{\theta\theta}^* \\ \varepsilon_{rr}^* \\ \varepsilon_{r\theta}^* \end{bmatrix}, \quad [\varepsilon_{ij}^0] = \begin{bmatrix} \varepsilon_{\theta\theta}^0 \\ \varepsilon_{rr}^0 \\ \varepsilon_{r\theta}^0 \end{bmatrix}, \quad [\varepsilon_{ij}] = \begin{bmatrix} \varepsilon_{\theta\theta} \\ \varepsilon_{rr} \\ \varepsilon_{r\theta} \end{bmatrix} \end{aligned} \quad (5)$$

Fig. 2. (a) The inhomogeneous circular inclusion problem in a finite circular domain, (b) The corresponding homogeneous circular inclusion problem. The original problem in Fig. 2(a) is transformed into the homogeneous inclusion problem in Fig. 2(b).

3.1 Homogeneous circular inclusion

Here, we first present the relation between the total strain ε_{ij} and the equivalent eigenstrain ε_{ij}^0 before solving the extended equivalent equation by studying the homogeneous inclusion problem in Fig. 2(b). Assume the equivalent eigenstrain $\varepsilon_{ij}^0(\rho)$ is distributed in the form of $\varepsilon_{ij}^0(\rho) = \delta_{ij}\varepsilon^0(\rho)$.

As discussed in Section 2, the problem can be decomposed into two sub-problems. For the homogeneous circular inclusion with the same eigenstrain in an infinite matrix, the stress and strain components at any point (x, y) outside the circular inclusion, i.e., $\rho = \sqrt{x^2 + y^2} > \rho_1$ are [43],

$$\begin{bmatrix} \sigma_{\theta\theta}^0(\rho) \\ \sigma_{rr}^0(\rho) \\ \sigma_{r\theta}^0(\rho) \end{bmatrix} = \frac{2\mu}{(1-\nu)\rho^2} \int_0^{\rho_1} \varepsilon_0(\rho_0) \rho_0 d\rho_0 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad (\rho = \sqrt{x^2 + y^2} > \rho_1) \quad (6)$$

$$\begin{bmatrix} \varepsilon_{\theta\theta}^0(\rho) \\ \varepsilon_{rr}^0(\rho) \\ \varepsilon_{r\theta}^0(\rho) \end{bmatrix} = \frac{1}{(1-\nu)\rho^2} \int_0^{\rho_1} \varepsilon_0(\rho_0) \rho_0 d\rho_0 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad (\rho = \sqrt{x^2 + y^2} > \rho_1) \quad (7)$$

And at those points inside the inclusion, i.e., $\rho < \rho_1$, they are expressed as,

$$\begin{bmatrix} \sigma_{\theta\theta}^0(\rho) \\ \sigma_{rr}^0(\rho) \\ \sigma_{r\theta}^0(\rho) \end{bmatrix} = -\frac{2\mu}{(1-\nu)\rho^2} \int_0^\rho \varepsilon_0(\rho_0) \rho_0 d\rho_0 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad (\rho < \rho_1) \quad (8)$$

$$\begin{bmatrix} \varepsilon_{\theta\theta}^0(\rho) \\ \varepsilon_{rr}^0(\rho) \\ \varepsilon_{r\theta}^0(\rho) \end{bmatrix} = -\frac{(1-2\nu)}{(1-\nu)\rho^2} \int_0^\rho \varepsilon_0(\rho_0) \rho_0 d\rho_0 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \varepsilon_0(\rho) \\ \varepsilon_0(\rho) \\ 0 \end{bmatrix}, \quad (\rho < \rho_1) \quad (9)$$

For the specific boundary value problem, the boundary loads are given by,

$$\sigma_{rr}^1|_{\rho=\rho_2} = \sigma_{rr}^0|_{\rho=\rho_2} = -\frac{2\mu}{(1-\nu)\rho_2^2} \int_0^{\rho_1} \varepsilon_0(\rho_0) \rho_0 d\rho_0, \quad \sigma_{r\theta}^1|_{\rho=\rho_2} = 0 \quad (10)$$

Then, its stress and strain components in Fig. 2(b) are obtained:

$$\begin{bmatrix} \sigma_{\theta\theta}^1(\rho) \\ \sigma_{rr}^1(\rho) \\ \sigma_{r\theta}^1(\rho) \end{bmatrix} = \frac{2\mu}{(1-\nu)\rho_2^2} \int_0^{\rho_1} \varepsilon_0(\rho_0) \rho_0 d\rho_0 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad (11)$$

$$\begin{bmatrix} \varepsilon_{\theta\theta}^1(\rho) \\ \varepsilon_{rr}^1(\rho) \\ \varepsilon_{r\theta}^1(\rho) \end{bmatrix} = \frac{(1-2\nu)}{(1-\nu)\rho_2^2} \int_0^{\rho_1} \varepsilon_0(\rho_0) \rho_0 d\rho_0 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad (12)$$

By superposing the solutions above, the stress and strain distribution in Fig. 2(a) is given by as follows. For $\rho_1 < \rho < \rho_2$,

$$\begin{bmatrix} \sigma_{\theta\theta}(\rho) \\ \sigma_{rr}(\rho) \\ \sigma_{r\theta}(\rho) \end{bmatrix} = \frac{2\mu}{(1-\nu)\rho^2} \int_0^{\rho_1} \varepsilon_0(\rho_0) \rho_0 d\rho_0 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \frac{2\mu}{(1-\nu)\rho_2^2} \int_0^{\rho_1} \varepsilon_0(\rho_0) \rho_0 d\rho_0 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad (\rho_1 < \rho < \rho_2) \quad (13)$$

$$\begin{bmatrix} \varepsilon_{\theta\theta}(\rho) \\ \varepsilon_{rr}(\rho) \\ \varepsilon_{r\theta}(\rho) \end{bmatrix} = \frac{1}{(1-\nu)\rho^2} \int_0^{\rho_1} \varepsilon_0(\rho_0) \rho_0 d\rho_0 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \frac{(1-2\nu)}{(1-\nu)\rho_2^2} \int_0^{\rho_1} \varepsilon_0(\rho_0) \rho_0 d\rho_0 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad (\rho_1 < \rho < \rho_2) \quad (14)$$

For $\rho < \rho_1$,

$$\begin{bmatrix} \sigma_{\theta\theta}(\rho) \\ \sigma_{rr}(\rho) \\ \sigma_{r\theta}(\rho) \end{bmatrix} = -\frac{2\mu}{(1-\nu)\rho^2} \int_0^\rho \varepsilon_0(\rho_0) \rho_0 d\rho_0 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{2\mu}{(1-\nu)\rho_2^2} \int_0^{\rho_1} \varepsilon_0(\rho_0) \rho_0 d\rho_0 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad (\rho < \rho_1) \quad (15)$$

$$\begin{bmatrix} \varepsilon_{\theta\theta}(\rho) \\ \varepsilon_{rr}(\rho) \\ \varepsilon_{r\theta}(\rho) \end{bmatrix} = -\frac{(1-2\nu)}{(1-\nu)\rho^2} \int_0^\rho \varepsilon_0(\rho_0) \rho_0 d\rho_0 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{(1-2\nu)}{(1-\nu)\rho_2^2} \int_0^{\rho_1} \varepsilon_0(\rho_0) \rho_0 d\rho_0 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \varepsilon_0(\rho) \\ \varepsilon_0(\rho) \\ 0 \end{bmatrix}, \quad (\rho < \rho_1) \quad (16)$$

The stress and strain fields of homogeneous inclusion in a finite domain are now derived. It is clearly shown that the solution is a further extended and also can be degenerated into the case in our previous researches [43] simply by letting $\rho_2 \rightarrow +\infty$.

3.2 Inhomogeneous circular inclusion

Via the modified equivalent eigenstrain equation (see Eq. (1)), ε_{ij}^0 in Fig. 2(b) satisfies,

$$C_{ijkl}\varepsilon_{kl}^0 = (C_{ijkl} - C_{ijkl}^*)\varepsilon_{kl} + C_{ijkl}^*\varepsilon_{kl}^*, \quad (\rho < \rho_1) \quad (17)$$

Due to the symmetry of both geometry and eigenstrain distribution, the total strain $\varepsilon_{ij}(\rho)$ and the equivalent strain $\varepsilon_{ij}^0(\rho)$ must have the form of $\varepsilon_{ij}(\rho) = \delta_{ij}\varepsilon(\rho)$, $\varepsilon_{ij}^0(\rho) = \delta_{ij}\varepsilon_0(\rho)$.

Substituting Eqs. (5), (16) into Eq. (17), we find

$$\varepsilon_0(\rho) + \chi \frac{1}{\rho^2} \int_0^\rho \varepsilon_0(\rho_0) \rho_0 d\rho_0 + c_1 = \varepsilon^*(\rho), \quad (\rho < \rho_1) \quad (18)$$

Where

$$\chi = \frac{\mu(1-2\nu^*) - \mu^*(1-2\nu)}{\mu^*(1-\nu)} \quad (19)$$

$$c_1 = -\frac{\chi}{\rho_2^2} \int_0^{\rho_1} \varepsilon_0(\rho_0) \rho_0 d\rho_0$$

Its solution is

$$\varepsilon_0(\rho) = \frac{1}{\rho^{\chi+2}} \left\{ \int \left[\rho^\chi \frac{d}{d\rho} (\rho^2 \varepsilon^*(\rho)) - 2c_1 \rho^{\chi+1} \right] d\rho + c_2 \right\}, \quad (\rho < \rho_1) \quad (20)$$

The constant c_2 can be determined by inserting Eq. (20) into Eq. (18). Now that the equivalent strain $\varepsilon_{ij}^0(\rho)$ is derived, the stress and the strain distributions in both inhomogeneous inclusion and matrix zone can be calculated by Eqs. (13) to (16).

It is worthy to note that the constant c_1 indicates the boundary effect of the matrix and χ represents the disparity of material properties between the matrix and the inclusion. As seen in Eq. (19), c_1 tends to zero when the radius of matrix ρ_2 extends to infinity. In such case, the solutions obtained above are degenerated to the solution for the inhomogeneous inclusion problem in an infinite elastic plane [43]. Moreover, when the inclusion and the matrix share the same material, thus, χ becomes zero, the solutions are directly degenerated into those for homogeneous inclusion problems, which prove our conclusions in Section 2.

4. A representative example: Polynomial series distributions of eigenstrains

In the previous two sections, the elastic field to both homogeneous and inhomogeneous circular inclusion with a non-uniform eigenstrain were obtained. And, from the perspective of understanding material properties and designing new material structures, the elastic interactions between inclusions and matrix has to be fully understand. Thus, in this section, we will apply the solutions to consider typical polynomial series distributions of eigenstrains to analyze the influence of eigenstrain distributions, the material mismatch and the size of the inclusion on stress distribution.

Assuming the eigenstrains are distributed in terms of polynomial series as,

$$\varepsilon^*(\rho) = \sum_{n=0}^N A_n (\rho/\rho_1)^n, \quad (\rho < \rho_1) \quad (21)$$

The equivalent eigenstrain ε_{ij}^0 is solved by Eq. (20),

$$\varepsilon_0(\rho) = \sum_{n=0}^N A_n \left(\frac{n+2}{n+2+\chi} \frac{\rho^n}{\rho_1^n} - \frac{2}{n+2+\chi} \frac{H}{H+1} \right), \quad (\rho < \rho_1) \quad (22)$$

Where $H = -\frac{\chi}{\chi+2} \frac{\rho_1^2}{\rho_2^2}$.

Inserting Eq. (22) into Eq. (13), we obtain the stress distribution in the zone $\rho_1 < \rho < \rho_2$,

$$\begin{cases} \sigma_{rr}(\rho) = \sum_{n=0}^N \frac{2A_n \mu \rho_1^2}{(1-\nu)(n+2+\chi)(H+1)} \left(-\frac{1}{\rho^2} + \frac{1}{\rho_2^2} \right) \\ \sigma_{\theta\theta}(\rho) = \sum_{n=0}^N \frac{2A_n \mu \rho_1^2}{(1-\nu)(n+2+\chi)(H+1)} \left(\frac{1}{\rho^2} + \frac{1}{\rho_2^2} \right) \\ \sigma_{r\theta}(\rho) = 0 \end{cases} \quad (23)$$

And the stress components in the zone $\rho < \rho_1$ by Eq. (15) are

$$\begin{cases} \sigma_{rr}(\rho) = \sigma_{\theta\theta}(\rho) = \sum_{n=0}^N \frac{2A_n \mu}{(1-\nu)(n+2+\chi)} \left(\frac{1}{H+1} \frac{\rho_1^2}{\rho_2^2} - \frac{\rho^n}{\rho_1^n} + \frac{H}{H+1} \right) \\ \sigma_{r\theta}(\rho) = 0 \end{cases} \quad (24)$$

It can be found from Eqs. (23) and (24) that the value of radial stress inside the inclusion is equal to the hoop stress at the same position, while they are not equal outside.

For the sake of simplification and clarity, in all cases below, we choose the material such that $\nu = \nu^* = 0.3$. Moreover, soft inclusion $\mu / \mu^* = 10$ is set in the first case, and the others include homogeneous inclusion $\mu / \mu^* = 1$ and stiff inclusion $\mu / \mu^* = 1/10$. Corresponding to each material parameter, $\varepsilon^*(\rho) = 1$, $\varepsilon^*(\rho) = \rho / \rho_1$, $\varepsilon^*(\rho) = (\rho / \rho_1)^2$ represent the uniform, linear and nonlinear eigenstrain distribution, respectively.

Fig. 3 shows the dimensionless radial stress $\bar{\sigma}_{rr} = \sigma_{rr} / \mu$ and hoop stress $\bar{\sigma}_{\theta\theta} = \sigma_{\theta\theta} / \mu$ against dimensionless radius ρ / ρ_2 for these three cases when $\rho_1 / \rho_2 = 0.2$. It can be observed that:

- (i) The radial stress $\bar{\sigma}_{rr}$ is continuous at the interface of the inclusion and the matrix, and $\bar{\sigma}_{rr}(1) = 0$, while the hoop stress $\bar{\sigma}_{\theta\theta}$ is discontinuous at the position and $\bar{\sigma}_{\theta\theta} > 0$ in the matrix;
- (ii) Under given eigenstrain distribution, the stiffer inclusion it is, the bigger amplitude stress distribution occurs;
- (iii) The curves of the stress distribution are polynomials of degree n inside the inclusion and parabolas outside the inclusion.
- (iv) The radial and hoop stress keeps constant inside the inclusion and variable outside the inclusion under the case of $n=0$, which shows the boundary effect of matrix. This result is consistent with the result [44].

Fig. 3. The dimensionless radial and hoop stress distribution along radius corresponding to soft inclusion $\mu/\mu^*=10$, homogeneous inclusion $\mu/\mu^*=1$ and stiff inclusion $\mu/\mu^*=1/10$ when $\rho_1/\rho_2=0.2$.

When the inclusion covers a volume fraction of 50%, i.e., $\rho_1/\rho_2=0.5$, the dimensionless radial stresses $\bar{\sigma}_{rr}=\sigma_{rr}/\mu$ and hoop stress $\bar{\sigma}_{\theta\theta}=\sigma_{\theta\theta}/\mu$ are shown in Fig. 4. Similar trends can be seen but the discontinuity of the hoop stress at the interface of the inclusion and the matrix increases, thus enhancing the boundary effect as the fraction of inclusion becomes larger.

Fig. 4. The dimensionless radial and hoop stress distribution along radius corresponding to soft inclusion $\mu/\mu^*=10$, homogeneous inclusion $\mu/\mu^*=1$ and stiff inclusion when $\rho_1/\rho_2=0.5$.

5. Conclusion

In this paper, we mainly study the inhomogeneous circular inclusion problem with a non-uniform eigenstrain in a finite matrix, which is motivated by the researches in studying the finite matrix containing the homogeneous inclusion with uniform eigenstrains. First, we develop the procedure to deal with the general cases of the inhomogeneous inclusion problems based on the equivalent eigenstrain principle. Its equation is then extended to unify both cases of the finite and infinite matrix. Then, the elastic fields of both homogeneous and inhomogeneous circular inclusion problem are directly obtained. The two key parameters, affected by the finite boundary of the matrix and the material difference, respectively, have been found.

Besides, we use a typical example to illustrate the effects of eigenstrain distributions, material

mismatch and the inclusion ratio on stress fields. We find that the hoop stress at the interface is discontinuous, and the discontinuity is enhanced because of increasement on the stiffness and inclusion domain.

This paper only considers the two-dimensional inclusion problem in the framework of plane strain, but three-dimensional cases with non-uniform eigenstrains also surely can be analyzed through the same way. All the results derived in this paper can help engineers to design and evaluate materials and predict their failures.

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