Generalized Multiscale Approximation of Mixed Finite Elements with Velocity Elimination for Subsurface Flow

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Abstract

A frame work of the mixed generalized multiscale finite element method (GMsFEM) for solving Darcy's law in heterogeneous media is studied in this paper. Our approach approximates pressure in multiscale function space that is between fine-grid space and coarse-grid space and solves velocity directly in the fine-grid space. To construct multiscale basis functions for each coarse-grid element, three types of snapshot space are raised. The first one is taken as the fine-grid space for pressure and the other two cases need to solve a local problem on each coarse-grid element. We describe a spectral decomposition in the snapshot space motivated by the analysis to further reduce the dimension of the space that is used to approximate the pressure. Since the velocity is directly solved in the fine-grid space, in the linear system for the mixed finite elements, the velocity matrix can be approximated by a diagonal matrix without losing any accuracy. Thus it can be inverted easily. This reduces computational cost greatly and makes our scheme simple and easy for application. Comparing to our previous work of mixed generalized multiscale finite element method [E. T. Chung, Y. Efendiev, and C. S. Lee. Mixed generalized multiscale finite element methods and applications. *Multiscale Modeling and Simulation*, 13(1):338-366, 2015.], both the pressure and velocity space in this approach are bigger. As a consequence, this method enjoys better accuracy. While the computational cost does not increase because of the good property of velocity matrix. Moreover, the proposed method preserves the local mass conservation property that is important for subsurface problems. Numerical examples are presented to illustrate the good properties of the proposed approach. If offline spaces are appro-

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propiately selected, one can achieve good accuracy with only a few basis functions per coarse element according to the numerical results.

Keywords: multiscale, porous media, mixed finite elements, local mass conservation, two-phase flow

1. Introduction

Multiscale phenomena extensively exist in subsurface flows. The subsurface property permeability can span multiple scales and be of several orders of magnitude. For example, in the fractured and shale reservoir, the permeability in the fractures and shale layers is much bigger than the one in the matrix. And the thicknesses of these layers are much smaller than the domain size. Moreover, these subsurface flow problems often need to be solved repeatedly with different boundary conditions and source terms. As a consequence, the model reduction technique that can reduce the degrees of freedom significantly is required by such subsurface problems. A traditional approach is to solve local problems in each coarse block and to compute the effective permeability [18, 11, 37]. This method can be extended to multi-phase flow [28, 38, 35, 39] by upscaling relative permeabilities [6, 19]. An alternative method to reduce the model is the multiscale method [25, 22, 26, 21, 13, 27], in which the local solutions on a coarse element are employed as basis functions in the simulation. Since local mass conservation is important for subsurface problems, mixed multiscale finite element methods [12, 1, 2, 3, 9, 10, 16], multiscale finite volume methods [27, 24, 30, 17, 29], mortar multiscale methods [33, 4, 32, 15], and various post-processing approaches are proposed to obtain mass conservative velocity fields [8, 31].

The mixed multiscale finite element method is a widely used mass conservative multiscale approach [12, 1, 2, 3], in which the multiscale technique is mainly used for velocity. Multiscale basis functions for the velocity field are found by solving a local problem with Neumann boundary conditions with support to be two neighbouring coarse elements sharing a common edge. Piecewise constant basis functions, whose support is a single coarse element, are used for the pressure equations. In these methods, spatially distributed fluxes are imposed on internal edges to reduce the subgrid effects and for each edge, only one multiscale basis function is employed. Recently, a mixed generalized multiscale finite element method has been raised to solve the multiscale and high-contrast problems [14, 23, 13, 20]. The main advantage of the mixed GMsFEM is that multiple multiscale basis functions for the velocity can be used in two coarse blocks sharing a common edge. To compute the multiple velocity basis functions, local cell problems are solved in two coarse-block regions to construct the snapshot spaces and then the snapshot spaces are further reduced by solving local spectral problems designed according to the analysis. Typical local cell problems consist of the same governing equations as the original problem together with prescribed Neumann boundary conditions on the common edge between two coarse blocks. Ordering the eigenvalues in increasing order, the eigenvectors corresponding to small eigenvalues are selected to construct offline basis functions. The convergence analysis of these methods gives a spectral convergence $1/\Lambda$, where Λ is the smallest eigenvalue corresponding to the unused eigenvector [14].

In this paper, a frame work of the mixed generalized multiscale finite element method (GMsFEM) for subsurface problems is proposed. Different from the previous approaches, multiple multiscale basis functions for the pressure are computed in single coarse element and the velocity is approximated in the fine grid space. To construct multiscale basis functions for each coarse-grid element, three types of snapshot space are raised. The first one is taken as the fine-grid space for pressure and the other two cases need to solve a local problem on each coarse-grid element. We describe a spectral decomposition in the snapshot space motivated by the analysis to further reduce the dimension of the space that is used to approximate the pressure. Although the velocity is directly solved on fine mesh, with the help of trapezoidal quadrature rule, the stiffness matrix for velocity is diagonal and can be inverted easily. Thus we only need to solve a linear system in multiscale finite element space for pressure. If offline spaces are appropriately selected, one can achieve good accuracy with only a few basis functions per coarse element according to our numerical results.

The rest of this paper is arranged as follows. Section 2 introduces the basic model of subsurface flows, the definition of notations and fine-scale discretization. Section 3 illustrates a systematic way to construct the snapshot space and offline space. A convergence analysis for the proposed mixed generalized multiscale finite element method is presented in section 4. Numerical results and discussions are shown in section 5. The paper ends with a conclusion.

2. Preliminaries

We consider the following flow problem in mixed formulation with high-contrast coefficient:

$$\begin{aligned} k^{-1}\mathbf{u} + \nabla p &= 0 & \text{in } \Omega, \\ \nabla \cdot \mathbf{u} &= f & \text{in } \Omega, \end{aligned} \quad (2.1)$$

with nonhomogeneous boundary condition

$$\mathbf{u} \cdot \mathbf{n} = g \quad \text{on } \partial\Omega, \quad (2.2)$$

where k is a high-contrast heterogeneous permeability, $\mathbf{u}$ is the Darcy velocity, p is the pressure, f is the source term, g is the given normal component of Darcy velocity on the boundary, Ω is the computational domain and $\mathbf{n}$ is the outward unit norm vector on the boundary.

To illustrate the general solution framework of the proposed mixed GMsFEM, the notations of coarse and fine grids are introduced as follows. Let $\mathcal{T}^H$ be a conforming partition of the computational domain Ω into finite elements (quadrilaterals, parallelepipeds, triangles, tetrahedrals, etc.) with coarse-block size H and $\mathcal{T}^h$ be the fine-grid partition with mesh size h . We use $\mathcal{E}^H := \bigcup_{i=1}^{N_e} E_i$ to denote the set of all edges in the coarse mesh $\mathcal{T}^H$, where N_e is the number of coarse edges. We also define the element in coarse mesh as T_i , thus $\Omega := \bigcup_{i=1}^{N_t} T_i$ (N_t is the number of elements in the coarse mesh). The element in the fine mesh is defined as t_i , thus $\Omega := \bigcup_{i=1}^{M_t} t_i$ with M_t being

the number of elements in the fine mesh. The set of edges in the fine grid is denoted by $\mathcal{E}^h := \bigcup_{i=1}^{M_e} e_i$, where M_e is the number of fine edges.

Define

$$L^2(\Omega) = \{v : v \text{ is defined on } \Omega \text{ and } \int_{\Omega} v^2 dx < \infty\}.$$

We also use the space

$$\mathbf{H}(\text{div}, \Omega) = \{\mathbf{v} = (v_1, v_2) \in (L^2(\Omega))^2 : \nabla \cdot \mathbf{v} \in L^2(\Omega)\}.$$

Define

$$\mathbf{V} = \mathbf{H}(\text{div}, \Omega), \quad W = L^2(\Omega).$$

Recall $\mathcal{T}^h$ is the fine partition of Ω into non-overlapping (open) elements. Define the mixed finite element spaces (we take the mixed finite element spaces on quadrilaterals for illustration).

$$\mathbf{V}_h = \{\mathbf{v}_h \in \mathbf{V} : \mathbf{v}_h|_t = (b_t x_1 + a_t, d_t x_2 + c_t), a_t, b_t, c_t, d_t \in \mathbb{R}, t \in \mathcal{T}^h\},$$

$$W_h = \{w_h \in W : w_h \text{ is a constant on each element in } \mathcal{T}^h\}.$$

The normal components of $\mathbf{v}_h \in \mathbf{V}_h$ are continuous across the interior edges in $\mathcal{T}^h$. Thus, the fine-grid solution $(\mathbf{u}_h, p_h) \in (\mathbf{V}_h, W_h)$ satisfies

$$\begin{aligned} \int_{\Omega} k^{-1} \mathbf{u}_h \cdot \mathbf{v}_h - \int_{\Omega} \text{div}(\mathbf{v}_h) p_h &= 0 & \forall \mathbf{v}_h \in \mathbf{V}_h^0, \\ \int_{\Omega} \text{div}(\mathbf{u}_h) w_h &= \int_{\Omega} f w_h & \forall w_h \in W_h, \end{aligned} \quad (2.3)$$

where $\mathbf{u}_h \cdot \mathbf{n} = g_h$ on $\partial\Omega$ with g_h being the average of function g on each fine boundary edge and $\mathbf{V}_h^0 = \{\mathbf{v}_h \in \mathbf{V}_h : \mathbf{v}_h \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}$. The above system can be recast in matrix form

$$\begin{aligned} M_{\text{fine}} \mathbf{U}_h + B_{\text{fine}} \mathbf{P}_h &= \mathbf{0}, \\ B_{\text{fine}} \mathbf{U}_h &= \mathbf{F}_h. \end{aligned} \quad (2.4)$$

The velocity will be approximated directly in the fine-grid space $\mathbf{V}_h$. The approximation of the pressure will be found in the multiscale finite element space which is between the coarse-grid space and fine-grid space. A set of multiscale basis functions for the pressure are defined in each coarse-grid element with the support to be the corresponding coarse element. Let $\{\Psi_j\}$ be the set of multiscale basis functions for the coarse element T_i . The multiscale space for the pressure p is defined as the linear span of all local basis functions, which can be denoted as

$$W_H = \bigoplus_{\mathcal{T}^H} \{\Psi_i\}.$$

The mixed GMSFEM is to find $(\mathbf{u}_H, p_H) \in (\mathbf{V}_h, W_H)$ such that

$$\begin{aligned} \int_{\Omega} k^{-1} \mathbf{u}_H \cdot \mathbf{v}_H - \int_{\Omega} \operatorname{div}(\mathbf{v}_H) p_H &= 0 & \forall \mathbf{v}_H \in \mathbf{V}_h^0, \\ \int_{\Omega} \operatorname{div}(\mathbf{u}_H) w_H &= \int_{\Omega} f w_H & \forall w_H \in W_H, \end{aligned} \quad (2.5)$$

where $\mathbf{u}_H \cdot \mathbf{n} = g_H$ on $\partial\Omega$ and for each coarse edge on the boundary and g_H is the average of function g on the corresponding coarse edge.

3. The construction of multiscale basis functions

In this section, we will illustrate a systematic way to construct the multiscale space W_H for approximating the pressure p . The constructions of snapshot spaces are firstly introduced. There are three possible ways to do this. The first case simply takes the fine-grid space for pressure W_h to be the snapshot space. The other two contain an extensive set of basis functions formed by the solutions of local problems with Dirichlet and Neumann boundary conditions, respectively. Then the snapshot space is further reduced by solving a local spectral problem to obtain the dominant modes. The spectral problem is carefully designed to reduce the residual rapidly. The linear span of the obtained dominant modes is called the offline space. Since the problem we considered is parameter independent, the offline space is the same as the online space.

3.1. Snapshot space

There are three ways to construct the snapshot space. We illustrate them one by one.

Case I: The snapshot space W_{snap} can be simply taken as the fine-grid space for the pressure W_h . That is

$$W_{\text{snap}} = \{\Psi_i^{\text{snap}} \in W : \Psi_i^{\text{snap}} \text{ is a constant on each element } t_i \text{ in } \mathcal{T}^h\}.$$

Case II: Let $T_i \in \mathcal{T}^H$ be a coarse element in Ω . We will find $(\mathbf{u}_j^{(i)}, p_j^{(i)}) \in (\mathbf{V}_h, W_h)|_{T_i}$ by solving the following problem on the coarse element T_i :

$$\begin{aligned} k^{-1} \mathbf{u}_j^{(i)} + \nabla p_j^{(i)} &= 0 & \text{in } T_i, \\ \operatorname{div}(\mathbf{u}_j^{(i)}) &= 0 & \text{in } T_i, \end{aligned} \quad (3.1)$$

where $(\mathbf{V}_h, W_h)|_{T_i}$ is the restriction of $(\mathbf{V}_h, W_h)$ on the coarse-grid element T_i . Notice that the boundary of coarse-grid element can be expressed as a union of fine-grid edges, i.e., $\partial T_i = \bigcup_{j=1}^{J_i} e_j$, where J_i is the total number of fine-grid edges on the boundary of coarse-grid element T_i . Let $\delta_j^{(i)}$ be a piecewise constant function defined on ∂T_i with respect to the fine grid such that it has value 1 on e_j and value 0 on the other fine-grid

edges; That is,

$$\delta_j^{(i)} = \begin{cases} 1 & \text{in } e_j, \\ 0 & \text{on other fine-grid edges on } \partial T_i, \end{cases} \quad j = 1, 2, \dots, J_i. \quad (3.2)$$

Thus the boundary condition on the boundary of coarse-grid element T_i for the local problem (3.1) is taken as

$$p_j^{(i)} = \delta_j^{(i)} \quad \text{on } \partial T_i. \quad (3.3)$$

The above local problem (3.1) together with the Dirichlet boundary condition (3.3) can be solved numerically on fine grid T_i by the lowest-order Raviart-Thomas elements, so that the resulting pressure $p_j^{(i)} \in W_h$ (we keep the same notation for the discrete solution $p_j^{(i)}$ for simplicity).

Case III: Let $T_i \in \mathcal{T}^H$ be a coarse element in Ω . We will find $(\mathbf{u}_j^{(i)}, p_j^{(i)}) \in (\mathbf{V}_h, W_h)|_{T_i}$ by solving the following problem on the coarse element T_i :

$$\begin{aligned} k^{-1} \mathbf{u}_j^{(i)} + \nabla p_j^{(i)} &= 0 & \text{in } T_i, \\ \operatorname{div}(\mathbf{u}_j^{(i)}) &= \alpha_j^{(i)} & \text{in } T_i, \end{aligned} \quad (3.4)$$

together with the Neumann boundary condition on the boundary of coarse-grid element T_i

$$\frac{\partial p_j^{(i)}}{\partial \mathbf{n}_i} = \delta_j^{(i)} \quad \text{on } \partial T_i, \quad (3.5)$$

where $\mathbf{n}_i$ is an outward unit norm vector on ∂T_i and $\alpha_j^{(i)}$ is chosen so that the compatibility condition $\int_{T_i} \alpha_j^{(i)} = \int_{\partial T_i} \delta_j^{(i)}$ is satisfied. In this case, p is unique up to an additive constant. Similarly, the above local problem (3.4) together with the Neumann boundary condition (3.5) can be solved numerically on fine grid T_i by the lowest-order Raviart-Thomas elements, so that the resulting pressure $p_j^{(i)} \in W_h$. We remark that the velocity satisfies $\mathbf{u} \cdot \mathbf{m}_i = \delta_j^{(i)}$ on ∂T_i in this case.

The collection of the solutions of above local problems (Case II or Case III) generates the snapshot space. We let $\Psi_j^{i, \text{snap}} := p_j^{(i)}$ be the snapshot fields and define the snapshot space W_{snap} by

$$W_{\text{snap}} = \operatorname{span}\{\Psi_j^{i, \text{snap}} : 1 \leq j \leq J_i, 1 \leq i \leq N_t\}.$$

To simplify expressions, we use the single-index notation

$$W_{\text{snap}} = \operatorname{span}\{\Psi_i^{\text{snap}} : 1 \leq i \leq M_{\text{snap}}\},$$

where $M_{\text{snap}} = \sum_{i=1}^{N_t} J_i$ is the total number of snapshot fields.

Remark 3.1. For Case I, there is no extra computational cost for the local problem

and the dimension of the snapshot space in coarse element T_i is the number of fine elements contained in T_i . For Case II and III, we need to solve local problems in each coarse-grid element and the dimension of the snapshot space is reduced to the number of fine edges e_j on the boundary of coarse element T_i .

Remark 3.2. For the boundary element that has two or more boundary edges (for example corner elements), all the edges belong to this element are considered as a single boundary edge for the definition of the function $\delta_j^{(i)}$, so that all Ψ_i^{snap} in the snapshot space W_{snap} are linearly independent.

Remark 3.3. For Case III, since the Neumann boundary condition for pressure is used in the local problem, the solution p is unique up to an additive constant. As a consequence, we emphasize that the piecewise constant basis on the coarse-grid element should be included in the snapshot space W_{snap} .

Each multiscale basis function Ψ_i^{snap} can be represented as a linear combination of fine-grid basis functions in W_h . Thus, we can use the coefficients vector ψ_i^{snap} to represent the multiscale basis function Ψ_i^{snap} . We define

$$R_{\text{snap}} = [\psi_1^{\text{snap}}, \dots, \psi_{M_{\text{snap}}}^{\text{snap}}],$$

which maps from the snapshot space to the fine space. Specially, for Case I, R_{snap} is an identity matrix.

3.2. Offline space

In this subsection, we further reduce the snapshot space by solving some local spectral problems and we term the solutions of local spectral problems the offline space. The offline space contains the important modes in the snapshot space and can approximate the solution efficiently. We consider the spectral problem of finding a real number λ and a function $p \in W_{\text{snap}}$ such that

$$a(p, w) = \lambda s(p, w), \quad \forall w \in W_{\text{snap}}, \quad (3.6)$$

where $a(p, w)$ and $s(p, w)$ are positive semidefinite symmetric bilinear forms defined on the snapshot space $W_{\text{snap}} \times W_{\text{snap}}$.

It is inefficient to solve the global spectral problem (3.6). Thus we solve it in each coarse-grid element T_i . We let $W_{\text{snap}}^{(i)}$ be the snapshot space corresponding to the coarse element T_i , which is defined by

$$W_{\text{snap}}^{(i)} = \text{span}\{\Psi_j^{i, \text{snap}} : 1 \leq j \leq J_i\}.$$

The following is the local spectral problem: Find a real number $\lambda \geq 0$ and a function $p \in W_{\text{snap}}^{(i)}$ such that

$$a_i(p, w) = \lambda s_i(p, w) \quad \forall w \in W_{\text{snap}}^{(i)}. \quad (3.7)$$

For each coarse element T_i , we take

$$a_i(p, w) = \sum_e k[p][w], \quad s_i(p, w) = \int_{T_i} kpw, \quad (3.8)$$

where $[p]$ and $[w]$ are the jump of functions p and w , respectively. e is an interior fine edge in coarse-grid element T_i .

Remark 3.4. *The definition of spectral problem (3.7) and (3.8) is suitable for all the three cases of the snapshot space. For case II and III, where a local problem is solved, the spectral problem (3.7) and (3.8) is equivalent to*

$$a_i(p, w) = \lambda s_i(p, w) \quad \forall w \in W_{\text{snap}}^{(i)}.$$

where

$$a_i(p, w) = \int_{T_i} k \nabla p \cdot \nabla w = \int_{T_i} k^{-1} \mathbf{u} \cdot \mathbf{v}, \quad s_i(p, w) = \int_{T_i} kpw.$$

We arrange the eigenvalues of (3.7) in increasing order,

$$\lambda_1^{(i)} < \lambda_2^{(i)} < \dots < \lambda_{J_i}^{(i)}, \quad (3.9)$$

where $\lambda_k^{(i)}$ denotes the k th eigenvalue for the coarse-grid element T_i . The corresponding eigenvector is expressed as $Z_k^{(i)} = (Z_{kj}^{(i)})_{j=1}^{J_i}$ with $Z_{kj}^{(i)}$ being the j th component of the vector $Z_k^{(i)}$. The first l_i eigenfunctions are selected to form the offline space. Using the eigenfunctions, the offline basis functions can be constructed as

$$\Psi_k^{i, \text{off}} = \sum_{j=1}^{J_i} Z_{kj}^{(i)} \Psi_j^{i, \text{snap}}, \quad k = 1, 2, \dots, l_i.$$

Then the global offline space is

$$W_{\text{off}} = \text{span}\{\Psi_k^{i, \text{off}} : 1 \leq k \leq l_i, 1 \leq i \leq N_t\}$$

Using the single-index notation to simplify expression, we get

$$W_{\text{off}} = \text{span}\{\Psi_k^{\text{off}} : 1 \leq k \leq M_{\text{off}}\}$$

where $M_{\text{off}} = \sum_{i=1}^{N_t} l_i$ is the total number of offline basis functions. We will employ this space to approximate the pressure, i.e. $W_H = W_{\text{off}}$ in the mixed GMsFEM system (2.5). The above eigenvalue problem (3.6) can be recast in matrix form

$$A_{\text{snap}}^{(i)} Z_k^{(i)} = \lambda_k^{(i)} S_{\text{snap}}^{(i)} Z_k^{(i)}, \quad (3.10)$$

where

$$\begin{aligned} A_{\text{snap}}^{(i)} &= a_i(\Psi_m^{i,\text{snap}}, \Psi_n^{i,\text{snap}}) = R_{\text{snap}}^T A_{\text{fine}}^{(i)} R_{\text{snap}} \\ S_{\text{snap}}^{(i)} &= s_i(\Psi_m^{i,\text{snap}}, \Psi_n^{i,\text{snap}}) = R_{\text{snap}}^T S_{\text{fine}}^{(i)} R_{\text{snap}}. \end{aligned}$$

The notations $A_{\text{snap}}^{(i)}$ and $S_{\text{snap}}^{(i)}$ express the fine-scale matrices using fine-grid basis functions in coarse element T_i . Each Ψ_k^{off} can be expressed by a vector ψ_k^{off} containing the coefficients in the expansion of Ψ_k^{off} in the fine-grid basis functions. Thus we have

$$R_{\text{off}} = [\psi_1^{\text{off}}, \dots, \psi_{M_{\text{off}}}^{\text{off}}],$$

which maps from the offline space to the fine space. In terms of matrix expressions, the mixed GMsFEM system (2.5) can be represented as

$$\begin{aligned} M_{\text{fine}} \mathbf{U}_H + B_{\text{fine}}^T R_{\text{off}} \mathbf{P}_H &= 0, \\ R_{\text{off}}^T B_{\text{fine}} \mathbf{U}_H &= R_{\text{off}}^T \mathbf{F}_H, \end{aligned} \tag{3.11}$$

where the matrix M_{fine} is symmetric, positive definite, and sparse. $\mathbf{U}_H$ and $\mathbf{P}_H$ are unknown velocity and pressure vectors in the spaces $\mathbf{V}_h$ and W_H , respectively. As a consequence, to implement the proposed mixed GMsFEM, we need to construct the fine-grid matrices M_{fine} , B_{fine} and the offline matrix R_{off} .

Since the velocity is directly solved on the fine mesh, in the case that k is a diagonal tensor, the matrix M_{fine} can be approximated by a diagonal matrix $\widehat{M}_{\text{fine}}$ when the trapezoidal quadrature rule is used [34, 5]. The lowest order Raviart-Thomas mixed finite element method is equivalent to a finite difference scheme with a five-point stencil in two dimensional space and the modified scheme has the same convergence rate as the unmodified scheme. In other words, we can replace M_{fine} with a diagonal matrix $\widehat{M}_{\text{fine}}$ without losing any accuracy. Thus $\widehat{M}_{\text{fine}}$ is easy to invert and we can solve the system (3.11) in following way

$$-R_{\text{off}}^T B_{\text{fine}} \widehat{M}_{\text{fine}}^{-1} B_{\text{fine}}^T R_{\text{off}} \mathbf{P}_H = R_{\text{off}}^T \mathbf{F}_H.$$

In this way, the original mixed formulation is approximated by a positive definite and sparse linear system for the pressure unknowns and there are only several pressure unknowns per coarse element. As a consequence, the modified system is substantially smaller and easier to solve than the hybrid system used in [14].

4. Convergence of the mixed GMsFEM

The convergence analysis of the proposed mixed GMsFEM is presented in this section. We first introduce a projection of the fine-grid velocity field $\mathbf{u}_h$ by solving a carefully designed local problem and then derive an error estimate between this projection and fine-grid solution. Next, An estimate between the mixed GMsFEM solution

and the projection is derived. Combining these two estimations, we have an estimate between the mixed GMsFEM and fine-grid solutions.

Recall that $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times W_h$ is the fine-grid solution obtained in (2.3). We will define a projection $\hat{\mathbf{u}} \in \mathbf{V}_h$ as follows. Let T be a coarse-grid block and let $\bar{f}$ be L^2 projection of the source term f onto the space W_H

$$\int_T (f - \bar{f})w = 0, \quad w \in W_H.$$

Then, the restriction of $\hat{\mathbf{u}}$ on T is obtained by solving the following problem

$$\begin{aligned} k^{-1}\hat{\mathbf{u}} + \nabla \hat{p} &= 0 & \text{in } T, \\ \operatorname{div}(\hat{\mathbf{u}}) &= \bar{f} & \text{in } T, \end{aligned} \quad (4.1)$$

subject to the following conditions

$$\hat{\mathbf{u}} \cdot \mathbf{n} = \mathbf{u}_h \cdot \mathbf{n}, \quad \text{on } \partial T \quad \text{and} \quad \int_T \hat{p} = \int_T p_h. \quad (4.2)$$

The above problem (4.1)-(4.2) is solved on the fine grid, thus we have $\hat{\mathbf{u}} \in \mathbf{V}_h$.

We introduce a weighted L^2 norm that will be used in the following analysis. Let T be an open set and $\mathbf{u}$ be a vector field. Then the weighted L^2 norm is defined as $\|\mathbf{u}\|_{k^{-1}, T} = (\int_T k^{-1} |\mathbf{u}|^2)^{\frac{1}{2}}$.

Next, we prove the following estimate for $\hat{\mathbf{u}}$.

Lemma 4.1. *Let $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times W_h$ be the fine-grid solution obtained in (2.3) and $\hat{\mathbf{u}} \in \mathbf{V}_h$ be the solution of (4.1)-(4.2). We have*

$$\int_{\Omega} k^{-1} |\mathbf{u}_h - \hat{\mathbf{u}}|^2 + \int_{\Omega} (\operatorname{div}(\mathbf{u}_h - \hat{\mathbf{u}}))^2 \preceq \max_{T \in \mathcal{T}^H} (k_{\min, T}^{-1}) \sum_{i=1}^{N_t} \|f - \bar{f}\|_{L^2(T_i)}^2, \quad (4.3)$$

where $k_{\min, T}$ is the minimum of k over T .

Proof. Let $T \in \mathcal{T}^H$ be a given coarse-grid element. First, subtracting (2.3) by the variational form of (4.1), we get

$$\begin{aligned} \int_T k^{-1} (\mathbf{u}_h - \hat{\mathbf{u}}) \cdot \mathbf{v}_h - \int_T \operatorname{div}(\mathbf{v}_h) (p_h - \hat{p}) &= 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h^0(T), \\ \int_T \operatorname{div}(\mathbf{u}_h - \hat{\mathbf{u}}) w_h &= \int_T (f - \bar{f}) w_h, \quad \forall w_h \in W_h(T), \end{aligned} \quad (4.4)$$

where $W_h(T)$ is the restriction of W_h on T and $\mathbf{V}_h^0(T)$ is the restriction of $\mathbf{V}_h$ on T containing vector fields with zero normal component on ∂T . Taking $\mathbf{v}_h = \mathbf{u}_h - \hat{\mathbf{u}}$ and $w_h = p_h - \hat{p}$ in (4.4), and summing up the resulting equations, we obtain

$$\int_T k^{-1} (\mathbf{u}_h - \hat{\mathbf{u}}) \cdot (\mathbf{u}_h - \hat{\mathbf{u}}) = \int_T (f - \bar{f}) (p_h - \hat{p}). \quad (4.5)$$

Recall that the Raviart-Thomas elements satisfy the following inf-sup condition [7]:

$$\|w_h\|_{L^2(T)} \preceq \sup_{\mathbf{v}_h \in \mathbf{V}_h(T)} \frac{\int_T \operatorname{div}(\mathbf{v}_h) w_h}{\|\mathbf{v}_h\|_{H(\operatorname{div}; T)}}, \quad \forall w_h \in W_h(T), \quad (4.6)$$

where $\mathbf{V}_h(T)$ is the restriction of $\mathbf{V}_h$ on T . Using the inf-sup condition (4.6) and the error equation (4.4), we have

$$\|p_h - \hat{p}\|_{L^2(T)} \preceq k_{\min, T}^{-\frac{1}{2}} \|\mathbf{u}_h - \hat{\mathbf{u}}\|_{k^{-1}, T}.$$

Finally, by (4.5), we obtain

$$\|\mathbf{u}_h - \hat{\mathbf{u}}\|_{k^{-1}, T} \preceq k_{\min, T}^{-\frac{1}{2}} \|f - \bar{f}\|_{L^2(T)}.$$

Also, by (4.4), we have

$$\|\operatorname{div}(\mathbf{u}_h - \hat{\mathbf{u}})\|_{L^2(T)} = \sup_{w_h \in W_h(T)} \frac{\int_T \operatorname{div}(\mathbf{u}_h - \hat{\mathbf{u}}) w_h}{\|w_h\|_{L^2(T)}} = \sup_{w_h \in W_h(T)} \frac{\int_T (f - \bar{f}) w_h}{\|w_h\|_{L^2(T)}} \leq \|f - \bar{f}\|_{L^2(T)}.$$

Collecting results for all coarse-grid elements, we have the desired estimate (4.3). $\square$

Now the convergence theorem for the mixed GMsFEM (2.5) is expressed and proved in the following.

Theorem 4.2. *Let $\mathbf{u}_h$ be the fine-grid solution obtained in (2.3) and $\mathbf{u}_H$ be the mixed GMsFEM solution obtained in (2.5). Then, the following estimate holds*

$$\int_{\Omega} k^{-1} |\mathbf{u}_h - \mathbf{u}_H|^2 \preceq \Lambda^{-1} \sum_{i=1}^{N_t} a_i(p_h, p_h) + \max_{T \in \mathcal{T}^H} \left(k_{\min, T}^{-1} \right) \sum_{i=1}^{N_t} \|f - \bar{f}\|_{L^2(T_i)}^2, \quad (4.7)$$

where $\Lambda = \min_{1 \leq i \leq N_t} \lambda_{l_i+1}^{(i)}$ and $\hat{\mathbf{u}}$ is the projection of $\mathbf{u}_h$ defined in (4.1)-(4.2).

Proof. Subtracting (2.3) by (2.5), and using the fact that $W_H \subset W_h$, we have

$$\begin{aligned} \int_{\Omega} k^{-1} (\mathbf{u}_h - \mathbf{u}_H) \cdot \mathbf{v}_H - \int_{\Omega} \operatorname{div}(\mathbf{v}_H) (p_h - p_H) &= 0, \quad \forall \mathbf{v}_H \in \mathbf{V}_h^0, \\ \int_{\Omega} \operatorname{div}(\mathbf{u}_h - \mathbf{u}_H) w_H &= 0, \quad \forall w_H \in W_H. \end{aligned} \quad (4.8)$$

By (4.4), for each coarse-grid element T , we have

$$\int_T \operatorname{div}(\mathbf{u}_h - \hat{\mathbf{u}}) w_H = \int_T (f - \bar{f}) w_H = 0, \quad \forall w_H \in W_H$$

by definition of $\bar{f}$. Thus, (4.8) can be written as

$$\begin{aligned} \int_{\Omega} k^{-1}(\mathbf{u}_h - \mathbf{u}_H) \cdot \mathbf{v}_H - \int_{\Omega} \operatorname{div}(\mathbf{v}_H)(\hat{p} - p_H) &= \int_{\Omega} \operatorname{div}(\mathbf{v}_H)(p_h - \hat{p}), \quad \forall \mathbf{v}_H \in \mathbf{V}_h^0, \\ \int_{\Omega} \operatorname{div}(\hat{\mathbf{u}} - \mathbf{u}_H)w_H &= 0, \quad \forall w_H \in W_H. \end{aligned} \quad (4.9)$$

Taking $\mathbf{v}_H = \hat{\mathbf{u}} - \mathbf{u}_h$ in (4.9), we obtain

$$\int_{\Omega} k^{-1}(\mathbf{u}_h - \mathbf{u}_H) \cdot (\hat{\mathbf{u}} - \mathbf{u}_h) = \int_{\Omega} \operatorname{div}(\hat{\mathbf{u}} - \mathbf{u}_h)(p_h - \hat{p}) \quad (4.10)$$

Note that we can replace $\hat{p}$ by any $\tilde{p} \in W_H$,

$$\int_{\Omega} k^{-1}(\mathbf{u}_h - \mathbf{u}_H) \cdot (\hat{\mathbf{u}} - \mathbf{u}_h) = \int_{\Omega} \operatorname{div}(\hat{\mathbf{u}} - \mathbf{u}_h)(p_h - \tilde{p}) \quad (4.11)$$

So,

$$\int_{\Omega} k^{-1}|\mathbf{u}_h - \mathbf{u}_H|^2 \preceq \int_{\Omega} k^{-1}|\mathbf{u}_h - \hat{\mathbf{u}}|^2 + \|\operatorname{div}(\hat{\mathbf{u}} - \mathbf{u}_h)\|_{L^2(\Omega)} \|p_h - \tilde{p}\|_{L^2(\Omega)}. \quad (4.12)$$

Next, we choose $\tilde{p}$ to be a linear combination of the modes for the eigenvalue problem (3.7) that corresponds to the eigenvalues $\lambda_1^{T_i}, \dots, \lambda_{l_i}^{T_i}$. Then we have

$$\|p_h - \tilde{p}\|_{L^2(T_i)}^2 \preceq s_i(p_h - \tilde{p}, p_h - \tilde{p}) \preceq \frac{1}{\lambda_{l_i+1}^{T_i}} a_i(p_h - \tilde{p}, p_h - \tilde{p}) = \frac{1}{\lambda_{l_i+1}^{T_i}} \int_{T_i} k|\nabla(p_h - \tilde{p})|^2 \quad (4.13)$$

Considering $\int_{T_i} k|\nabla(p_h - \tilde{p})|^2 \preceq \int_{T_i} k|\nabla p_h|^2$ and collecting the result of Lemma 4.1, (4.12) and (4.13), we obtain the desired result (4.7). $\square$

5. Numerical examples

In this section, some numerical examples are presented to illustrate the performance of the proposed mixed GMsFEM (2.5) for approximating the subsurface flow problem (2.1). In the following examples, the computational domain is set to be $\Omega = [0, 1]^2$. The $N \times N$ uniform mesh is chosen to be the coarse grid $\mathcal{T}^H$ and the fine grid $\mathcal{T}^h$ is the $n \times n$ uniform mesh. The permeability field k we considered here is shown in Figure 1, which represents fractured porous media. The fine-mesh size is chosen to have the same resolution as the permeability field. To shown the performance of the mixed GMsFEM, we define $(\mathbf{u}_f, p_f)$, $(\mathbf{u}_s, p_s)$, and $(\mathbf{u}_g, p_g)$ as the fine-mesh solution, snapshot solution, and mixed GMsFEM solution, respectively. The snapshot solution is the solution of the local problem (3.1) (Case II) and (3.4) (Case III) with all basis functions in the snapshot space selected. For Case I, the snapshot solution is the same as fine-mesh solution. The theoretic result shows that the mixed GMsFEM solution

Figure 1: The permeability field of the fractured porous media

contains both coarse-grid and spectral errors and the snapshot solution contains only the coarse-grid error. The following error quantities are defined for the pressure

$$E_{gf}(p) := \|p_g - p_f\|_{L^2(\Omega)} / \|p_f\|_{L^2(\Omega)}, \quad E_{gs} := \|p_g - p_s\|_{L^2(\Omega)} / \|p_s\|_{L^2(\Omega)}.$$

For the velocity, we define

$$E_{gf}(\mathbf{u}) := \|\mathbf{u}_g - \mathbf{u}_f\|_{k-1,\Omega} / \|\mathbf{u}_f\|_{k-1,\Omega}, \quad E_{gs} := \|\mathbf{u}_g - \mathbf{u}_s\|_{k-1,\Omega} / \|\mathbf{u}_s\|_{k-1,\Omega}.$$

5.1. Single-phase flow

In this subsection, we consider the single-phase flow in fractured porous media. In the simulation, the fine-grid size is set to be $n = 100$ and the coarse-grid size is $N = 10$. The permeability field is illustrated in Figure 1. The permeability is 1000 in the fractures and 1 in the matrix. The homogenous Neumann boundary condition $\frac{\partial p}{\partial \mathbf{n}} = 0$ is used on northern and southern boundary. The Dirichlet boundary condition $p = 100$ is put on the eastern boundary and $p = 0$ is used on western boundary. The numerical results of Case I and Case II are shown in Table 3 and the results of Case III are presented in Table 4. The term “dof per T ” means the number of basis functions used for that coarse-grid element. The convergence behaviors of the proposed mixed GmsFEM are shown in Table 3 and Table 4. Since each coarse-grid element is decomposed as 10×10 fine grids, there are 40 fine-grid boundary edges for each coarse-grid element. The two boundary edges belong to the same corner fine-grid element are considered as a single edge in the local problem, so the number of basis functions in the snapshot space is 36 for Case II and III. From Table 3 and Table 4, we can see that the error decreases as more basis functions are added into the offline space. For Case I, since we use the fine-grid space for pressure W_h as the snapshot space W_{snap} , the spectral error is the same as the total error. For Case II, both the total error and spectral error converge to the machine precision, which means the snapshot

Table 1: Error estimation and convergence of the offline solution for Case I and Case II.

dof per T	Case I		Case II			
	$E_{gf}(p)$	$E_{gf}(\mathbf{u})$	$E_{gf}(p)$	$E_{gf}(\mathbf{u})$	$E_{gs}(p)$	$E_{gs}(\mathbf{u})$
1	0.1280	2.6333	0.1280	2.6333	0.1280	2.6333
3	0.0548	0.5463	0.0533	0.5381	0.0533	0.5381
5	0.0189	0.3015	0.0144	0.2736	0.0144	0.2736
10	0.0132	0.2580	0.0062	0.1481	0.0062	0.1481
15	0.0051	0.1393	0.0029	0.1103	0.0029	0.1103
20	0.0041	0.1235	0.0012	0.0498	0.0012	0.0498
25	0.0021	0.0700	7.96e-04	0.0301	7.96e-04	0.0301
30	0.0017	0.0604	3.36e-04	0.0130	3.36e-04	0.0130
35	0.0015	0.0538	1.41e-04	0.0041	1.41e-04	0.0041
36	0.0015	0.0534	4.13e-16	4.25e-15	5.25e-16	4.53e-15
80	1.02e-04	0.0034	-	-	-	-
100	7.46e-16	3.32e-15	-	-	-	-

Table 2: Error estimation and convergence of the offline solution for Case III.

dof per T	Case III			
	$E_{gf}(p)$	$E_{gf}(\mathbf{u})$	$E_{gs}(p)$	$E_{gs}(\mathbf{u})$
1	0.1280	2.6333	0.1281	2.6330
3	0.0556	0.5439	0.0556	0.5439
5	0.0150	0.2754	0.0149	0.2752
10	0.0068	0.1609	0.0067	0.1607
15	0.0029	0.1092	0.0028	0.1089
20	0.0014	0.0519	0.0013	0.0513
25	7.6503e-04	0.0251	5.9956e-04	0.0237
30	5.6132e-04	0.0159	3.4960e-04	0.0136
35	4.4009e-04	0.0095	1.4973e-04	0.0045
36	4.2434e-04	0.0090	1.1871e-15	1.1077e-13

space is very close to the fine-grid space. For Case III, the spectral error converges to the machine precision.

The reciprocals of the eigenvalues in a particular coarse element for three cases are shown in Figure 2. Please note that the first 0 eigenvalue is not included in figures. It is obvious that there is a rapid decay for the first several eigenvalues and the eigenvalues decrease all the time. This eigenvalue behaviors is corresponding to the error estimations shown in Table 3 and Table 4.

The pressure and velocity solved on fine and coarse meshes are shown in Figure 3 and 4, respectively. The mixed GMsFEM solutions of Case I, II and III are illustrated in Figure 5, 6 and 7 with “dof per T ” being 40, 25 and 36, respectively. It’s obvious that all the cases have a competitive performance of fine-grid solutions.

Figure 2: Inverse of eigenvalue behavior for three cases. Top left: Case I. Top right: Case II. Bottom: Case III

Figure 3: Fine grid solution. left: pressure. right: magnitude of velocity.

Figure 4: Coarse grid solution. left: pressure. right: magnitude of velocity.

Figure 5: Mixed GMsFEM solution for Case I. left: pressure. right: magnitude of velocity.

Figure 6: Mixed GMsFEM solution for Case II. left: pressure. right: magnitude of velocity.

Figure 7: Mixed GMsFEM solution for Case III. left: pressure. right: magnitude of velocity.

5.2. Single-phase flow and transport problems

In this subsection, we employ the proposed mixed GMsFEM to simulate the single-phase flow and transport problems. Specifically, the following problem with homogeneous Neumann boundary condition is considered:

$$\begin{aligned} k^{-1}\mathbf{u} + \nabla p &= 0 & \text{in } \Omega, \\ \nabla \cdot \mathbf{u} &= f & \text{in } \Omega, \\ \mathbf{u} \cdot \mathbf{n} &= 0 & \text{on } \partial\Omega, \end{aligned}$$

together with the saturation equation

$$S_t + \mathbf{u} \cdot \nabla S = r,$$

where S represents the saturation and r is the source term. f is zero except the bottom left and top right fine-grid elements, where f is 1 and -1 , respectively. r is zero except the bottom left fine-grid element, where r is 1. Time step size is 0.01. We use the mixed GMsFEM to solve the flow equation and the saturation equation is directly solved on fine mesh explicitly.

Since the velocity solution of mixed GMsFEM preserves the mass conservation property only in the coarse mesh but not in the fine mesh, we need to do a postprocessing procedure to make it compatible with the transport in the fine mesh [36]. We illustrate the postprocessing procedure as following. First we calculate the residual on each coarse-grid element

$$R_T(\mathbf{u}_H) = \sum_{i=1}^n \left(\int_{t_i} f - \int_{\partial t_i} \mathbf{u}_H \cdot \mathbf{n} \right),$$

where $\mathbf{u}_H$ is the velocity solution of mixed GMsFEM; t_i is a fine-grid element contained in coarse-grid element T , and n is the number of fine-grid elements contained in coarse-grid element T . f is the source term in flow equation.

Then solving the following local problem, we get a correction of velocity

$$\begin{aligned} \nabla \cdot \mathbf{u}^c &= R_T(\mathbf{u}_H) & \text{in } T \\ \mathbf{u}^c &= -\nabla \Phi & \text{in } T \end{aligned} \tag{5.1}$$

with homogenous boundary condition

$$\mathbf{u}^c \cdot \mathbf{n} = 0 \quad \text{on } \partial T,$$

where $\mathbf{u}^c$ is a correction of velocity field and Φ is a supplementary variable. The corrected solution of velocity $\widehat{\mathbf{u}}_H = \mathbf{u}_H + \mathbf{u}^c$ is locally conservative in fine mesh, i.e. $R_T(\widehat{\mathbf{u}}_H)$ is of machine precision. Then $\widehat{\mathbf{u}}_H$ is used for solving transport equation explicitly. Let Δt be the time step size, and S_i^n be the value of S on the fine grid t_i .

Figure 8: Snapshots of saturation S on the fine mesh calculated by time step $\Delta t = 0.01$ at time $t = 100, 300, 400, 500, 600, 700$.

The saturation S_i^n is updated by the following discretized equation explicitly.

$$|t_i| \frac{S_i^{n+1} - S_i^n}{\Delta t} + \int_{\partial t_i} \hat{S}^n(\widehat{\mathbf{u}}_H \cdot \mathbf{n}) = r_i |t_i|, \quad (5.2)$$

where $\hat{S}^n(\widehat{\mathbf{u}}_H \cdot \mathbf{n})$ is the upwind flux; $|t_i|$ is the area of the fine element t_i and r_i is the averaged value of r on t_i .

We present the solutions of saturation on fine mesh in Figure 8 as a reference. The saturations for Case I, II and III are shown in Figure 9, Figure 10 and Figure 11, respectively. Obviously, all the cases we presented have a competitive performance of fine-grid solutions.

5.3. Two-phase flow and transport problems

In this subsection, the performance of the proposed mixed GMsFEM for two-phase flow and transport problems is reported. The following problem with homogenous

Figure 9: Snapshots of saturation S for Case I calculated by time step $\Delta t = 0.01$ at time $t = 100, 300, 400, 500, 600, 700$ with “dof per T ” being 40.

Figure 10: Snapshots of saturation S for Case II calculated by time step $\Delta t = 0.01$ at time $t = 100, 300, 400, 500, 600, 700$ with “dof per T ” being 25.

Figure 11: Snapshots of saturation S for Case III calculated by time step $\Delta t = 0.01$ at time $t = 100, 300, 400, 500, 600, 700$ with “dof per T ” being 25.

Neumann boundary condition is considered:

$$\begin{aligned} -\eta(S)k\nabla p &= \mathbf{u} & \text{in } \Omega, \\ \nabla \cdot \mathbf{u} &= f & \text{in } \Omega, \\ \mathbf{u} \cdot \mathbf{n} &= 0 & \text{on } \partial\Omega, \end{aligned}$$

where

$$\eta(S) = \frac{k_{rw}(S)}{\mu_w} + \frac{k_{ro}(S)}{\mu_o}$$

is the total mobility and

$$k_{rw}(S) = S^2, \quad k_{ro}(S) = (1 - S)^2, \quad \mu_w = 1, \quad \mu_o = 5.$$

The transport equation is

$$S_t + \mathbf{u} \cdot \nabla F(S) = r,$$

where S represents the saturation; r is the source term and

$$F(S) = \frac{k_{rw}(S)/\mu_w}{k_{rw}(S)/\mu_w + k_{ro}(S)/\mu_o}.$$

Similar to the previous case, we employ a postprocessing procedure to make the velocity solution of mixed GMsFEM compatible with the transport in the fine mesh. Then the corrected velocity solution $\widehat{\mathbf{u}}_H$ is used for solving transport equation explicitly.

$$|t_i| \frac{S_i^{n+1} - S_i^n}{\Delta t} + \int_{\partial t_i} F(\hat{S}^n)(\widehat{\mathbf{u}}_H \cdot \mathbf{n}) = r_i |t_i|,$$

where the same source terms f and r as in the single-phase flow case are used.

In the single-phase flow case, we only solve the flow equations once and the velocity field doesn't change in the whole simulation. However, the velocity field needs to be updated for the two-phase flow case. We carefully set the time step sizes to save the computational cost. Since the pressure changes less rapidly in time than the saturation in porous media and the constraint on time steps is primarily used in the explicit calculation of saturation. Furthermore, the multiscale pressure space changes more slowly than the pressure solution itself. All these reasons motivate us to set the time step sizes as following.

$$\begin{aligned} \Delta t_p &= N_p \Delta t_s, \\ \Delta t_g &= N_g \Delta t_p. \end{aligned}$$

At time step Δt_g we solve the local cell problem to obtain the snapshot space and then solve the spectral problem to construct the multiscale pressure space. At time step Δt_p we solve the flow equations in the multiscale space and correct the velocity

Figure 12: Snapshots of saturation S on the fine mesh calculated by time step $\Delta t_s = 0.01$, $N_p = 100$ and $N_g = 1$ at time $t = 100, 300, 400, 500, 600, 700$.

solution. At every times step Δt_s we employ the corrected velocity solution to update the saturation explicitly. N_p and N_g are positive integers to be chosen. So, totally we have three levels of time steps, the fine time step Δt_s , the medium time step Δt_p and the coarse time step Δt_g . This set of time steps will save the computational time greatly.

We present the solutions of saturation on fine mesh in Figure 12 as a reference. The saturations for Case I and II are shown in Figure 13 and Figure 14, respectively. For Case I, we don't need to solve the local cell problem on the coarse time step Δt_g , so we set $N_p = 100$ and $N_g = 1$. $N_p = 10$ and $N_g = 10$ are chosen for the Case II. It is obvious that both cases give a competitive performance of fine-grid solutions.

5.4. Comparison between the proposed method and the previous method

The convergence results of the proposed method have the similar form of our previous work [14]. This is because both theoretical conclusions are obtained by the spectral analysis. The error bounds depend on the reciprocal of the smallest unused eigenvalue.

Figure 13: Snapshots of saturation S for Case I calculated by time step $\Delta t_s = 0.01$, $N_p = 100$ and $N_g = 1$ at time $t = 100, 300, 400, 500, 600, 700$ with “dof per T ” being 40.

Figure 14: Snapshots of saturation S for Case II calculated by time step $\Delta t_s = 0.01$, $N_p = 10$ and $N_g = 10$ at time $t = 100, 300, 400, 500, 600, 700$ with “dof per T ” being 25.

Figure 15: The permeability field κ_1 of the porous media [14].

However, since both the pressure and velocity are approximated in bigger spaces, the new method enjoys better accuracy than the previous method, especially for the pressure. This can be illustrated by the following numerical comparisons. We resimulate the numerical example 1 “Single-phase flow” in [14] and compare the error bounds of both algorithms.

The permeability field κ_1 is shown in Figure 15. For the ease of comparison, we use two different coarse-mesh sizes with $N = 10$ and $N = 20$, called case 1 and case 2, which is same as the previous work [14]. Notice that cases 1 and 2 decompose each coarse-grid block as 20×20 and 10×10 grids, respectively. The error bounds of both methods for the permeability field κ_1 are shown in Table 3 and 4 for case 1 and case 2, respectively. For the previous method, the multiscale base of velocity on the coarse edge and the piecewise constant function in the coarse grid for pressure are used as freedoms. The new method employs the multiscale base of pressure in the coarse grid as unknowns. As a consequence, “dof per T ” = $2 \times$ “dof per E ” + 1, where “dof per T ” is the number of basis functions used for that coarse-grid element T and “dof per E ” is the number of basis functions used for that coarse edge E . From these tables, we see clearly the convergence of both methods when basis functions are added to the offline space. The new method enjoys better accuracy than the previous method, especially for the pressure.

6. conclusions

In this paper, we propose a frame work of the mixed generalized multiscale finite element method for solving subsurface flow problem in heterogeneous media. The main contribution of this work lies on constructing a systematic enrichment for multiscale basis functions for the pressure. We raise three types of local cell problems to construct the snapshot space. We describe a spectral decomposition in the snapshot space motivated by the analysis to further reduce the dimension of the space that is used to approximate the pressure. The convergence of the proposed method is analysed. The

Table 3: Comparisons of error bounds between the new method and our previous method, $n = 200$, $N = 10$.

	The new method		The previous method		
dof per T	$E_{gf}(p)$	$E_{gf}(\mathbf{u})$	$E_{gf}(p)$	$E_{gf}(\mathbf{u})$	dof per E
1	0.0851	3.1963	-	-	-
3	0.0630	0.6651	0.0903	0.1331	1
7	0.0020	0.0371	0.0896	0.0569	3
11	6.9365e-04	0.0192	0.0898	0.0308	5
15	3.8857e-04	0.0136	0.0898	0.0236	7
19	2.5469e-04	0.0106	0.0898	0.0210	9
23	1.6394e-04	0.0080	0.0898	0.0208	11
41	3.5303e-05	0.0029	0.0898	0.0208	20

Table 4: Comparisons of error bounds between the new method and our previous method, $n = 200$, $N = 20$.

	The new method		The previous method		
dof per T	$E_{gf}(p)$	$E_{gf}(\mathbf{u})$	$E_{gf}(p)$	$E_{gf}(\mathbf{u})$	dof per E
1	0.0477	2.4040	-	-	-
3	0.0395	0.2734	0.0601	0.1788	1
5	8.7752e-04	0.0363	0.0486	0.0460	2
7	4.9249e-04	0.0164	0.0486	0.0251	3
9	2.5304e-04	0.0107	0.0486	0.0115	4
11	1.6364e-04	0.0081	0.0486	0.0054	5
21	3.7923e-05	0.0030	0.0486	0.0054	10

resulting linear system of our method is easy to solve because of the good property of velocity matrix. Numerical examples including single-phase and two-phase flow with transport problems are tested to verify the performance of our proposed method.

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