Numerical Investigation of the POD Reduced-Order Model for Fast Predictions of Two-Phase Flows in Porous Media

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Abstract

In this paper, we present an efficient IMPES algorithm based on a global model order reduction method, Proper Orthogonal Decomposition (POD), to achieve the fast solution and prediction of two-phase flows in porous media. The key point of the proposed algorithm is to establish an accurate POD reduced-order model (ROM) for two-phase porous flows. To this end, two projection methods including projecting the original governing equations (method I) and projecting the discrete form of original governing equations (method II) are respectively applied to construct the POD-ROM, and their distinctions are compared and analyzed in detail. It is found that the POD-ROM established by method I is inapplicable to multiphase porous flows due to its failed introduction of fluid saturation and permeability that locate on the edge of grid cell, which would lead to unphysical results. By using method II, an efficient IMPES algorithm that can substantially speed up the simulation of two-phase porous flows is developed based on the POD-ROM. The computational efficiency and numerical accuracy of the proposed algorithm are validated through three numerical examples, simulation results illustrate that the proposed algorithm displays satisfactory computational speed-up ($1 \sim 2$ orders of magnitude) without sacrificing numerical accuracy obviously when comparing to the standard IMPES algorithm that without any acceleration technique. In addition, the determination of POD modes number, the relative errors of wetting phase pressure and saturation, as well as the influence of POD modes number on the overall performances of the proposed algorithm, are investigated.

Keywords: POD-ROM, Two-phase flow, Porous media, IMPES algorithm, Numerical methods

1. Introduction

The multiphase fluid flow in porous media is one of the most fundamental phenomena in various physical processes, some prominent examples include oil/gas flows in reservoir [1], CO₂ sequestration, subsurface pollution dispersion, and chemical process, etc. Fig. 1 displays a basic

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schematic of two-phase fluid flows in porous media, it always involves the displacement of a non-wetting fluid from the porous media by a wetting fluid. In most of the real-life processes, other phenomena such as mass transfer [2], heat transfer [3], and deformation [4, 5], are always coupled with the multiphase fluid flow, thus the solution of multiphase flow plays a basic role in these problems. Due to its crucial role, the efficient and accurate solution and prediction of multiphase flows in porous media are highly required in practical engineering and mechanism studies, which has been a research hot spot with increasing interest in recent years.

Figure 1: Illustration of two-phase flows in porous media

Modeling of the multiphase fluid flow is the first step to describe flow behaviors in porous media. A set of nonlinear partial differential equations (PDEs) have been proposed based on Darcy's law that cover different mechanisms of porous flows to represent a class of porous fluid flow problems in various engineering applications. Due to the strong nonlinearity implicated in the multiphase porous flow models, it has brought significant difficulties for the computation and analysis, where iterative methods, like Newton's Method, have been widely used to handle the nonlinear system. However, these methods require successively updating the solution at each iteration, thus if the discrete matrix of the model is in large-scale and ill-conditioned, the computational efforts are always extremely large and even unaffordable. Furthermore, the high contrast properties is an enormous technical challenge in solving this type of porous flow problems. For example, the permeability in Darcy-type pressure equation is always represented as a high contrast coefficient, which complicates the simulation difficulty especially for high-fidelity prediction and decision-making process, as well as uncertainty quantifications or sensitivity analyses. In addition, it should be noted that the saturation calculation for multiphase porous flows is a huge challenge. As it is the key parameter that directly influences the mass conservation property of the whole algorithm, the time step for the computation is restricted strictly. Therefore, the demand to develop high-efficient and accurate numerical methods for multiphase porous flows which could significantly reduce the computational complexities at the same time conserve main relevant physical properties, is highly addressed in engineering and academic communities [6, 7].

Many efforts have been paid on the simulation acceleration of multiphase flows in porous media, including upscaling methods [8, 9, 10, 11], mixed finite volume method (FVM)/finite element method (FEM) [12, 13, 14, 15, 16, 17], and parallel computing [18, 19], etc. All above acceleration methods can be divided into two groups: local model order reduction and global model order reduction. For example, the mixed FEM/FVM is the most popular scheme for local model order reduction, with many accurate reduced-order methods have been introduced, e.g. MsFVM [20], mortar multiscale methods [21], and Mixed MsFEM [22]. In recent years, the mixed FEM has been extended to allow for the systematic enrichment of coarse solution spaces in order to converge to the fine-grid solution [23]. Generally speaking, however, the local model order reduction method is always more complex to implement or sometimes hard to control the numerical accuracy in practical engineering applications.

The global model order reduction is an unmissable alternative for engineering problems. As for this type of model order reduction method, two common techniques namely Dynamic Mode Decomposition (DMD) and Proper Orthogonal Decomposition (POD) have received extensive attentions in recent years [24], detailed comparison of the DMD and POD can be found in [25]. Compared to DMD, POD is powerful to construct a problem-specific set of basis functions with global support that capture the dominant characteristics and extract the energetic structures from the system of interest. It can largely reduce the dimension of the target original full-order system by projecting the high-dimensional system into a lower-dimensional manifold using a set of orthonormal basis functions (POD modes) constructed from the snapshots [26]. Thus, POD is considered as one of the best known global model order reduction methods. For example, a typical application of POD method is to compress image data at the same time reserving the main information with different POD modes that selected for the reconstruction of image.

POD with Galerkin projection has been proved to be an efficient model order reduction technique in a large number of applications, such as turbulent flow [27], heat transfer [28], oil transportation [29], etc. However, the application of POD to porous flow problems is still at its initial stage and more efforts need to paid on. Here we briefly review several representative works as follows. Vermeulen et al. [30] applied the POD to high-dimensional model inversion of transient nonlinear groundwater flow in order to save computational resources. The POD methods were demonstrated robustly and efficiently for a wide range of prior estimates. Siade et al. [31] proposed a strategy to choose optimal snapshots for groundwater model reduction using POD. Ghommam et al. [25] applied the DMD and POD methods to flows in highly-heterogeneous porous media and derived ROMs via Galerkin projection, the robustness of DMD- and POD-based ROMs with regard to variations in initial conditions, permeability fields, and forcing terms were discussed. Alotaibi et al. [32] developed a global-local nonlinear model reduction method to reduce the computational complexity associated with nonlinear flows in highly-heterogeneous porous media. In their work, the POD was used for global model reduction at online stage. Wang et al. [33] established a POD-ROM for single-phase porous flows, in their study high precision (relative deviation $1.0 \times 10^{-4}\% \sim 2.3 \times 10^{-1}\%$) and large acceleration (speed-up 880~98454

times) obtained by the POD model were observed. Wang et al. [34] also studied the ROM for gas flow in dual-porosity dual-permeability porous media using the POD method. A new POD modeling approach was proposed considering the mass conservation of the whole matrix fracture system.

Although scholars have conducted some researches on the simulation of fluid flows in porous media using POD method, more efforts are still needed especially in the presence of multiphase porous flows. In addition, for the popular IMPES algorithm [35], which has been widely used in the simulation of multiphase flows in porous media, the POD based model reduction method is still in its infancy and an exhaustive investigation of their potential is lacking. Therefore, there remains a lot of work to fill the gap and utilize the enriched experience in IMPES algorithm in the application of POD method to handle large-scale, high-fidelity multiphase porous flows in realistic engineering problems. Especially in petroleum engineering and subsurface pollution engineering, the efficient predictions of two-phase flows using the simple IMPES algorithm or its variants are always highly demanded and preferred, where the POD method can be exploited and combined.

In this paper, we aim to develop an efficient POD-ROM for incompressible two-phase flows in porous media to achieve their fast solution and prediction in related engineering problems, e.g. oil production from the petroleum reservoir. Especially, we will detailedly analyze the differences of two POD-ROM construction methods and point out the shortcomings of the traditional projection method that directly projects the original governing equations in handling the multiphase porous flow. In addition, the challenges existing in the POD-ROM based IMPES algorithm for two-phase porous flows would be discussed, which will help to clarify the reason why POD-ROM based solver is always difficult to be applied to multiphase porous flow simulations. The main contribution of this study is to present an efficient IMPES algorithm based on an accurate POD-ROM for incompressible two-phase fluid flows in porous media, and clarifying the distinctions of two projection approaches as well as the difficulties in the application of POD-ROM to multiphase porous flow predictions. Though the present work is still at initial stage and more efforts need to be paid, as far as the authors knowledge, there is few work focusing on the POD-ROM in multiphase flows especially in porous media.

The structure of this paper is organized as follows. We devote Section 2 to a brief introduction of the flow problem and standard IMPES algorithm. The establishment of POD-ROM using two different projection methods and their distinctions are detailedly discussed in Section 3. Section 4 outlines the implementation of our proposed POD-ROM based IMPES algorithm. Computational performances (such as reconstruction accuracy, computational speed-up, prediction accuracy, etc.) of the proposed algorithm and related analyses are presented in Section 5. The main concluding remarks of this study are reported in Section 6.

2. Preliminaries

2.1. Incompressible two-phase flows in porous media

In this study, the incompressible two-phase flow in porous media is considered to illustrate our proposed fast IMPES algorithm based on the framework of POD-ROM. Assume the flow in two spatial dimensions and in time, the governing equations of incompressible two-phase flows in porous media reads,

Mass conservation law,

$$\frac{\partial(\phi S_\alpha)}{\partial t} + \nabla \cdot \mathbf{u}_\alpha = q_\alpha - k_{w,\alpha} p_\alpha \text{ in } \Omega, \alpha = n, w \quad (1)$$

Darcy's law,

$$\mathbf{u}_\alpha = -\frac{k_{r,\alpha}(S_w)}{\mu_\alpha} \mathbf{k}(\nabla p_\alpha - \rho_\alpha \mathbf{g}) \text{ in } \Omega, \alpha = n, w \quad (2)$$

Saturation summation constraint,

$$S_w + S_n = 1 \quad (3)$$

Capillary pressure equation,

$$p_n - p_w = p_c(S_w) \quad (4)$$

where the subscript α is the phase index, n and w represent the non-wetting and wetting phases, respectively; Ω represents the computational domain; S_α , p_α and $\mathbf{u}_\alpha$ denote the fluid saturation, phase pressure, and Darcy velocity, respectively; $\mathbf{k}$ and $k_{r,\alpha}$ are absolute permeability and relative permeability, respectively, $k_{r,\alpha} = \mathbf{k}_\alpha \setminus \mathbf{k}$; μ_α is the dynamic viscosity; ρ_α is the mass density; $\mathbf{g}$ is the gravitational acceleration; ϕ is the porosity; q_α and $k_{w,\alpha}$ are well injection rate and well parameter, respectively, where (a) for rate-specified wells, $k_{w,\alpha}$ can be set as 0 and q_α represents the specified net injection rate, (b) for bottom-hole pressure (BHP) imposed wells, $k_{w,\alpha}$ can be set as $k_{r,\alpha} \frac{k^{WI}}{\mu_\alpha}$ and $q_\alpha = k_{w,\alpha} p_{bh}$ with p_{bh} is the imposed BHP and k^{WI} is the well index.

The right-hand-side (RHS) source/sink term can be used to represent various boundary conditions. In this study, the following velocity boundary condition and initial saturation condition are adopted to close the governing equations (1) ~ (4),

$$\mathbf{u}_\alpha \cdot \mathbf{n} = 0 \text{ on } \partial\Omega, \alpha = n, w \quad (5)$$

$$S_\alpha = S_{\text{init},\alpha} \text{ at } t = t_0, \alpha = n, w \quad (6)$$

where $\mathbf{n}$ represents the outward normal direction; the boundary velocity $\mathbf{u}_\alpha$ and the initial saturation $S_{\text{init},\alpha}$ are given parameters.

2.2. IMPES algorithm

The IMPES (Implicit Pressure, Explicit Saturation) algorithm is one of the most popular methods to solve the two-phase flows in porous media, which was originally developed by Sheldon et al. [35]. The main idea of IMPES algorithm is splitting the coupled system into a pressure equation and a saturation equation, then the pressure and saturation equations are solved using implicit iterations and explicit time approximation approaches, respectively. A large number of studies have shown the IMPES algorithm offers advantages such as (a) the IMPES formulation is more convenient to implement than the fully implicit scheme; (b) the IMPES algorithm successfully decouples pressure from the saturation; (c) the total velocity has a continuous normal component, which is retained in the standard IMPES formulation; (d) the IMPES scheme can guarantee non-negative wetting phase saturation if the time step is smaller than a certain value. In this study, the IMPES algorithm is applied to the two-phase porous flow simulations, thus here we first introduce this algorithm briefly.

Suppose the wetting phase pressure p_w and wetting phase saturation S_w are taken as the primary variables, summing the conservation law for the two phases gives,

$$\frac{\partial \phi(S_w + S_n)}{\partial t} + \nabla \cdot (\mathbf{u}_w + \mathbf{u}_n) = q_w - k_{w,w}p_w + q_n - k_{w,n}p_n \quad (7)$$

Define the source term $q_{\text{src}} = \sum_{\alpha} (q_{\alpha} - k_{w,\alpha}p_{\alpha})$ and the total velocity $\mathbf{u}_t = \mathbf{u}_w + \mathbf{u}_n$, recall the saturation summation contrast Eq. (3), then Eq. (7) can be written as,

$$\nabla \cdot \mathbf{u}_t = q_{\text{src}} \quad (8)$$

Introduce the phase mobility and total phase mobility as follows, respectively,

$$\lambda_{\alpha}(S_w) = \frac{k_{r,\alpha}(S_w)}{\mu_{\alpha}} \quad (9)$$

$$\lambda_t(S_w) = \lambda_w(S_w) + \lambda_n(S_w) \quad (10)$$

Then substitute Eq. (9) into Darcy velocity (2), the total velocity reads,

$$\mathbf{u}_t = -\lambda_w(S_w)\mathbf{k}(\nabla p_w - \rho_w\mathbf{g}) - \lambda_n(S_w)\mathbf{k}(\nabla p_n - \rho_n\mathbf{g}) \quad (11)$$

Consider the capillary pressure Eq. (4) and the total phase mobility (10), Eq. (11) can be reformulated as,

$$\mathbf{u}_t = -\mathbf{k}(\lambda_t(S_w)\nabla p_w + \lambda_n(S_w)\nabla p_c - (\lambda_t(S_w)\rho_w\mathbf{g} + \lambda_n(S_w)\rho_n\mathbf{g})) \quad (12)$$

Substituting Eq. (12) into Eq. (8) gives the wetting phase pressure equation,

$$-\nabla \cdot [\lambda_t(S_w)\mathbf{k}\nabla p_w] = q_{\text{src}} + q_{\text{grav}} + q_{\text{cap}} \quad (13)$$

where the first term of the RHS q_{src} represents the source/sink term, $q_{\text{src}} = \sum_{\alpha} (q_{\alpha} - k_{w,\alpha} p_{\alpha}) = \sum_{\alpha} (q_{\alpha}^{\text{rate}} + \lambda_{\alpha}(S_w) k^{\text{WI}} (p_{bh} - p_{\alpha}))$; the second term of the RHS q_{grav} denotes the contribution of gravity to the total Darcy velocity, $q_{\text{grav}} = -\nabla \cdot (\mathbf{k}(\lambda_w(S_w)\rho_w + \lambda_n(S_w)\rho_n)\mathbf{g})$, $q_{\text{grav}} = 0$ if the influence of gravity is neglected; the third term q_{cap} represents the contribution of capillary pressure gradient to the total Darcy velocity, $q_{\text{cap}} = \nabla \cdot (\lambda_n(S_w)\mathbf{k}\nabla p_c(S_w))$, $q_{\text{cap}} = 0$ if the capillary pressure is ignored.

For brevity, the implementation of standard IMPES algorithm is presented in Table 1, readers of interest can refer to [35] for more details. It is worth noting that in Table 1 the superscript k and $k+1$ represent the k th and $k+1$ th time layer, respectively. The first term of the RHS q_{src} is computed by $q_{\text{src}} = \sum_{\alpha} (q_{\alpha}^{(k+1)} - k_{w,\alpha}^{(k)} p_{\alpha}^{(k+1)}) = \sum_{\alpha} (q_{\alpha}^{\text{rate}(k+1)} + \lambda_{\alpha}(S_w) k^{\text{WI}} (p_{bh} - p_{\alpha}^{(k+1)}))$ implicitly, the other two terms of the RHS are computed by using the parameters of the k th time layer explicitly.

Table 1: Standard IMPES algorithm

Algorithm 1: IMPES	
Step1:	Solve the pressure equation implicitly $-\nabla \cdot [\lambda_t(S_w)\mathbf{k}\nabla p_w] = q_{\text{src}} + q_{\text{grav}} + q_{\text{cap}}$
Step2:	Compute the Darcy velocity explicitly $\mathbf{u}_{\alpha} = -\frac{k_{r,\alpha}(S_w)}{\mu_{\alpha}}\mathbf{k}(\nabla p_{\alpha} - \rho_{\alpha}\mathbf{g})$
step3:	Solve the wetting phase saturation explicitly $\phi \frac{S_w^{(k+1)} - S_w^{(k)}}{t^{(k+1)} - t^{(k)}} + \nabla \cdot \mathbf{u}_w = q_w - k_{w,w} p_w$

From Table 1, it can be easily found the computational cost of standard IMPES algorithm mainly stems from the implicit solution of the wetting phase pressure equation. To speed up the computation, a popular global model order reduction method namely proper orthogonal decomposition is applied in our study.

3. Establishment of POD-ROM for two-phase porous flows

3.1. POD method

POD is a global model order reduction method for constructing a low-dimensional approximation of a subspace in Hilbert space, it can efficiently extract the main characteristics of the expected solutions [36]. Suppose a set of snapshots $\{\mathbf{s}_1, \dots, \mathbf{s}_{ns}\} \subset \mathbf{R}^n$ is given, let $s = \text{span}\{\mathbf{s}_1, \dots, \mathbf{s}_{ns}\} \subset \mathbf{R}^n$ and $d = \dim(s)$. A POD basis of dimension $m < d$ is a set of orthonormal vectors $\{\phi\}_{i=1}^m \subset \mathbf{R}^n$ whose linear span best approximates the space s . The basis set $\{\phi\}_{i=1}^m$ solves the below minimization problem,

$$\min_{\{\phi\}_{i=1}^m} \sum_{j=1}^{ns} \left\| \mathbf{s}_j - \sum_{i=1}^m (\mathbf{s}_j^T \phi_i) \phi_i \right\|_2^2 \quad (14)$$

where $\phi_i^T \phi_j = \delta_{ij}$, $i, j = 1, \dots, m$.

The solution of Eq. (14) can be provided by the set of left singular vectors of the snapshot matrix $\mathbf{S} = [\mathbf{s}_1, \dots, \mathbf{s}_{ns}] \in \mathbf{R}^{n \times ns}$. Suppose the singular value decomposition of $\mathbf{S}$ is,

$$\mathbf{S} = \mathbf{V} \mathbf{\Sigma} \mathbf{W}^T \quad (15)$$

where $\mathbf{V} = [\mathbf{v}_1, \dots, \mathbf{v}_d] \in \mathbf{R}^{n \times d}$ and $\mathbf{W} = [\mathbf{w}_1, \dots, \mathbf{w}_d] \in \mathbf{R}^{ns \times d}$ are orthogonal, and $\mathbf{\Sigma} = \text{diag}(\sigma_1, \dots, \sigma_d) \in \mathbf{R}^{d \times d}$ with the singular value is arranged in descending order $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_d > 0$. The rank of $\mathbf{S}$ is $d < \min(n, ns)$. Then the POD basis or the optimal solution of Eq. (14) is $\{\Phi_i\}_{i=1}^m$. The minimum L_2 norm error stemming from the approximation of snapshots using the POD basis can be computed by,

$$\sum_{j=1}^{ns} \left\| \mathbf{s}_j - \sum_{i=1}^m (\mathbf{s}_j^T \Phi_i) \Phi_i \right\|_2^2 = \sum_{i=m+1}^d \sigma_i^2 \quad (16)$$

3.2. Establishment of POD-ROM

Two intuitive projection methods can be used to establish the POD-ROM for the wetting phase pressure Eq. (13), of which one is directly projecting the original governing Eq. (13) into the low-dimensional space spanned by selected basis functions using Galerkin method, we name it the method I; the other is projecting the discrete form of original governing Eq. (13) into the basis function space, we call it the method II. Detailed procedures of these two methods are presented below. Especially, the distinctions of these two projection methods and the underlying issue of the method I raising in the two-phase porous flows would be pointed out and clarified.

3.2.1. Method I

Suppose the wetting phase pressure can be expressed as a combination of the spectral coefficient and the selected basis functions as follows,

$$p_w = \sum_{n=1}^M c_n \phi^n \quad (17)$$

where c_n is the spectral coefficient; ϕ^n is the basis function.

Substituting Eq. (17) into the wetting phase pressure Eq. (13) yields,

$$-\nabla \cdot \left[\lambda_t(S_w) \mathbf{k} \nabla \left(\sum_{n=1}^M c_n \phi^n \right) \right] + \tilde{q}_{\text{src}} = \hat{q}_{\text{src}} + q_{\text{grav}} + q_{\text{cap}} \quad (18)$$

where $\hat{q}_{\text{src}} = \sum_{\alpha} [q_{\alpha}^{\text{rate}} + \lambda_{\alpha}(S_w) k^{\text{WI}} p_{bh}] - \lambda_n(S_w) k^{\text{WI}} p_c(S_w)$; $\tilde{q}_{\text{src}} = \lambda_t(S_w) k^{\text{WI}} \left(\sum_{n=1}^M c_n \phi^n \right)$.

The Eq. (18) can be further reformulated as follows because the spectral coefficient is only a

function of time,

$$\left(\sum_{n=1}^M c_n \right) \{ \lambda_t(S_w) k^{\text{WI}} \phi^n - \nabla \cdot [\lambda_t(S_w) \mathbf{k} \nabla \phi^n] \} = \hat{q}_{\text{src}} + q_{\text{grav}} + q_{\text{cap}} \quad (19)$$

Project Eq. (19) into the low-dimensional space spanned by the basis function $\Phi = \{\phi^m, m = 1, \dots, M\}$ using Galerkin method, the following ROM can be obtained,

$$\left(\sum_{n=1}^M c_n \right) \int_{\Omega} \{ \lambda_t(S_w) k^{\text{WI}} \phi^n - \nabla \cdot [\lambda_t(S_w) \mathbf{k} \nabla \phi^n] \} \phi^m d\Omega = \int_{\Omega} (\hat{q}_{\text{src}} + q_{\text{grav}} + q_{\text{cap}}) \phi^m d\Omega \quad (20)$$

Apply the Green's theorem to Eq. (20), the left hand side (LHS) can be further written as,

$$\begin{aligned} \text{LHS} &= \left(\sum_{n=1}^M c_n \right) \int_{\Omega} \left[\lambda_t(S_w) k^{\text{WI}} \phi^n - \frac{\partial}{\partial x} \left(\lambda_{t,x}(S_w) \mathbf{k} \frac{\partial \phi^n}{\partial x} \right) - \frac{\partial}{\partial y} \left(\lambda_{t,y}(S_w) \mathbf{k} \frac{\partial \phi^n}{\partial y} \right) \right] \phi^m d\Omega \\ &= - \left(\sum_{n=1}^M c_n \right) \int_{\partial\Omega} \left[\left(\lambda_{t,x}(S_w) \mathbf{k} \frac{\partial \phi^n}{\partial x} \phi^m dy \right) - \left(\lambda_{t,y}(S_w) \mathbf{k} \frac{\partial \phi^n}{\partial y} \phi^m dx \right) \right] \\ &\quad + \left(\sum_{n=1}^M c_n \right) \int_{\Omega} \left[\left(\lambda_{t,x}(S_w) \mathbf{k} \frac{\partial \phi^n}{\partial x} \frac{\partial \phi^m}{\partial x} \right) + \left(\lambda_{t,y}(S_w) \mathbf{k} \frac{\partial \phi^n}{\partial y} \frac{\partial \phi^m}{\partial y} \right) + \lambda_t(S_w) k^{\text{WI}} \phi^n \phi^m \right] d\Omega \end{aligned} \quad (21)$$

Then the POD-ROM can be established in the whole computational domain Ω according to Eqs. (20) $\sim$ (21),

$$\begin{aligned} &\left(\sum_{n=1}^M c_n \right) \int_{\Omega} \left[\left(\lambda_{t,x}(S_w) \mathbf{k} \frac{\partial \phi^n}{\partial x} \frac{\partial \phi^m}{\partial x} \right) + \left(\lambda_{t,y}(S_w) \mathbf{k} \frac{\partial \phi^n}{\partial y} \frac{\partial \phi^m}{\partial y} \right) + \lambda_t(S_w) k^{\text{WI}} \phi^n \phi^m \right] d\Omega \\ &- \left(\sum_{n=1}^M c_n \right) \int_{\partial\Omega} \left[\left(\lambda_{t,x}(S_w) \mathbf{k} \frac{\partial \phi^n}{\partial x} \phi^m dy \right) - \left(\lambda_{t,y}(S_w) \mathbf{k} \frac{\partial \phi^n}{\partial y} \phi^m dx \right) \right] \\ &= \int_{\Omega} (\hat{q}_{\text{src}} + q_{\text{grav}} + q_{\text{cap}}) \phi^m d\Omega \end{aligned} \quad (22)$$

The Eq. (22) can also be reformulated as a discrete algebraic form as follows,

$$\mathbf{A}_{\text{ROM}}^{\text{I}} \mathbf{c} = \mathbf{b}_{\text{ROM}}^{\text{I}} \quad (23)$$

where $\mathbf{A}_{\text{ROM}}^{\text{I}} \in \mathbf{R}^{M \times M}$ is the reduced-order discrete coefficient matrix; $\mathbf{b}_{\text{ROM}}^{\text{I}} \in \mathbf{R}^M$ is the reduced-order RHS term.

Suppose the original computational domain Ω is divided into $nx \times ny$ uniform grids with the grid size Δx and Δy in x and y coordinate directions. For POD-ROM (23), any element in $\mathbf{A}_{\text{ROM}}^{\text{I}}$ and $\mathbf{b}_{\text{ROM}}^{\text{I}}$ except for boundary nodes can be expressed as follows,

$$\begin{aligned} A_{m,n}^{\text{I}} &= \int_{\Omega} \left[\left(\lambda_{t,x}(S_w) \mathbf{k} \frac{\partial \phi^n}{\partial x} \frac{\partial \phi^m}{\partial x} \right) + \left(\lambda_{t,y}(S_w) \mathbf{k} \frac{\partial \phi^n}{\partial y} \frac{\partial \phi^m}{\partial y} \right) + \lambda_t(S_w) k^{\text{WI}} \phi^n \phi^m \right] d\Omega \\ &= \sum_{j=1}^{ny} \sum_{i=1}^{nx} \left[\left(\lambda_{t,x}(S_w) \mathbf{k}_{xx} \right)_{i,j} \left(\frac{\partial \phi_{i,j}^n}{\partial x} \frac{\partial \phi_{i,j}^m}{\partial x} \right) + \left(\lambda_{t,y}(S_w) \mathbf{k}_{yy} \right)_{i,j} \left(\frac{\partial \phi_{i,j}^n}{\partial y} \frac{\partial \phi_{i,j}^m}{\partial y} \right) + \left(\lambda_t(S_w) k^{\text{WI}} \right)_{i,j} \phi_{i,j}^n \phi_{i,j}^m \right] \Delta x \Delta y \end{aligned} \quad (24)$$

$$b_m^I = \sum_{j=1}^{ny} \sum_{i=1}^{nx} [(\hat{q}_{\text{src}} + q_{\text{grav}} + q_{\text{cap}}) \phi_{i,j}^m] \Delta x \Delta y \quad (25)$$

where $1 \leq m \leq M$, $1 \leq n \leq M$.

It can be easily found that for any interior node (i, j) in the computational domain, the phase mobility satisfies,

$$(\lambda_{t,x}(S_w))_{i,j} = (\lambda_{t,y}(S_w))_{i,j} = (\lambda_t(S_w))_{i,j} \quad (26)$$

Thus Eq. (24) can be simplified as,

$$A_{m,n}^I = \sum_{j=1}^{ny} \sum_{i=1}^{nx} \left[(\lambda_t(S_w) \mathbf{k}_{xx})_{i,j} \left(\frac{\partial \phi_{i,j}^n}{\partial x} \frac{\partial \phi_{i,j}^m}{\partial x} \right) + (\lambda_t(S_w) \mathbf{k}_{yy})_{i,j} \left(\frac{\partial \phi_{i,j}^n}{\partial y} \frac{\partial \phi_{i,j}^m}{\partial y} \right) + (\lambda_t(S_w) k^{\text{WI}})_{i,j} \phi_{i,j}^n \phi_{i,j}^m \right] \Delta x \Delta y \quad (27)$$

where the terms $(\lambda_t(S_w) \mathbf{k}_{xx})_{i,j}$ and $(\lambda_t(S_w) \mathbf{k}_{yy})_{i,j}$ thus are all computed on the center of grid cell, as the light blue dot shown in Fig. 2. It has nothing to do with the $\lambda_t(S_w)$ and $\mathbf{k}$ on the edges of grid cell.

Figure 2: Location of variables used in the discrete coefficients in method I

3.2.2. Method II

Firstly suppose the wetting phase pressure Eq. (13) is discretized using the finite difference method (FDM), then the discrete form of Eq. (13) on any interior node (i, j) in computational domain Ω reads,

$$a_{i,j} p_{i,j} = a_{i+1/2,j} p_{i+1/2,j} + a_{i-1/2,j} p_{i-1/2,j} + a_{i,j+1/2} p_{i,j+1/2} + a_{i,j-1/2} p_{i,j-1/2} + b_{i,j} \quad (28)$$

where $a_{i,j} = a_{i+1/2,j} + a_{i-1/2,j} + a_{i,j+1/2} + a_{i,j-1/2} + \lambda_t(S_w) k^{\text{WI}}$; $a_{i+1/2,j} = \frac{(\lambda_{t,x}(S_w^{(k)}) \mathbf{k}_{xx})_{i+1/2,j}}{\Delta x^2}$; $a_{i-1/2,j} = \frac{(\lambda_{t,x}(S_w^{(k)}) \mathbf{k}_{xx})_{i-1/2,j}}{\Delta x^2}$; $a_{i,j+1/2} = \frac{(\lambda_{t,y}(S_w^{(k)}) \mathbf{k}_{yy})_{i,j+1/2}}{\Delta y^2}$; $a_{i,j-1/2} = \frac{(\lambda_{t,y}(S_w^{(k)}) \mathbf{k}_{yy})_{i,j-1/2}}{\Delta y^2}$; $b_{i,j} = \hat{q}_{\text{src}} + q_{\text{grav}} + q_{\text{cap}}$; the subscript '1/2' existing in the discrete coefficient represents the location of cell edge.

The above Eq. (28) can be further arranged as an algebraic form,

$$\mathbf{A}\mathbf{p}_w = \mathbf{b} \quad (29)$$

where $\mathbf{A} \in \mathbf{R}^{N \times N}$ is the discrete coefficient matrix; $\mathbf{p}_w \in \mathbf{R}^N$ is the unknown wetting phase pressure; $\mathbf{b} \in \mathbf{R}^N$ is the RHS term; N is the number of unknown wetting phase pressure, $N = nx \times ny$.

Suppose $\mathbf{p}_w = \mathbf{c}\Phi$ with the low-dimensional space spanned by the basis function $\Phi = \{\phi_w^m, m = 1, \dots, M\}$, Eq. (29) can be written as,

$$\mathbf{A}\mathbf{c}\Phi = \mathbf{b} \quad (30)$$

To construct the POD-ROM, the algebraic Eq. (30) can be directly projected into the low-dimensional space spanned by Φ , then we can get,

$$\mathbf{A}_{\text{ROM}}^{\text{II}}\mathbf{c} = \mathbf{b}_{\text{ROM}}^{\text{II}} \quad (31)$$

where $\mathbf{A}_{\text{ROM}}^{\text{II}} = \Phi^T \mathbf{A} \Phi \in \mathbf{R}^{M \times M}$ is the reduced-order discrete coefficient matrix, $\mathbf{b}_{\text{ROM}}^{\text{II}} = \Phi^T \mathbf{b} \in \mathbf{R}^M$ is the reduced-order RHS term.

For any element in $\mathbf{A}_{\text{ROM}}^{\text{II}}$ and $\mathbf{b}_{\text{ROM}}^{\text{II}}$ corresponding to the low-dimensional space, we have,

$$A_{m,n}^{\text{II}} = \sum_{j=1}^{ny} \sum_{i=1}^{nx} (a_{i,j} \phi_{i,j}^n \phi_{i,j}^m - a_{i+1/2,j} \phi_{i+1,j}^n \phi_{i+1,j}^m - a_{i-1/2,j} \phi_{i-1,j}^n \phi_{i-1,j}^m - a_{i,j+1/2} \phi_{i,j+1}^n \phi_{i,j+1}^m - a_{i,j-1/2} \phi_{i,j-1}^n \phi_{i,j-1}^m) \quad (32)$$

$$b_m^{\text{II}} = \sum_{j=1}^{ny} \sum_{i=1}^{nx} [(\hat{q}_{\text{src}} + q_{\text{grav}} + q_{\text{cap}}) \phi_{i,j}^m] \quad (33)$$

where the discrete coefficients $a_{i+1/2,j}$, $a_{i-1/2,j}$, $a_{i,j+1/2}$, $a_{i,j-1/2}$ are all computed on the edges of grid cell in the grid system, as the red and purple arrows shown in Fig. 3. Therefore, the $\lambda_t(S_w)$ and $\mathbf{k}$ on the edges of grid cell are used naturally.

3.2.3. Comparison of methods I and II

As illustrated in Table 2, the above two methods for the establishment of POD-ROM are compared in detail:

(a) It can be obviously seen that the RHS terms of POD-ROMs (23) and (31) satisfy $\mathbf{b}_{\text{POD}}^{\text{I}} = \mathbf{b}_{\text{POD}}^{\text{II}}$;

(b) For the reduced-order discrete coefficient matrices $\mathbf{A}_{\text{POD}}^{\text{I}}$ and $\mathbf{A}_{\text{POD}}^{\text{II}}$, however, there are some significant distinctions. As mentioned in method I, the terms $(\lambda_t(S_w)\mathbf{k}_{xx})_{i,j}$ and $(\lambda_t(S_w)\mathbf{k}_{yy})_{i,j}$ are all computed on the cell center in the grid system, thus the values of $\mathbf{k}_{xx}$, $\mathbf{k}_{yy}$ and S_w on the cell center are used. In method II, however, the discrete coefficients $a_{i+1/2,j}$, $a_{i-1/2,j}$, $a_{i,j-1/2}$, $a_{i,j+1/2}$ are all computed on the cell edge in the grid system, the values of $\mathbf{k}_{xx}$,

Figure 3: Location of variables used in the discrete coefficients in method I

$\mathbf{k}_{yy}$ and S_w on the cell edge instead of cell center are used. For single-phase flow in homogeneous porous media, the values of $\mathbf{k}_{xx}$ and $\mathbf{k}_{yy}$ are identical both on cell center and cell edge, the method I and method II would be equivalent which means the two POD-ROMs are the same with each other. However, for heterogeneous porous media and/or multiphase flows, the two POD-ROMs are different: (i) for heterogeneous porous media, the permeability $\mathbf{k}_{xx}$ and $\mathbf{k}_{yy}$ that on cell center may be different from those on cell edge, the permeability on cell edge is commonly calculated by using the harmonic average method. (ii) for multiphase flows, the phase saturation S_w that on cell center are different from that on cell edge. The phase saturation S_w on cell edge is commonly calculated using the upwind scheme, which is influenced by the flow direction; while the phase saturation S_w on cell center is calculated using its own value, it has no relation with the flow direction.

Table 2: Comparison of two establishment methods for POD-ROM

Reduced-order coefficient matrix $\mathbf{A}_{\text{ROM}}$	Reduced-order RHS $\mathbf{b}_{\text{ROM}}$
$\mathbf{A}_{\text{ROM}}^{\text{I}} \neq \mathbf{A}_{\text{ROM}}^{\text{II}}$ in heterogeneous porous media and/or multiphase flows because the elements of $\mathbf{A}_{\text{ROM}}$ are computed on different locations in the grid system	$\mathbf{b}_{\text{ROM}}^{\text{I}} = \mathbf{b}_{\text{ROM}}^{\text{II}}$

From above comparison, it can be concluded that the method II by projecting the discrete form of the original governing equation is a more general approach to construct the POD-ROM, and the method I by directly projecting the original governing equation is limited to some special cases, such as homogeneous single-phase flow.

4. IMPES algorithm based on the POD-ROM

As mentioned in Subsection 2.2, the computational cost of standard IMPES algorithm mainly depends on the implicit solution of wetting phase pressure equation. Thus to speed up the

computation, an efficient IMPES algorithm is developed on the POD-ROM framework in this Section. The method II we introduced in last Section is adopted to establish the POD-ROM for wetting phase pressure equation.

The cornerstone of POD-ROM based IMPES algorithm is to establish and solve the reduced-order model of wetting phase pressure equation instead to solve the original full-order wetting phase pressure equation, which could reduce the degree of freedom (DOF) of unknown wetting phase pressure substantially. Owing to the low-dimension of the POD-ROM, the computational burden could be reduced significantly compared with directly solving the original problem in high-dimensional space. Especially, in most cases the advantage of the POD-ROM is much more attractive with the increase of mesh density, problem complexity, etc. Here we introduce the detailed implementation of the proposed IMPES algorithm based on POD-ROM. As shown in Fig. 4, the implementation procedures include two stages: offline stage and online stage.

Figure 4: Implementation of the IMPES algorithm based on POD-ROM

4.1. Offline stage

At offline stage, the input is the snapshot ensemble of the wetting phase pressure and the output is the extracted orthogonal basis functions. First, to obtain representative snapshots, the original wetting phase pressure equation can be solved under various calculation conditions that

can reflect representative properties and characteristics of the two-phase flows in porous media by FDM, FEM, etc. Suppose the snapshots of wetting phase pressure are given as follows,

$$snap_1 = \begin{bmatrix} p_{11}^w \\ p_{21}^w \\ \vdots \\ p_{N1}^w \end{bmatrix}, snap_2 = \begin{bmatrix} p_{12}^w \\ p_{22}^w \\ \vdots \\ p_{N2}^w \end{bmatrix}, \dots, snap_i = \begin{bmatrix} p_{1i}^w \\ p_{2i}^w \\ \vdots \\ p_{Ni}^w \end{bmatrix}, \dots, snap_L = \begin{bmatrix} p_{1L}^w \\ p_{2L}^w \\ \vdots \\ p_{NL}^w \end{bmatrix} \quad (34)$$

where N represents the DOF of the unknown wetting phase pressure in the whole computational domain; L represents the number of snapshots. Generally, $N \gg L$.

The above snapshot can be written as,

$$\mathbf{S} = [snap_1, snap_2, \dots, snap_i, \dots, snap_L] \in \mathbf{R}^{N \times L} \quad (35)$$

Based on the snapshot ensemble $\mathbf{S}$, a kernel matrix $\mathbf{C} \in \mathbf{R}^{L \times L}$ for the eigenvalue decomposition can be constructed as follows,

$$\mathbf{C} = \frac{1}{N} \int_{\Omega} \mathbf{S}^T \mathbf{S} d\Omega = \frac{\Delta x \Delta y}{N} \mathbf{S}^T \mathbf{S} \quad (36)$$

Perform the eigenvalue decomposition for the kernel to obtain eigenvalues and eigenvectors,

$$\mathbf{C}\mathbf{V} = \lambda\mathbf{V} \quad (37)$$

where λ and $\mathbf{V}$ are eigenvalues and corresponding eigenvectors. The significance of $\mathbf{V}$ is related to the value of corresponding λ ; a larger λ implies a more significant effect of $\mathbf{V}$.

Then the POD basis function (POD mode) can be computed by,

$$\Phi = \frac{\mathbf{S}\mathbf{V}}{\|\mathbf{S}\mathbf{V}\|} \quad (38)$$

It should be noticed that the choice of snapshot ensemble is significant for the construction of POD basis functions, it can greatly affect the POD prediction accuracy of the original full-order problem.

4.2. Online stage

When both the POD basis functions and the discrete form of wetting phase pressure equation are all constructed, at online stage, the POD-ROM can be established by projecting the discrete governing equation into a low-dimensional space spanned by the basis functions. With the constructed POD-ROM, the spectral coefficients can be computed efficiently due to the low DOF of the POD-ROM. Then the wetting phase pressure of the original full-order problem can be reconstructed using the spectral coefficient and basis functions according to Eq. (17). The

reconstructed wetting phase pressure is used to calculate the Darcy velocity and to solve the wetting phase saturation equation.

In Table 3, the proposed IMPES algorithm based on POD-ROM is presented and summarized step by step.

Table 3: Implementation of the proposed IMPES algorithm based on POD-ROM

Algorithm 2: IMPES algorithm based on POD-ROM		
Offline stage	Step 1:	Solve the original wetting phase pressure equation under different calculation conditions using FDM, FEM, etc., to obtain the representative snapshot ensemble $\mathbf{S}$ of the wetting phase pressure
	Step2:	Compute the kernel $\mathbf{C}$ and solve the eigenvalue problem using SVD
	Step3:	Compute the basis functions Φ
Online stage	Step 4:	Project the discrete equation $\mathbf{A}\mathbf{p}_w = \mathbf{b}$ into the low-dimensional space spanned by selected POD basis functions using the method II, and establish the POD-ROM $\mathbf{A}_{\text{ROM}}^{\text{II}}\mathbf{c} = \mathbf{b}_{\text{ROM}}^{\text{II}}$
	Step 5:	Compute the spectral coefficient $\mathbf{c}$ by solving the POD-ROM
	Step 6:	Reconstruct the wetting phase pressure $\mathbf{p}_w = \mathbf{c}\Phi$ using the spectral coefficient and basis functions
	Step 7:	Compute the Darcy velocity explicitly $\mathbf{u}_\alpha = -\frac{k_{r,\alpha}(S_w)}{\mu_\alpha}\mathbf{k}(\nabla p_\alpha - \rho_\alpha\mathbf{g})$
	Step 8:	Solve the saturation equation explicitly $\phi \frac{S_w^{(k+1)} - S_w^{(k)}}{t^{(k+1)} - t^{(k)}} + \nabla \cdot \mathbf{u}_w = q_w - k_{w,w}p_w$

5. Results and discussion

In this Section, we first validate the reconstruction accuracy and computational speed-up of the proposed IMPES algorithm based on POD-ROM through numerical examples 1 ~ 2. The 1-D and 2-D cases are considered in these two examples, and the influence of POD modes number on computational performance of the proposed algorithm is also assessed against the standard IMPES algorithm using FDM without any acceleration technique. Then the advantages of the proposed algorithm are further demonstrated by numerical example 3, in which the randomly distributed permeability field is taken into account. In this example, the prediction accuracy, applicability, and computational efficiency of the proposed algorithm are tested. It should be noted that all computations in this study are performed on an Intel Core i7-5500U, CPU2.40 GHz PC with 8.00 GB of RAM using the in-house code.

It is worth pointing out a notable problem associated with POD-ROM that the reconstruction accuracy of POD-ROM strongly depends on the quality and the number of snapshots obtained from the original full-order problem. Representative and optimal snapshots which contain as much information as possible about the original problem would yield high-quality POD basis

functions. However, it is always not straightforward to select representative snapshots. Siade et al. [31] proposed an exponential method for snapshots selection that assumes fixed initial and final snapshot times. With a preselected fixed snapshot number, the optimal snapshot times are determined by maximizing the smallest eigenvalue of the covariance matrix. However, this snapshot selection technique is only applicable to the confined aquifers that subject to constant pumping and reach approximate steady state in an exponential manner. The selection of snapshots is still seriously empirical and problem-dependent, studies on the perfect way for the optimal snapshot choice in the presence of practical engineering applications are still in their infancy and an exhaustive investigation is lacking [31, 37]. One straightforward way to select representative snapshots containing dominant characteristics of the physical problems is to cover various and different boundary conditions and parameter settings. In this study, to remove the impacts of snapshot number and snapshot quality on the validation of reconstruction accuracy of the proposed POD-ROM, in numerical examples 1 ~ 2 we just use the same calculation conditions for computing the snapshots and for solving the original problem. With this simplification, we only need to control the POD modes number to reconstruct the solution in low-dimensional space for the original problem. In numerical example 3, to validate the applicability of the proposed algorithm, we will use different calculation conditions to compute the snapshots and to predict the original problem, which is the key objective of the present study.

5.1. Numerical example 1

(1) Example detail

In this example, the classic Buckley–Leverett problem in 1-D horizontal homogeneous domain initially saturated with oil is considered. Water is injected with a constant flow rate at left end to displace oil to the right end. The pressure is kept constant at the production end, both the gravitational acceleration and capillary pressure are neglected. The domain is divided into 80×3 uniform grids, and the time step $\Delta t = 0.5$ day. For the setting of boundary and initial conditions, the Darcy velocities on boundaries and the initial wetting phase saturation are all set as 0. Other parameters used in this example are presented in Table 4. Especially, the nonlinear relative permeability functions are assumed as follows [38],

$$k_{r,w} = k_{wn} S_e^m, \quad k_{r,n} = (1 - S_e)^m \quad (39)$$

where $m = 2$; S_e is the normalized saturation, $S_e = \frac{S_w - S_{r,w}}{1 - S_{r,w} - S_{r,n}}$, $S_{r,w}$ and $S_{r,n}$ are residual saturations of wetting phase and non-wetting phase, respectively.

(2) Results discussion

In this example, 2000 snapshots are selected to construct the POD basis functions. Before computations, the number of POD modes should be determined initially. Generally the most energetic POD modes are preferred to be chosen in the computation, which can be characterized

Table 4: Calculation parameters in numerical example 1

Calculation parameter	Value
Domain dimensions	300m×100m
Rock properties	$\phi = 0.2$, $k = 1\text{md}$
Fluid density	$\rho_w = \rho_n = 1000\text{kg/m}^3$
Fluid viscosity	$\mu_w = 1\text{cP}$, $\mu_n = 1.5\text{cP}$
Relative permeability	Quadratic function, Eq.(39)
Residual saturations	$S_{r,w} = 0$, $S_{r,n} = 0.2$, $k_{wn} = 0.6$
Injection rate	$5 \times 10^{-4}\text{PV/year}$
Well parameter	$k^{\text{WI}} = 100\text{md}$, $p_{bh} = 1\text{atm}$

by the accumulative energy contribution as follows,

$$E_n = \sum_{j=1}^n \lambda_j \setminus \sum_{i=1}^N \lambda_i \times 100\% \quad (40)$$

where λ represents the eigenvalue obtained from the eigenvalue decomposition (37) and is sorted in descending order; E_n is the accumulative energy contribution of the first n POD modes to the energy of the total N POD modes.

Theoretically, all information and characteristics of the original full-order problem would be recovered by the top n POD modes if E_n is equal to 100%. In practice, however, it is uneconomical to apply a large number of POD modes to capture all the information. In this study, we would truncate the POD modes number when the accumulative energy contribution is approximately 100% instead of exactly equal to 100%. As shown in Fig. 5, the cumulative energy contribution is approximately 100% when the POD modes number reaches 40. Therefore, it can be confirmed that a high reconstruction accuracy can be achieved if the top 40 POD modes are adopted in computations.

First we qualitatively validate the reconstruction accuracy of the proposed algorithm. In Figs. 6 ~ 7, the evolutions of wetting phase pressure and saturation calculated by IMPES algorithm based on POD-ROM using 40 POD modes and standard IMPES using FDM at different production time are presented, respectively. From these comparisons, it can be clearly observed that the results of the two quantities of interest obtained by the proposed algorithm display an identical evolution trend with these obtained by standard algorithm, it is hard to see any difference between these two groups of results. The variations of wetting phase pressure and saturation can be well approximated by the proposed algorithm with 40 POD modes. Thus it can be qualitatively confirmed that the proposed algorithm illustrates good reconstruction accuracy for both the wetting phase pressure and wetting phase saturation.

In order to validate the reconstruction accuracy of the proposed algorithm quantitatively,

Figure 5: Cumulative energy contribution with different POD modes numbers in numerical example 1

average relative errors for wetting phase pressure and saturation are defined respectively,

$$\varepsilon_p = \frac{\sum_{j=1}^{ny} \sum_{i=1}^{nx} \left| \frac{p_{w,POD} - p_{w,FDM}}{p_{w,FDM}} \right|}{nx \times ny} \times 100\% \quad (41)$$

$$\varepsilon_S = \frac{\sum_{j=1}^{ny} \sum_{i=1}^{nx} \left| \frac{S_{w,POD} - S_{w,FDM}}{S_{w,FDM}} \right|}{nx \times ny} \times 100\% \quad (42)$$

where $p_{w,POD}$ and $p_{w,FDM}$ represent the wetting phase pressure calculated by the IMPES based on POD-ROM and the standard IMPES using FDM, respectively; $S_{w,POD}$ and $S_{w,FDM}$ denote the wetting phase saturation calculated by the IMPES based on POD-ROM and the standard IMPES using FDM, respectively; nx and ny are grid numbers along x and y coordinate directions.

For brevity but without loss of generality, in Table 5 relative errors of the wetting phase pressure and saturation with different POD modes numbers are presented at the production time $t=1000$ days, and the corresponding variation trend of the relative errors is displayed in Fig. 8 as well. From Table 5 and Fig. 8, two characteristics regarding the error results can be clearly observed. First, with the increase of POD modes number, relative errors of the wetting phase pressure and saturation all decrease substantially, which indicates that more POD modes can capture more detailed information of the original problem. Thus to improve the reconstruction accuracy of POD-ROM, more POD modes are expected to be employed in computations. Second, with the same POD modes number, the relative error of wetting phase saturation is much larger

Figure 6: Evolution of wetting phase pressure respectively calculated by IMPES algorithm based on POD-ROM (left side, 40 POD modes) and standard IMPES algorithm using FDM (right side) in numerical example 1

than that of the wetting phase pressure in most cases. It can be explained from the perspective of physical nature and flow characteristics of multiphase flows in porous media. In multiphase porous flows, the fluid pressure usually change slightly while the coupled fluid saturation is sensitive to the pressure change, thus a tiny pressure change may cause an obvious variation of the saturation, in some cases the saturation even overshoots or undershoots beyond its physical value. For instance, in this example the relative error of wetting phase pressure is only 0.578% with 15 POD modes, however, the error value of wetting phase saturation is as large as 1.413%, which is nearly 3 times of that of wetting phase pressure.

From Table 5 and Fig. 8, it can be confidently confirmed that the proposed POD-POM for two-phase flows in porous media possesses high reconstruction accuracy. For instance, when 20 POD modes are applied to the computation, the relative errors of wetting phase pressure and saturation are all below 1%; and when 40 POD modes are adopted, the reconstruction accuracy

Figure 7: Evolution of wetting phase saturation calculated by IMPES algorithm based on POD-ROM (left side, 40 POD modes) and standard IMPES algorithm using FDM (right side) in numerical example 1

Table 5: Relative errors with different POD modes numbers in numerical example 1 ($t=1000$ days)

POD mode	5	10	15	20	25	30	35	40
ε_p	18.873%	3.741%	0.578%	0.423%	0.245%	0.141%	0.063%	0.035%
ε_S	17.638%	2.538%	1.413%	0.747%	0.571%	0.332%	0.257%	0.158%

can up to 0.035% and 0.158% for wetting phase pressure and saturation, respectively. The achieved reconstruction accuracy is confidential to various engineering applications.

For the sake of vivid and intuitive comparison, contours of the wetting phase pressure (Fig. 9) and wetting phase saturation (Fig. 10) obtained from the IMPES algorithm based on POD-ROM (with POD modes number 10, 20, 30, 40) are compared to those of the standard IMPES algorithm using FDM, respectively. From the two figures, we can observe that the obvious results

Figure 8: Variation trend of relative errors with different POD modes numbers in numerical example 1 ($t=1000\text{days}$)

distinctions between the proposed algorithm and the standard algorithm can easily be found when 10 POD modes are applied in the computation. The other three groups of comparison results are in good agreement, especially when 30 and 40 POD modes are considered. It can also be clearly seen that the deviation of wetting phase saturation is larger than that of wetting phase pressure with the same POD modes number. The comparison shown in Figs. 9~ 10 can perfectly validate the high reconstruction accuracy of the proposed POD-ROM and the influence of POD modes number on the reconstruction accuracy.

The aim of this study is to improve the computational efficiency of the standard IMPES algorithm using POD-ROM without losing the numerical accuracy obviously, thus in the following context the acceleration of the proposed algorithm is evaluated by comparing to the standard algorithm without any acceleration technique. It is worth mentioning that the absolute computation time of the numerical example depends on specific computers and parameter settings, thus in Fig. 11 only the speed-up ratio is shown with the increase of POD modes number, in which the speed-up ratio is defined as,

$$\text{ratio} = \frac{t_{\text{FDM}}}{t_{\text{POD}}} \quad (43)$$

where t_{FDM} and t_{POD} represent the calculation time of the standard IMPES algorithm using FDM and the proposed IMPES algorithm based on POD-ROM, respectively.

From Fig. 11, it is exciting to see that compared to the standard IMPES algorithm, the

Figure 9: Comparison of wetting phase pressure respectively calculated by IMPES algorithm based on POD-ROM (red dashed line) and standard IMPES algorithm using FDM (blue solid line) in numerical example 1 ($t=1000$ days)

POD-ROM based IMPES algorithm can accelerate the computation significantly. For instance, in this example the largest speed-up ratio can reach up to about 100 when 5 POD modes are applied, and the speed-up ratio is still around 30 even when 20 POD modes are employed in the computation where the reconstruction accuracy can be guaranteed confidently. However,

Figure 10: Comparison of wetting phase pressure respectively calculated by IMPES algorithm based on POD-ROM (red dashed line) and standard IMPES algorithm using FDM (blue solid line) in numerical example 1 ($t=1000$ days)

it should be noted as well that the POD-ROM is a double-edged sword, the computational efficiency and the reconstruction accuracy cannot be ensured at the same time. This hints us that in real applications of the proposed IMPES algorithm based on POD-ROM, how many POD modes should be used depends on the specific requirement on the numerical accuracy and

Figure 11: Speed-up ratio of the IMPES algorithm based on POD-ROM to the standard IMPES algorithm using FDM with different POD modes numbers in numerical example 1 ($t=1000\text{days}$)

computational cost because the acceleration of POD-ROM is problem-dependent.

5.2. Numerical example 2

(1) Example detail

In this example, a 2-D square homogeneous domain initially saturated with oil is considered. As shown in Fig. 12, water is injected with a constant flow rate at left bottom corner to displace oil to the right top corner. Similar to numerical example 1, the pressure is kept constant at the production end, both the capillary pressure and the gravitational acceleration are ignored. In this study, the domain is divided into 40×40 uniform grids, and the time step $\Delta t = 0.002$ year. The boundary and initial conditions are set as same with example 1, other relevant parameters are provided in Table 6. In this example, 1000 snapshots are computed to construct the POD basis functions.

(2) Results discussion

Similar to numerical example 1, the cumulative energy contribution with different POD modes numbers is first shown in Fig. 13. It can be obviously seen from Fig. 13 that when approximate the top 32 POD modes are considered, the cumulative energy contribution E_n is approaching to 100%. Thus in the computation, more than 30 POD modes are preferred to be adopted in order to ensure the high reconstruction accuracy of the POD-ROM based IMPES algorithm.

To ensure good reconstruction accuracy, 35 POD modes are used in this example. Figs. 14 ~ 15 present the evolutions of wetting phase pressure and saturation calculated by IMPES

Figure 12: Computational domain of numerical example 2

Table 6: Calculation parameters in numerical example 2

Calculation parameter	Value
Domain dimensions	500m×500m
Rock properties	$\phi = 0.2$, $k_{xx} = k_{yy} = 100\text{md}$
Fluid density	$\rho_w = 1000\text{kg/m}^3$, $\rho_n = 660\text{kg/m}^3$
Fluid viscosity	$\mu_w = 1\text{cP}$, $\mu_n = 0.45\text{cP}$
Relative permeability	Quadratic function, Eq.(39)
Residual saturations	$S_{r,w} = S_{r,n} = 0$, $k_{wn} = 1$
Injection rate	0.1PV/year
Well parameter	$k^{\text{WI}} = 10000\text{md}$, $p_{bh} = 1\text{atm}$

algorithm based on POD-ROM and standard IMPES using FDM at different production time, respectively. From these two figures, it is difficult to see any distinctions between the numerical results of the proposed algorithm and these of the standard algorithm. The proposed POD-ROM based algorithm can accurately predict the evolutions of wetting phase pressure and saturation with no doubt. Therefore, it can be said that the proposed algorithm offers perfect reconstruction accuracy for the wetting phase pressure and saturation.

For further quantitative analysis, relative errors of the wetting phase pressure and saturation with different POD modes numbers at $t=2$ years are illustrated in Table 7 and Fig. 16. It can be easily found the variation trend of relative errors for both wetting phase pressure and saturation are almost same with that in numerical example 1, which indicates the proposed

Figure 13: Cumulative energy contribution with different POD modes numbers in numerical example 2

IMPES algorithm is applicable to more complex problems as well. Compared to the results in Table 5, it can also be observed that the relative error of the wetting phase pressure is much smaller than that in example 1, but the error value of the wetting phase saturation is still very large compared to that of the wetting phase pressure. The main reason is that although the wetting phase pressure changes slightly, the wetting phase saturation would still change greatly due to the sensitive relation with pressure change. This is one major difference between the multiphase flows and single-phase flows, which would lead to a great challenge in the application of POD-ROM to the multiphase porous flow solvers by the Galerkin projection method.

Table 7: Relative errors with different POD modes numbers in numerical example 2 ($t=2\text{year}$)

POD mode	5	10	15	20	25	30	35
ε_p	1.119%	0.195%	0.102%	0.054%	0.024%	0.012%	0.008%
ε_S	13.748%	2.666%	1.265%	0.623%	0.270%	0.108%	0.045%

Based on above observations and results analyses, one issue needing to be addressed here is that with the increase of problem complexity, e.g., from one-dimension to multi-dimension, from single-phase to multi-phase, from homogeneity to heterogeneity, etc., to ensure the POD-ROM possesses good reconstruction accuracy, many more POD modes are required in complex problems than that in simple problems. It has been well proved in many previous studies, for instance, Wang et al. [33, 39] obtained high fidelity results by applying only 2 POD modes

Figure 14: Evolution of wetting phase pressure respectively calculated by IMPES algorithm based on POD-ROM (left side, 35 POD modes) and standard IMPES algorithm using FDM (right side) in numerical example 2

for an incompressible single-phase flow in homogeneous porous medium, and when performing the gas reservoir simulation based on POD-ROM for ideal gases, at least 10 POD modes were needed to ensure good simulation precision; Han et al. [40] adopted the POD-ROM to solve steady laminar flows on a simple irregular domain with at least 16 POD modes. Another issue that should be pointed out is that actually there are many factors which may exert impacts on the reconstruction accuracy of POD-ROM, especially the choice of representative snapshots used to construct POD basis functions. However, in most cases it is always difficult to obtain the optimal snapshots because we always have no idea about which snapshots are representative in

Figure 15: Evolution of wetting phase saturation respectively calculated by IMPES algorithm based on POD-ROM (left side, 35 POD modes) and standard IMPES algorithm using FDM (right side) in numerical example 2

advance for a complex or new problem. Thus in the application of POD-ROM to multiphase flows in porous media, except for the special attentions paid for the establishment of POD-ROM, it should be noted as well that appropriate POD modes number should be determined to guarantee a good reconstruction precision.

Similar to Figs. 9 ~ 10, the contours of wetting phase pressure and saturation corresponding to 10, 20, 30, 35 POD modes are plotted in Figs. 17 ~ 18, respectively. From these comparisons, we can see the wetting phase pressure respectively computed by the POD-ROM based algorithm

Figure 16: Variation trend of relative errors with different POD modes numbers in numerical example 2 ($t=2\text{years}$)

and the standard algorithm match well with each other even when only 10 POD modes are used. For the wetting phase saturation, except for the case with 10 POD modes, the good agreement of other three groups of results can be observed. With the increase of POD modes number, the gap existing in the wetting phase saturation between the proposed algorithm and the standard algorithm becomes narrow. We can confidently confirm that the POD-ROM based IMPES algorithm proposed in this study works well in this complex 2-D example.

In Fig. 19, the computational speed-up ratio of the proposed algorithm is displayed with different POD modes numbers. It is noticeable that comparing to the 1-D case in example 1, the speed-up ratio is much larger in current 2-D problem, even when 35 POD modes are applied, the computation of the proposed algorithm is still 2 orders of magnitude faster than the standard algorithm. Thus again we can definitely confirm that the acceleration efficiency of the POD-ROM depends on the specific problem. It is reasonable that with the increase of problem complexity, the POD-ROM would show much better acceleration, which means that the advantage of POD-ROM is more obvious in complex problems. The problem-dependent characteristic of POD-ROM reminds us it is significant and meaningful to seek a balance between the computational efficiency and reconstruction accuracy.

5.3. Numerical example 3

(1) Example detail

Through numerical examples 1 ~ 2, it can be confidently confirmed that the proposed IMPES algorithm based on POD-ROM possesses a desirable reconstruction accuracy and computational

Figure 17: Comparison of wetting phase pressure respectively calculated by IMPES algorithm based on POD-ROM (red dashed line) and standard IMPES algorithm using FDM (blue solid line) in numerical example 2 (t=2year)

speed-up with appropriate POD modes number. In numerical example 3, to validate the robustness and applicability of the proposed algorithm, we adopt a random permeability field caused by the uncertainties stemming from the inherent spatial heterogeneity of media properties. In this example, we use two sets of parameter settings, where the first one is used to compute the snapshots and construct the POD basis functions at offline stage, and the other set is applied to predict the incompressible two-phase flows in porous media using the POD-ROM based IMPES algorithm at online stage.

Suppose the random permeability field K to be lognormally distributed that can guarantee

Figure 18: Comparison of wetting phase saturation respectively calculated by IMPES algorithm based on POD-ROM (red dashed line) and standard IMPES algorithm using FDM (blue solid line) in numerical example 2 (t=2year)

the generated permeability is positive, i.e., $\log K$ follows the Gaussian distribution ,

$$\log K \sim N(\mu, \sigma^2) \quad (44)$$

Assume the logarithm of K to be a wide-sense (weakly) stationary Gaussian random field, then we can get,

$$E [\log K(\mathbf{x})] = \text{const} \quad (45)$$

$$\text{Cov} [\log K(\mathbf{x}_1), \log K(\mathbf{x}_2)] = f(\mathbf{x}_1 - \mathbf{x}_2) \quad (46)$$

Figure 19: Speed-up ratio of the IMPES algorithm based on POD-ROM to the standard IMPES algorithm using FDM with different POD modes numbers in numerical example 2 (t=2years)

where in this study, the const is set as $\log(10)$.

If the field is imposed to be isotropic, then,

$$Cov[\log K(\mathbf{x}_1), \log K(\mathbf{x}_2)] = f(\|\mathbf{x}_1 - \mathbf{x}_2\|_p) = f(d) \quad (47)$$

In this study, we adopt the L_1 norm to compute the distance between two locations, thus $p = 1$.

There are many choices for the covariance function, such as Matrn covariance. In this study, the following covariance function is applied,

$$Cov(d) = \sigma^2 \exp\left(-\frac{\pi}{4} \sum_{i=1}^2 \frac{d_i}{\xi_i}\right) \quad (48)$$

where the variance $\sigma^2 = 1$; ξ is the correlation length, ξ_1 and ξ_2 represent the correlation length along x and y directions respectively with $\xi_1 = 0.01$ and $\xi_y = 0.001$. In this study, the correlation lengths are rotated by an angle $\pi/12$ clockwise to obtain much more complex permeability field.

Several methods have been put forward for the random field generation [41], such as matrix decomposition, Karhunen-Loève expansion, moving average, sequential simulation, etc. In this study, the simple Cholesky decomposition method is adopted to generate the random permeability field K (unit: mD), as shown in Fig. 20.

The computational domain and production conditions of this example are similar to example 2. The pressure is kept constant at the production end, and both the capillary pressure and the

Figure 20: Random permeability field in numerical example 3

gravitational acceleration are neglected. The domain is divided into 50×50 uniform grids. The boundary and initial conditions are set as same with example 2, other calculation parameters are presented in Table 8. In this example, we first compute the snapshots and construct the POD basis functions at the offline stage. To obtain the representative POD basis functions, we collect three sets of snapshots with the parameters shown in Table 9. Then we use the extracted POD basis functions to predict the incompressible two-phase porous flows with another two sets of parameters at online stage.

Table 8: Calculation parameters in numerical example 3

Calculation parameter	Value
Domain dimensions	500m $\times$ 500m
Rock properties	$\phi = 0.2$, $k_{xx} = k_{yy} = k(x, y)$ md
Fluid density	$\rho_w = 1000\text{kg/m}^3$, $\rho_n = 660\text{kg/m}^3$
Fluid viscosity	$\mu_w = 1\text{cP}$, $\mu_n = 0.45\text{cP}$
Relative permeability	Quadratic function, Eq.(39)
Residual saturations	$S_{r,w} = S_{r,n} = 0$, $k_{wn} = 1$
Well parameter	$k^{\text{WI}} = 100k_{xx}$, $p_{bh} = 1\text{atm}$

(2) Results discussion

In this part, the first 4 POD modes of the wetting phase pressure are shown in Fig. 21. From

Table 9: Parameter settings for calculating the snapshots and for prediction

Parameter	Offline stage			Online stage	
	Snapshot I	Snapshot II	Snapshot III	Prediction I	Prediction II
Injection rate	0.1PV/year	0.2PV/year	0.3PV/year	0.15PV/year	0.25PV/year
Time step	0.004 year	0.002 year	1/750 year	0.002 year	0.001 year
Calculation time	6 years	3 years	2 years	1, 2, 3, 4 year	1, 2 year

the counters, it can be easily seen that the values of the 4 POD modes have the same order of magnitude. In addition, due to the importance of the 1st POD mode, it can depict the main configuration of the wetting phase pressure in this example, and we will validate this in later prediction part.

To choose appropriate POD modes in our prediction at online stage, we first show the cumulative energy contribution with different POD modes numbers in Fig. 22. It can be found that if approximately 40 POD modes are adopted in the computation, the cumulative energy contribution E_n is about 100%. Thus in the prediction, we choose the first 40 POD modes to predict the two-phase flows in porous media with the parameters shown in Table 9. It should be noted that compared with the homogenous permeability field in example 2, we can infer the randomly distributed permeability field is the main reason for applying more POD modes than that in example 2. Due to the introduction of random permeability field, the coefficient of the wetting phase pressure will show high contrast property, which would increase the computational difficulty.

Based on above analysis, we apply 40 POD modes to ensure good prediction accuracy in online computations. Figs. 23 ~ 24 illustrate the evolutions of wetting phase pressure and saturation computed by the proposed IMPES algorithm based on POD-ROM with the injection rate 0.15PV/year. To assess the prediction accuracy of the proposed algorithm, corresponding results obtained by the standard IMPES algorithm are depicted by the black line in the same figure as well. From the comparison of wetting phase pressure, it can be clearly seen that the predicted results by the proposed algorithm are consistent with those of the standard algorithm at different production time. Only tiny derivation between these two results can be observed in a small area located at the left top region. For the comparison results of wetting phase saturation, the results respectively obtained by the proposed algorithm and the standard algorithm are in good agreement too. Different from the wetting phase pressure, the predicted wetting phase saturation shows much obvious deviations at specific local locations. This is because the saturation variation is sensitive to the pressure change, if there are some errors existing in the wetting phase pressure, the wetting phase saturation would show the corresponding much larger errors as well. However, the overall counters of the wetting phase saturation are still perform good prediction accuracy. Thus it can be concluded that the proposed IMPES algorithm based

Figure 21: Top four POD modes of wetting phase pressure in numerical example 3

on POD-ROM is capable of predicting the two-phase flows accurately by using 40 POD modes with consideration of randomly distributed permeability field.

To comprehensively validate the prediction accuracy of the POD-ROM based IMPES algorithm, the calculated wetting phase pressure and saturation with injection rate 0.25PV/year are presented in Figs. 25 ~ 26 as well. It can be easily found that the wetting phase pressure obtained by proposed algorithm match well with those obtained by standard algorithm. Similar to Fig. 24, the predicted wetting phase saturation calculated by the proposed algorithm show small derivations compared to results of standard algorithm. However, it can still be confidently confirmed that the proposed IMPES algorithm based on POD-ROM offers satisfactory prediction accuracy regarding the comparison. Through the two online predictions, it can be concluded that the proposed POD-ROM based IMPES algorithm show good applicability in more complex problems, it can accurately handle complicated calculation conditions such as the random

Figure 22: Cumulative energy contribution with different POD modes numbers in numerical example 3

permeability field, etc.

Last but not least, under the premise of good prediction accuracy, the quality of most interest in this example is the computational speed-up of the proposed POD-ROM based IMPES algorithm compared to the standard IMPES algorithm. As mentioned before, the computational speed-up of the proposed algorithm is problem-dependent and is easily influenced by many factors. In this example, we can control all calculation conditions of the two algorithm to be same except for the solver, one is the direct solver in POD-ROM based IMPES algorithm and the other is the iterative solver in standard IMPES algorithm due to the large-scale size of the original full-order problem. Thus in comparison of the speed-up ratio of the proposed algorithm to the standard algorithm, we have to set specific convergence criterion for the iterative solver in standard algorithm. As shown in Table 10, two sets of computational speed-up results are presented with regard to two convergence criteria for the standard algorithm, in which the average relative error of the wetting phase pressure is set as 10^{-9} (FDM1) and 10^{-10} (FDM2), respectively. From Table 10, it can be clearly observed that the speed-up ratio ranges from 60.16 to 270.55, indicating that the POD-ROM based IMPES algorithm proposed in this study can markedly accelerate the computations compared to the standard IMPES algorithm using FDM. It can be also easily found that under the same calculation condition, more strict convergence criterion would yield more obvious speed-up ratio. We can also see that with the increasing of the production time from 1 year to 4 years with injection rate 0.15PV/year, or from 1 year to 2 years with injection rate 0.25PV/year, the computational speed-up would increase as well. Overall, it is confident to say that the proposed efficient IMPES algorithm based on POD-ROM can sub-

Figure 23: Evolution of wetting phase pressure respectively predicted by the proposed IMPES algorithm based on POD-ROM (40 POD modes) at different production time with injection rate 0.15PV/year in numerical example3 (black line represents the results obtained by standard IMPES algorithm)

stantially accelerate the computations of two-phase porous flows without losing the prediction accuracy obviously.

6. Conclusions

The multiphase flow in porous media is one of the most fundamental phenomena in various engineering applications. Fast simulation and prediction of multiphase porous flows is of great significance for the engineering practice. As one of the most popular model order reduction methods, in this study the POD-ROM is applied to develop an efficient solver for two-phase flows in porous media in combination with the IMPES algorithm. The establishment of the proposed IMPES algorithm based on POD-ROM is introduced in detail, and the computational

Figure 24: Evolution of wetting phase saturation respectively predicted by the proposed IMPES algorithm based on POD-ROM (40 POD modes) at different production time with injection rate 0.15PV/year in numerical example3 (black line represents the results obtained by standard IMPES algorithm)

Table 10: Computational speed-up of the POD-ROM based IMPES algorithm in numerical example 3

Conditions	0.15 PV/year				0.25 PV/year	
	1 year	2 years	3 years	4 years	1 year	2 years
FDM1 (s)	133.10	295.16	597.86	896.05	239.45	738.33
FDM2 (s)	461.91	931.44	1519.28	2148.15	872.67	1875.25
POD (s)	2.08	4.25	6.17	7.94	3.98	8.06
Speed-up1	63.99	69.45	96.90	112.85	60.16	91.60
Speed-up2	222.07	219.16	246.24	270.55	219.26	232.65

Figure 25: Evolution of wetting phase pressure respectively predicted by the proposed IMPES algorithm based on POD-ROM (40 POD modes) at different production time with injection rate 0.25PV/year in numerical example3 (black line represents the results obtained by standard IMPES algorithm)

Figure 26: Evolution of wetting phase saturation respectively predicted by the proposed IMPES algorithm based on POD-ROM (40 POD modes) at different production time with injection rate 0.25PV/year in numerical example3 (black line represents the results obtained by standard IMPES algorithm)

performance of the proposed algorithm is evaluated by comparing to the standard IMPES without any acceleration technique. Then the proposed algorithm is applied to predict the two-phase flows with different parameter settings at online stage. Based on above studies, the main concluding remarks of this work can be summarized as follows,

(1) Two projection methods are respectively applied to establish the POD-ROM for incompressible two-phase flows in porous media. It is found the POD-ROM established by directly projecting the original governing equation possesses the demerit that it is incapable of intro-

ducing the fluid saturation and permeability located on the grid cell edge, which would lead to unphysical results. Thus in this study, we project the discrete governing equation to construct the POD-ROM for the incompressible two-phase porous flow.

(2) Computational performances of the proposed IMPES algorithm based on POD-ROM are validated by three examples (1-D and 2-D cases, random permeability field case). Results show that the proposed algorithm displays a satisfactory reconstruction accuracy and acceleration, especially in the 2-D case, the computational speed of the proposed algorithm is 2 orders of magnitude faster than standard IMPES algorithm using FDM at the same time preserving good reconstruction accuracy. The proposed algorithm also show good applicability to complex problem with randomly distributed field. Due to the performance of POD-ROM based algorithm is problem-dependent, the number of POD modes has significant effect on the reconstruction accuracy and computational speed-up. It hints us that a balance between the computational speed-up and the reconstruction accuracy should be considered in the determination of POD modes number.

(3) The prediction accuracy of wetting phase saturation is slightly lower than that of the wetting phase pressure owing to the fact that the saturation is much sensitive to the pressure change in multiphase porous flows. Thus in the computation the numerical accuracy of the wetting phase pressure should be guaranteed strictly to ensure the wetting phase saturation can be predicted accurately. In addition, the proposed POD-ROM based algorithm has good applicability to various problems, such as 1-D or 2-D, homogeneous permeability field or randomly distributed permeability field, etc. The IMPES algorithm based on POD-ROM proposed in this work is a good choice for the fast prediction of incompressible two-phase flows in porous media.

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