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Highlights

- A parallel and fully implicit simulator for the black oil model based on the variational inequality (VI) framework has been established.
- Unlike most existing simulators in literature, the developed simulator ensure the predicted solution to stay within the physical range.
- An inexact Newton method with the domain decomposition based linear preconditioners is proposed to solve the arising resultant nonlinear system.
- The robustness and efficiency of the developed simulator are demonstrated using a number of large-scale and heterogeneous test cases.

A fully implicit constraint-preserving simulator for the black oil model of petroleum reservoirs

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Abstract

Due to the rapid advancement of supercomputing resource, there is a growing interest in developing parallel algorithms for the large-scale reservoir simulation. In this paper, we present a parallel and fully implicit simulator for the black oil model based on the variational inequality (VI) framework, which can be used to enforce important mathematical and physical properties to obtain accurate constraint-preserving solutions. In other words, this framework ensures the predicted solution to stay within the physical range. In the proposed approach, the black oil model is reformulated as a variational inequality system that naturally satisfies the basic boundedness requirement of the solution, and then a fully implicit finite volume method is applied to discretize the model equations. In addition to that, a number of nonlinear and linear fast solver technologies, including a variant of inexact Newton methods and the domain decomposition based preconditioners, are employed to guarantee the robustness and parallel scalability of the simulator. A particular emphasis of the proposed framework is placed on the parallel and algorithmic performance of the variational inequality approach across large-scale and heterogeneous problems. Several numerical results pertaining to the problems in one, two and three dimensions are presented to illustrate the efficiency, robustness, and the overall performance of the fully implicit constraint-preserving simulator.

Keywords: Black oil model, Fully implicit method, Variational inequality, Active-set reduced-space, Parallel computing

1. Introduction

Numerical simulation plays a critical role in maximizing field productions by accurate prediction of reservoir performance. In the last few decades, several mathematical models

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have been proposed to describe oil recovery processes of different mechanisms. An extensive study of numerical methods has been done to approximate solutions with higher accuracy and efficiency. Based on the classification of fluid models, reservoir simulators can be classified into two categories: black oil simulator and compositional simulator.

The black oil model is most widely used in reservoir simulation and mainly used in cases where recovery processes are insensitive to compositional changes [12]. Apart from forecast of oil and gas recovery, it is also applied to estimate different effects on oil recovery, including well pattern and spacing, well completion intervals, gas/water coning as a function of rate, producing rate, infill drilling, and gas vs. water vs. water-alternating-gas (WAG) injection. [14]. Although it is not recommended to use black oil simulator for gas injection [18], most primary recovery mechanisms, such as water drive, solution gas drive, gas cap expansion, gravity drainage, and capillary imbibition, as well as water flooding which belongs to the secondary recovery stage, can be simulated by the black oil model with proper PVT data [12]. In the black oil model, two pseudo-components are used to represent oil and gas phases, and water explicitly represents water phase. The mathematical formulation of the black oil model includes thermodynamic constraints which governs mass redistribution, equation of state, Darcy's law, and mass conservation equation for each component [36]. Compared to the traditional multiphase flow model which assumes no mass transfer between phases, the black oil model is more physically meaningful because it relaxes this assumption. Specifically, the pseudo oil and gas component are allowed to transfer between hydrocarbon phases so that phase state can be changed. In some cases if the PVT data are generated properly, the black oil model has enough accuracy but much less computationally expensive than the fully compositional model.

The main contribution of this paper is to present a parallel computational framework that is applicable to the black oil model and can automatically satisfy the boundedness requirement of the solution. The framework is built by reformulating the original problem as a dynamic system of differential variational inequality (VI) that encompasses partial differential equations with variational inequalities or complementarity problems [30, 37]. The variational inequality system, which involves both dynamics and inequality constraints, is derived as the first variation of the involved potential energy and has to be solved for all possible values of a given variable that usually belongs to a convex set. The variational inequality theory originates from a problem posed by Antonio Signorini in [33] and was later more accurately described by Fichera in [19], in which the existence and uniqueness of solutions was established. In the recent years, the variational inequality methods are becoming a very powerful tool for studying a wide range of problems arising in many diverse fields of pure and applied sciences, and has proven to be very efficient in a variety of applications [3, 4, 20, 22, 31]. For reservoir simulation, with the help of the variational inequality technique, some progresses have been made to preclude the formation of over- and under-shoots in the proximity of sharp gradients of the solution [10, 24, 27, 41]. In particular, the approach can be combined with certain variants of nonlinear and linear solvers in a robust and efficient way to significantly improve the physical feasibility of the reservoir simulation, which greatly expands the scope of application of these simulators [28, 42]. However, to the best of the authors' knowledge, very little work has been done to address the challenge of

obtaining accurate constraint-preserving solutions for the black oil model within the variational inequality framework, due in large part to the much complicated dynamics of the black oil model that may cause issues in the solver.

The computational efficiency and robustness of the new variational inequality framework depend heavily on sophisticated nonlinear and linear solvers, which are suitable for high performance computers with a large number of processors accompanied by the increase of the size of problems. It is therefore of great importance to study highly efficient nonlinear and linear solvers of the algebraic systems to accelerate the simulation at large scale in a reasonable time. In this study, we propose a parallel solver based on the framework of a variant of inexact Newton methods, i.e., the active-set reduced-space (ASRS) method [2, 10, 29, 40, 42], to guarantee the nonlinear consistency. In the active-set reduced-space method, an index set is partitioned into active and inactive parts, based on a criterion specifying a certain active set method, and then a reduced linear system of equations is solved to calculate a direction of the gradient descent. The descent direction of the inactive set is approximated, and the solution is updated using a projected line search, while the descent direction of the active set is set to zero. Unlike the classical cutting-off procedure, this method is variationally consistent because only the gradient descent is clipped and not the actual solution itself [10, 27, 40]. Within each nonlinear iteration, the linear Jacobian systems are solved with a preconditioned Krylov subspace method [32], and the overlapping additive Schwarz type domain decomposition methods [7, 8, 34, 35] are used to build the preconditioner, based on the fact that it can substantially reduce the condition number of the linear system and is suitable to massively parallel computing. Several key parameters in the preconditioner, including the type of the Schwarz preconditioner, the size of the overlap, and the subdomain solvers, are discussed and tested to achieve the optimal performance. A particular emphasis of the proposed framework is placed on its ability of combining with any spatial discretization schemes and numerical formulations, and enforcing bounded constraints to naturally satisfy the mass conservation. In addition to that, the framework is suitable to parallel computing, which will be illustrated that it is efficient, robust and scalable on a supercomputer.

The paper is organized as follows. In Section 2, the variational inequality formulation for the black oil model is presented, following which a fully implicit scheme is provided for the temporal and spatial discretization. In Section 3, we present a nonlinear constraint-preserving solver, including the active-set reduced-space method and the domain decomposition based preconditioner, for the resultant nonlinear system. We examine the efficiency and the parallel performance of the proposed method by carrying out numerical results on several test cases in Section 4. The paper is concluded in Section 5.

2. Description of black oil model

In the standard black oil model, the reservoir flow involves three pseudo components (oil, water and gas) and three phases. The oil component exists in the oil phase, the gas component can be transferred between the oil and gas phases, and the water component only exists in the water phase. For each phase α ($\alpha = o, w, g$), let S_α be the saturation, then

the black oil model can be written as follows [11, 21, 38]

$$\begin{cases} \frac{\partial (\phi \rho_o^o S_o)}{\partial t} &= \nabla \cdot \left(\frac{\kappa_{ro} \rho_o^o}{\mu_o} \mathbf{K} \nabla \Theta_o \right) + q_o, \\ \frac{\partial (\phi \rho_w S_w)}{\partial t} &= \nabla \cdot \left(\frac{\kappa_{rw} \rho_w}{\mu_w} \mathbf{K} \nabla \Theta_w \right) + q_w, \\ \frac{\partial (\phi \rho_o^g S_o + \phi \rho_g S_g)}{\partial t} &= \nabla \cdot \left(\frac{\kappa_{ro} \rho_o^g}{\mu_o} \mathbf{K} \nabla \Theta_o \right) + \nabla \cdot \left(\frac{\kappa_{rg} \rho_g}{\mu_g} \mathbf{K} \nabla \Theta_g \right) + q_o^g + q_g, \end{cases} \quad (1)$$

where ϕ is the porosity of the medium, ρ_α denotes the density, q_α is the external mass flow rate, $\kappa_{r\alpha}$ denotes the relative permeability, and μ_α is the viscosity. $\mathbf{K}$ is the absolute permeability tensor and $\Theta_\alpha = p_\alpha + \rho_\alpha \zeta \nabla z$ is the flow potential with ζ the gravitational constant and z the reservoir depth, respectively. Moreover, ρ_o^g is the density of solution gas, and ρ_o^o is the density of the oil component. The relation between any two phase pressures is given by a capillary pressure. To simplify the treatment of injectors and producers, they are modeled as the inflow and outflow boundaries. All the other boundaries are assumed to be impervious.

2.1. Variational inequality formulation

In this study, we focus on the black oil model under the saturated state, i.e., all three phases coexist. Hence, the primary variables in this system are $X = (p_o, S_w, S_g, S_o)^T$ and an alternative way of formulating (1) is to define the multi-components system:

$$F(X) = \begin{bmatrix} F^{(p_o)}(X) \\ F^{(S_w)}(X) \\ F^{(S_g)}(X) \\ F^{(S_o)}(X) \end{bmatrix} = 0, \quad (2)$$

where each residual nonlinear function is defined as follows:

1. The residual nonlinear function for the variable p_o

$$F^{(p_o)}(X) = \frac{\partial (\phi \rho_o^o S_o)}{\partial t} - \nabla \cdot \left(\frac{\kappa_{ro} \rho_o^o}{\mu_o} \mathbf{K} \nabla \Theta_o \right) - q_o.$$

2. The residual nonlinear function for the variable S_w

$$F^{(S_w)}(X) = \frac{\partial (\phi \rho_w S_w)}{\partial t} - \nabla \cdot \left(\frac{\kappa_{rw} \rho_w}{\mu_w} \mathbf{K} \nabla \Theta_w \right) - q_w.$$

3. The residual nonlinear function for the variable S_g

$$F^{(S_g)}(X) = \frac{\partial (\phi \rho_o^g S_o + \phi \rho_g S_g)}{\partial t} - \nabla \cdot \left(\frac{\kappa_{ro} \rho_o^g}{\mu_o} \mathbf{K} \nabla \Theta_o \right) - \nabla \cdot \left(\frac{\kappa_{rg} \rho_g}{\mu_g} \mathbf{K} \nabla \Theta_g \right) - q_o^g - q_g.$$

4. The residual nonlinear function for the variable S_o

$$F^{(S_o)}(X) = S_o + S_w + S_g - 1. \quad (3)$$

In the system (2), the predicted saturations need to satisfy the underlying constraints $0 \leq S_\alpha \leq 1$, and the computed pressure belongs to the range $[p_\xi, p_\psi]$, where p_ξ (p_ψ) is the bottomhole pressure at the production (injection) well. Considering the above constraints as a potential obstacle of the solutions and introducing an L^2 -inner product, we obtain the variational inequality (VI) formulation of the black oil model (2); see references [3, 10, 17, 24, 27, 41] for more details. Here, we take the water saturation S_w for example. With the help of Lagrange multipliers for the inequality constraints $S_w \geq 0$ and $S_w \leq 1$, respectively, we have the following variational inequality

$$\begin{cases} S_w = 0, & \& F^{(S_w)}(X) = \frac{\partial(\phi\rho_w S_w)}{\partial t} - \nabla \cdot \left(\frac{\kappa_{rw}\rho_w}{\mu_w} \mathbf{K} \nabla \Theta_w \right) - q_w \geq 0, \\ S_w = 1, & \& F^{(S_w)}(X) = \frac{\partial(\phi\rho_w S_w)}{\partial t} - \nabla \cdot \left(\frac{\kappa_{rw}\rho_w}{\mu_w} \mathbf{K} \nabla \Theta_w \right) - q_w \leq 0, \\ S_w \in (0, 1), & \& F^{(S_w)}(X) = \frac{\partial(\phi\rho_w S_w)}{\partial t} - \nabla \cdot \left(\frac{\kappa_{rw}\rho_w}{\mu_w} \mathbf{K} \nabla \Theta_w \right) - q_w = 0. \end{cases}$$

Similarly, we also build a variational inequality formulation for the pressure p_o ,

$$\begin{cases} p_o = p_\xi, & \& F^{(p_o)}(X) \geq 0, \\ p_o = p_\psi, & \& F^{(p_o)}(X) \leq 0, \\ p_o \in (p_\xi, p_\psi), & \& F^{(p_o)}(X) = 0, \end{cases}$$

and the other two saturations S_g and S_o .

Remark 1 *The variational inequality for all the three saturations is a key element for the simulation of the black oil model under the saturated state. In the traditional simulator for (1), the primary variables in this system are the pressure and the two saturations, and then the third saturation can be computed by the relation (3) [11, 38]. However, this treatment can not be directly extended to the variational inequality approach, i.e., only two of the three saturations are carried out with the variational inequality technique. In such a situation, the solution of the remaining third saturation without variational inequality often jumps out of the physically meaningful range, even though the other two computed saturations stay within this range.*

2.2. Fully-implicit finite volume discretization

A finite volume method [11, 12, 21] is then applied to discretize the model equations for the spatial terms, and the implicit first order scheme (Backward Euler) [23, 38, 41, 42] is used for the time differencing approximation. The unknowns (p_o, S_w, S_g, S_o) are approximated by cell-wise constants. The permeability and mobilities are not well-defined at the interface,

and hence approximated separately by harmonic average and upstream weighting scheme [11]. The upwind direction for each phase α is determined using its phase velocity

$$u_\alpha = -\frac{\kappa_{r\alpha}\rho_\alpha}{\mu_\alpha} \mathbf{K} (\nabla p_\alpha + \rho_\alpha g \nabla z).$$

For a given time-stepping sequence $0 = t^{(0)} < t^{(1)} < t^{(2)} < \dots < t^{(M)} = T_f$, we define the time step size $\Delta t^{(m)} = t^{(m+1)} - t^{(m)}$ and use the superscript (m) to denote the discretized evaluation at time point $t = t^{(m)}$. Let Ω be the computational domain that is covered with the grid cell Ω_i , i.e., $\Omega = \cup \Omega_i$, the cell interface between two grid cell Ω_i and Ω_j is defined by $\Gamma_{i,j} = \partial\Omega_i \cap \partial\Omega_j$, and $\mathbf{n}_{i,j}$ denote the associated normal vectors pointing from cell i to cell j . Then, taking the integral over the cell Ω_i for the saturation equation, we obtain

$$\begin{aligned} & \frac{1}{|\Omega_i|} \int_{\Omega_i} \frac{\partial (\phi \rho_w^{(m+1)} S_w^{(m+1)})}{\partial t} d\Omega_i + \frac{1}{|\Omega_i|} \int_{\Omega_i} \nabla \cdot u_w d\Omega_i - \frac{1}{|\Omega_i|} \int_{\Omega_i} q_w^{(m+1)} d\Omega_i \\ &= \frac{1}{|\Omega_i|} \int_{\Omega_i} \frac{\partial (\phi \rho_w^{(m+1)} S_w^{(m+1)})}{\partial t} d\Omega_i + \frac{1}{|\Omega_i|} \sum_j \int_{\Gamma_{i,j}} u_w \cdot \mathbf{n}_{i,j} d\Gamma_{i,j} - q_{w,i}^{(m+1)}. \end{aligned}$$

We define the cell-averaged saturation $S_{w,i}^{(m)}$ at time $t = t^{(m)}$ as

$$S_{w,i}^{(m)} = \frac{1}{|\Omega_i|} \int_{\Omega_i} S_w d\Omega_i,$$

and apply the backward Euler scheme to approximate the derivative term with respect to the time. We obtain the fully implicit finite volume scheme for the saturation equation

$$\mathcal{F}_{S_w}(X) = \frac{\phi \rho_w^{(m+1)} S_w^{(m+1)} - \phi \rho_w^{(m)} S_w^{(m)}}{\Delta t^{(m)}} + \frac{1}{|\Omega_i|} \sum_j \int_{\Gamma_{i,j}} u_w^{(m+1)} \cdot \mathbf{n}_{i,j} d\Gamma_{i,j} - q_{w,i}^{(m+1)}.$$

Similarly, we also build the discretized formulations $\mathcal{F}_{p_o}(X)$, $\mathcal{F}_{S_g}(X)$ and $\mathcal{F}_{S_o}(X)$ for the variables p_o , S_g and S_o , respectively.

Let the solution X be organized by $X \equiv (p_o, S_w, S_g, S_o) = (X_1, \dots, X_n) \in \mathbb{R}^n$, the corresponding function be defined by

$$\mathcal{F}(X) \equiv (\mathcal{F}_{p_o}(X), \mathcal{F}_{S_w}(X), \mathcal{F}_{S_g}(X), \mathcal{F}_{S_o}(X)) = \mathcal{F}(X_1, \dots, X_n) \in \mathbb{R}^n,$$

where $\mathcal{F}_{S_\alpha}(X)$ is the residual function with respect to S_α , and the lower- and upper-bound vectors for the solution X be denoted as

$$\begin{cases} \Xi \equiv (\dots, p_\xi, 0, 0, 0, \dots) = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n, \\ \Psi \equiv (\dots, p_\psi, 1, 1, 1, \dots) = (\psi_1, \psi_2, \dots, \psi_n) \in \mathbb{R}^n. \end{cases} \quad (4)$$

Then the discretization of variational inequality for (1) is defined as follows: find $X \in \mathbb{R}^n$ such that one of the following holds for all $i \in \mathcal{S}$,

$$\begin{cases} X_i = \xi_i & \& \mathcal{F}_i(X) \geq 0, \\ X_i = \psi_i & \& \mathcal{F}_i(X) \leq 0, \\ X_i \in (\xi_i, \psi_i) & \& \mathcal{F}_i(X) = 0, \end{cases} \quad (5)$$

where the inequality holds componentwisely by Ξ and Ψ , i.e., only one of these three equations hold at a time.

3. Nonlinear constraint-preserving solver

The nonlinear system (5) is solved by a nonlinear constraint-preserving solver, which consists of three important components: a variant of inexact Newton methods as the nonlinear solver; then a Krylov iterative method as the linear solver at each Newton step; and an overlapping Schwarz method as the preconditioner.

3.1. Inexact Newton with Backtracking

We briefly review the classical inexact Newton method with backtracking (INB) [7, 15, 39] for a nonlinear system $\mathcal{F}(X) = 0$, which will be used as the basis of the proposed algorithm. Suppose X^k is the current approximate solution, then a new approximate solution X^{k+1} can be computed through the following steps:

1. Compute the inexact Newton direction d^k such that

$$\|\nabla\mathcal{F}(X^k)d^k + \mathcal{F}(X^k)\| \leq \max\{\eta_r\|\mathcal{F}(X^k)\|, \eta_a\}, \quad (6)$$

where $\nabla\mathcal{F}(X^k)$ is the Jacobian matrix, $\eta_r \in [0, 1)$ is a relative tolerance and $\eta_a \in [0, 1)$ is an absolute tolerance.

2. Compute the new approximate solution

$$X^{k+1} = X^k + \lambda^k d^k, \quad (7)$$

where the step length $\lambda^k \in (0, 1]$ is determined to satisfy

$$\|\mathcal{F}(X^k + \lambda^k d^k)\| \leq (1 - \alpha\lambda^k)\|\mathcal{F}(X^k)\|. \quad (8)$$

3. We continue the nonlinear iteration until the following convergence criterion is satisfied

$$\|\mathcal{F}(X^{k+1})\| \leq \max\{\varepsilon_r\|\mathcal{F}(X^0)\|, \varepsilon_a\},$$

where ε_r (ε_a) is the relative (absolute) solver tolerance for the nonlinear iteration.

In general, the family of inexact Newton algorithms could be a good choice as the numerical solver of the nonlinear system, since it is easy to implement, general-purpose and has a rapid local convergence [5, 9]. However, most used Newton methods in practice do not guarantee the computed solution to sit within physically meaningful range. Hence, we propose the active-set reduced-space (ASRS) method [2, 10, 29, 40, 42] to deal with the difficulties.

3.2. Active-set reduced-space method

Let us discuss some details about the active-set reduced-space method for a given non-linear system. Let $\mathcal{S} = \{1, 2, \dots, n\}$ be an index set with respect to a vector $X \in \mathbb{R}^n$. The lower and upper active sets for the system (5) are respectively defined as

$$\mathcal{A}_\xi(X) = \{i \in \mathcal{S} \mid X_i = \xi_i \text{ \& } \mathcal{F}_i(X) \geq 0\}$$

and

$$\mathcal{A}_\psi(X) = \{i \in \mathcal{S} \mid X_i = \psi_i \text{ \& } \mathcal{F}_i(X) \leq 0\}.$$

Then the inactive set used within the reduced-space method is denoted as

$$\mathcal{I}(X) = \mathcal{S} \setminus (\mathcal{A}_\xi(X) \cup \mathcal{A}_\psi(X)). \quad (9)$$

Actually, the active sets denote the variables where the lower and upper bounds are active and the function value can be ignored, i.e.,

$$X_i = \begin{cases} \xi_i, & \text{if } i \in \mathcal{A}_\xi(X), \\ \psi_i, & \text{if } i \in \mathcal{A}_\psi(X). \end{cases}$$

And the inactive set contains the remainder of the variables associated with a reduced linear system.

The class of active-set reduced-space methods for solving (5) can be described as follows. Starting from an initial guess $X^0 \in \mathbb{R}^n$, suppose X^k is the current approximate solution, then a new approximate solution X^{k+1} can be computed through

$$X^{k+1} = \pi [X^k + \alpha^k d^k], \quad (10)$$

where the operator π is a projection onto the variable bounds $[\xi, \psi]$, d^k is a search direction, and α^k is the step length for line search. To be more precise, at every iteration of the reduced-space method, the search direction d^k is calculated by approximately solving the linear system equation

$$(\nabla \mathcal{F}(X^k))_{\mathcal{I}^k, \mathcal{I}^k} d_{\mathcal{I}^k} = -\mathcal{F}_{\mathcal{I}^k}(X^k), \quad (11)$$

and setting $d_{\mathcal{A}_\xi^k}$ and $d_{\mathcal{A}_\psi^k}$ to 0. The step size α^k is chosen to be the optimal value for residual reduction with the constraint of

$$\|\mathcal{F}_\Theta(\pi [X^k + \alpha^k d^k])\|_2 \leq (1 - \sigma \alpha^k) \|F_\Theta(X^k)\|_2, \quad (12)$$

where σ is employed to assure that the reduction of $\|F(X^k)\|_2$ is sufficient, and $F_\Theta(X)$ is defined component-wise as

$$[\mathcal{F}_\Theta(X)]_i = \begin{cases} \mathcal{F}_i(X), & \text{if } \xi_i < X_i < \psi_i, \\ \min\{\mathcal{F}_i(X), 0\}, & \text{others.} \end{cases} \quad (13)$$

We continue the nonlinear iteration until the following convergence criterion is satisfied

$$\|\mathcal{F}_\Theta(X^{k+1})\| \leq \max\{\varepsilon_r \|\mathcal{F}_\Theta(X^0)\|, \varepsilon_a\},$$

where ε_r (ε_a) is the relative (absolute) solver tolerance for the nonlinear iteration.

3.3. Schwarz preconditioner

The Krylov subspace iterative methods, such as the Generalized Minimal RESidual (GMRES) method [7, 32, 34], are usually employed as linear solvers for the solution of the corresponding linear system. However, these solvers do not yield a good convergence rate without a proper preconditioner. In large-scale parallel computing, the additive Schwarz (AS) preconditioner is the key to the success of the linear solver, since it can help to improve the convergence and meanwhile is beneficial to the scalability of the linear solver. The additive Schwarz method was first introduced for solving symmetric positive definite linear systems, and then extended to many other nonsymmetric systems [16, 35]. We assume that the computational domain Ω is covered by a non-overlapping and an overlapping partition as in ([8]). Let R_i^δ and R_i^0 be the restriction operator from Ω to its overlapping and non-overlapping subdomains with δ being the size of the overlap, respectively. Then the classical additive Schwarz preconditioner is defined as

$$M_{\delta,\delta}^{-1} = \sum_{i=1}^{N_p} (R_i^\delta)^T J_i^{-1} R_i^\delta, \quad (14)$$

with $J_i = R_i^\delta J (R_i^\delta)^T$, and N_p is the number of subdomains which is the same as the number of processors. In (14), neither a direct nor an approximate inverse of J_i is explicitly calculated in solving subdomain problems. Instead, only the matrix-vector multiplications of J_i^{-1} are required, which are obtained either directly by using a sparse LU factorization or approximately by using a sparse incomplete LU (ILU) factorization.

The restricted additive Schwarz (RAS) method, proposed by Cai and Sarkis in [8], is a modification of the additive Schwarz method to save communication and obtain better convergence, which greatly expands the scope of application of the domain decomposition algorithm. In fact, there are two modified approaches of the additive Schwarz preconditioner that may have some potential advantages for parallel computing. The first version is the left restricted additive Schwarz (left-RAS, [8]) method defined by

$$M_{0,\delta}^{-1} = \sum_{i=1}^{N_p} (R_i^0)^T J_i^{-1} R_i^\delta, \quad (15)$$

and the other modification to the original method is the right restricted additive Schwarz (right-RAS, [6]) preconditioner as follow:

$$M_{\delta,0}^{-1} = \sum_{i=1}^{N_p} (R_i^\delta)^T J_i^{-1} R_i^0. \quad (16)$$

The RAS preconditioners (15) and (16) can save half of the communication cost, since no data exchange with the neighbouring processors is involved after the subdomain problems are solved. Moreover, the RAS preconditioner has better performance with fewer iterations and less computation time. We remark that when the overlap is zero, all the Schwarz preconditioners degenerate to the block-Jacobi preconditioner. As a result, the performance of the three AS preconditioners is the same when $\delta = 0$. Some numerical results will be shown later to compare the performances of these Schwarz preconditioners, to provide more understanding of the Schwarz method for proposed nonlinear constraint-preserving solver.

4. Numerical experiments

In this section, we investigate the numerical behavior and parallel performance of the fully implicit constraint-preserving simulator for the black oil model. We begin with several numerical tests to validate the effectiveness of the proposed algorithm to naturally satisfy the basic boundedness requirement, and then investigate different performance-related parameters in the proposed simulator to obtain the best performance. We implement the algorithms studied in this paper using the open-source Portable, Extensible Toolkit for Scientific computation (PETSc) library [1]. The numerical tests are carried out on the Tianhe-2 supercomputer. The computing nodes of Tianhe-2 are interconnected via a proprietary high performance network, and there are two 12-core Intel Ivy Bridge Xeon CPUs and 24GB local memory in each node. The Intel Xeon Phi coprocessors are not utilized in our tests. In the numerical experiments, we use all 24 CPU cores in each node and assign one subdomain to each core.

In this paper, we conduct the numerical experiments based on the one-dimensional (1D), two-dimensional (2D) and three-dimensional (3D) test cases. In the 2D and 3D test cases, we use the following parameters if it is not specifically stated. The absolute permeability tensor is defined as $\mathbf{K} = \kappa \mathbf{I}$, where $\mathbf{I}$ is the identity matrix and κ is a positive real number. The saturations are normalized as follows:

$$S_{no} = \frac{S_o - S_{ro}}{1 - S_{rw} - S_{ro}}, \quad S_{nw} = \frac{S_w - S_{rw}}{1 - S_{rw} - S_{ro}}, \quad S_{ng} = \frac{S_g}{1 - S_{rw} - S_{ro}},$$

with S_{rw} being the irreducible water saturation and S_{ro} being the irreducible oil saturation. The relative permeability functions for the water and the gas phase are given by the following analytical expressions:

$$\kappa_{rw} = S_{nw}^2, \quad \kappa_{rg} = S_{ng}^2,$$

and the relative permeability of oil κ_{ro} is defined by Stone's model II [11]. Capillary pressure functions are neglected. The porosity of the medium $\phi = 0.2$, the viscosities $\mu_w = 0.3$ cP, $\mu_o = 3.0$ cP and $\mu_g = 0.1$ cP.

Fluid properties are adopted from the Ninth Society of Petroleum Engineers (SPE) Comparative Solution Project [26]. The density of oil, water and gas at standard condition ($P_{sc} = 1.0135$ bar and $T_{sc} = 288.7056$ K) is 720.6, 1009.5 and 1.0451 kg/m³, respectively. Table 1 shows values for PVT properties of the oil and gas phases in SI units, including solution gas-oil ratio R_s (Sm³/Sm³), formation volume factor B_o and B_g (Rm³/Sm³) and viscosity μ_o and μ_g (cP). We denote by Rm³/Sm³ the ratio of the reservoir cubic meter to the standard cubic meter. The gas formation volume factor is calculated using the formula

$$B_g = Z_g \frac{P_{sc}}{T_{sc}} \frac{T}{P_{sat}},$$

where Z_g is the gas compressibility factor, T is the reservoir temperature and P_{sat} is the saturation pressure. Considering the water formation volume factor is insensitive to pressure, B_w is assumed to be constant and equal to 1.0034 Rm³/Sm³.

Table 1: Pressure-volume-temperature (PVT) properties for the oil and gas phases.

P_{sat}	R_s	B_o	B_g
275.7904	267.1606	1.1200	0.0033
248.2114	247.5688	1.1100	0.0036
220.6323	226.1960	1.0985	0.0042
193.0533	201.2610	1.0870	0.0047
165.4742	175.4355	1.0750	0.0055
137.8952	147.4727	1.0630	0.0066
110.3162	118.4412	1.0510	0.0083
82.7371	89.0535	1.0380	0.0110
55.1581	59.6659	1.0255	0.0166
27.5790	29.3877	1.0120	0.0331
1.0135	0	1.0000	1.0769

The void space of medium is initially fully saturated with oil and gas (i.e., the initial conditions are $S_w = 0$, $S_o = 0.9$ and $S_g = 0.1$). In the computation, we assume the minimal value of saturations is 10^{-4} , which is used to replace the lower bound 0 in (4). Moreover, the lower-bound for the pressure is the bottomhole pressure at the production well, which is equal to an atmospheric pressure, and there is no restriction to the upper bound that denotes the bottomhole pressure at the injection well, i.e, $p_\xi = 1.0135$ bar and $p_\psi = +\infty$ in (4). In the following, we first validate the discretization scheme and then compare the performance of the proposed fully implicit solver with the use of the classical inexact Newton method with backtracking (INB, i.e., without the variational inequality) and the active-set reduced-space (ASRS, i.e., with variational inequality) method, as introduced in Section 3.

4.1. Validation by several test cases

In this subsection, we validate the discretization scheme and the fully implicit solver by running several test cases.

4.1.1. Validation in 1D test case

To validate the numerical model, we first present a 1D test case (Case-1) within the saturated region. A homogeneous reservoir with constant permeability is considered in this example. The length of the computational domain is 1000 ft and the absolute permeability is 100 millidarcies (md). In this example, gravity and capillarity are ignored. The injector and producer are placed at two ends of the reservoir (left and right respectively), whose flow rates are controlled by the bottomhole pressure. The pressure-volume properties, such as formation volume factors (B_o , B_g and B_w), solubility ratios (R_o , R_g and R_w), viscosities and porosity, as well as the relative permeability, can be found in [36]. In Figure 1, we show the saturation profiles of all three phases. The problem is solved on a 400 uniform mesh and the simulation ends at $t = 150$ days with the time step $\Delta t = 10^{-1}$ day. We compare our simulation result, shown as colour curves, to the result published by the reference [36].

The red curve represents the gas saturation and the blue curve is obtained by adding the oil and gas saturation together. Thus, the oil saturation is the difference value between the lower curve and the upper curve, and the water saturation is the value of the upper curve subtracted from one. It is clear the solid line in blue is consistent with the published results in [36] represented by upwards triangles. The dash line in red fits their results represented by downward triangles very well.

Figure 1: Saturation profiles by using ASRS for the 1D test case. In the figure, the upward triangles and downward triangles are data points digitized from the published results in [36].

As introduced in Section 3, the variational inequality of the black oil model is applied to make sure flow entirely within the saturated region. Table 2 presents the history of the saturation values at the first time step when the variational inequality is used or not. We have observed that the oil and gas saturations can be outside of the physically meaningful range by using the classical INB method. By the variational inequality approach, the evolution of the black oil model is successfully resolved and the boundedness requirement is never violated. Our result shows that the boundedness of oil and gas saturation is violated from the fifth iteration at the given cell due to the lack of the variational inequality.

4.1.2. Validation in 2D test cases

In this section, we present results from 2D test cases in a horizontal layer with heterogeneous permeabilities. With the media being horizontal, the effect of gravity is neglected. In the first test case (Case-2), we considered a $300\text{ m} \times 150\text{ m}$ domain discretized with a rectangular 100×50 mesh. We inject water from the left-hand side of the domain and the production end is the right-hand side. The injection rate is 0.1 PV/year and $S_{rw} = S_{ro} = 0$,

Table 2: The historical values of the pressure and the saturations by using ASRS and INB at the position $x = 23.75$ for the 1D test case.

Method	Iter	p	S_o	S_g	S_w
ASRS	0	124.1057	0.8922	0.0579	0.0499
	2	132.1381	0.9296	0.0205	0.0500
	4	135.0339	0.9410	0.0091	0.0499
	6	135.0546	0.9402	0.0094	0.0504
	8	135.0615	0.9402	0.0094	0.0504
INB	0	124.1057	0.8922	0.0579	0.0499
	2	135.5924	0.9459	0.0041	0.0500
	4	134.4133	0.9402	0.0094	0.0505
	6	136.4335	1.0048	-0.0538	0.0490
	8	135.9785	0.8702	0.0768	0.0530
	20	135.7578	1.6713	-0.7236	0.0524
	30	135.9785	1.6126	-0.6662	0.0537

and the domain is composed of layers of alternate permeabilities (1 md and 100 md), i.e.,

$$\kappa = \begin{cases} 1 \text{ md} & 50 \leq y \leq 100, \\ 100 \text{ md} & \text{otherwise,} \end{cases}$$

where (x, y) represents the position in the two-dimensional Cartesian coordinate system.

Figure 2 shows the plots of the pressure profile and the saturation distributions for the variational inequality of the black oil model. The simulation ends at $t = 0.6$ year with the time step size $\Delta t = 10^{-2}$ year. As expected, the flow behavior is influenced by the low permeability field, which causes the delay of displacement in the middle layer. We observe that the variational inequality approach successfully resolves the evolution of the black oil model, and do not violate the boundedness requirement of the solution. We would also like to point out that the undershoot of the solution is present by using the INB algorithm. Tables 3 and 4 gives the history of the values of the pressure and the saturations at the first time step when the variational inequality is used or not. Results clearly show the often appearance of overshoots and undershoots from the numerical solution in the absence of the variational inequality. As a result, the nonlinear residual norm of the classical Newton method stagnates around 10 without any progress after many nonlinear iterations and then it fails to converge, as shown in Figure 3.

In the second configuration (Case-3) of this section, we use a random distribution of permeabilities in the domain as shown in Figure 4. In this test case, the domain is $300 \text{ m} \times 300 \text{ m}$ and the random distribution of permeability belongs to the range $[0.0013, 400.8484]$, which are generated by the open source code MRST (MATLAB Reservoir Simulation Toolbox) [25]. Water is uniformly injected from the left hand side of the domain with the injection rate of 10^{-2} PV/year, while oil and gas are produced from the production end in the opposite side. Figure 4 displays the distributions for the pressure and the saturations by using

Figure 2: Pressure and the saturation profiles by using ASRS for Case-2. The problem is solved on a 100×50 mesh, and the time step size is fixed to $\Delta t = 10^{-2}$ year.

Table 3: The historical values of the pressure and the saturations by using ASRS and INB at the position $(x, y) = (292.5, 30.5)$ for Case-2.

Method	Iter	p	S_w	S_g	S_o
ASRS	0	5.000000	1.000000e-4	1.000000e-1	8.999000e-1
	1	2.286465	1.000000e-4	5.084369e-1	4.914631e-1
	3	2.024178	1.000000e-4	3.221537e-1	6.777463e-1
	5	1.546437	1.000000e-4	3.482839e-1	6.516161e-1
	7	1.544104	1.000000e-4	3.486468e-1	6.512532e-1
INB	0	5.000000	1.000000e-4	1.000000e-1	8.999000e-1
	1	2.286470	9.995014e-5	5.084362e-1	4.914639e-1
	3	1.643349	9.991064e-5	3.482845e-1	6.516156e-1
	5	1.396322	9.970734e-5	1.166381	-1.664804e-1
	7	1.376299	9.966899e-5	1.142625	-1.427247e-1
	30	1.452989	1.004199e-4	1.845371	-8.454715e-1
	50	1.454045	1.004345e-4	1.884514	-8.846148e-1

Table 4: The historical values of the pressure and the saturations by using ASRS and INB at the position $(x, y) = (298.5, 148.5)$ for Case-2.

Method	Iter	p	S_w	S_g	S_o
ASRS	0	5.000000	1.000000e-4	1.000000e-1	8.999000e-1
	1	2.117311	1.000000e-4	5.338976e-1	4.660024e-1
	3	1.561308	1.000039e-4	3.745666e-1	6.254234e-1
	5	1.344389	1.000000e-4	3.711413e-1	6.287588e-1
	7	1.342113	1.000000e-4	3.722036e-1	6.276964e-1
INB	0	5.000000	1.000000e-4	1.000000e-1	8.999000e-1
	1	2.117316	9.994703e-5	5.338969e-1	4.660032e-1
	3	1.397354	1.007621e-4	-2.114390e-1	1.211338
	5	1.211443	1.017103e-4	-6.304079e-1	1.630306
	7	1.201559	1.017893e-4	-7.175632e-1	1.717461
	30	1.193921	1.020302e-4	-1.112505	2.112403
	50	1.001772	1.041566e-4	-2.549002	3.548897

the variational inequality approach. The simulation is conducted on a 100×100 mesh and is ended after 200 time steps with the time step size $\Delta t = 10^{-2}$ year. Results clearly show the successful removal of overshoots and undershoots from the numerical solution when the variational inequality method is applied for the test case with random distributed permeabilities.

The Tenth SPE Comparative Project [13] is a classic and challenging benchmark problem in reservoir simulation, due to the highly heterogeneous permeability and porosity. In the third configuration (Case-4), we import porosity and permeability data from the 85th layer

Figure 3: The convergence history plots of the nonlinear residual norms by using ASRS and INB at the first time step for Case-2.

of the model, whose permeability ranges from 6.65×10^{-4} md to 2×10^4 md and the value of the porosity belongs to $[0, 0.5]$, as shown in Figure 6. Its dimensions are 2200 ft $\times$ 1200 ft. In the test, the injection well is located at the center of the domain with the injection rate of 10^{-2} PV/year, and production wells are located at the coordinates (0, 0), (2200 ft, 0), (0, 1200 ft) and (2200 ft, 1200 ft), respectively. In Figure 7, we display the distributions for the pressure and the saturations when the variational inequality method is used. The simulation is finished after 200 time steps on a 220×60 mesh and with the time step size $\Delta t = 10^{-3}$ year. Again, the numerical conservation of the ASRS method is also satisfactory, since the violation the boundedness requirement does not appear.

4.1.3. Validation in 3D test cases

The experiment is conducted to simulate the black oil model in three dimensional space. In the first test case (Case-5) of this section, we consider a 3D domain with dimension (300 m $\times$ 150 m $\times$ 150 m), which can be regarded as a 3D version of Case-2. The permeabilities are defined by

$$\kappa = \begin{cases} 100 \text{ md} & 50 \leq x \leq 150 \text{ and } 50 \leq y \leq 100 \text{ and } 50 \leq z \leq 100, \\ 1 \text{ md} & \text{otherwise,} \end{cases}$$

where (x, y, z) represents the position in the three-dimensional Cartesian coordinate system. The injection and production wells are located at the left and right sides, respectively. A uniform mesh comprised of $100 \times 50 \times 50$ elements is used, and the time step size is fixed to $\Delta t = 10^{-2}$ year. In Table 5, we present the history of the values of the pressure and the saturations at the first time step under the influence of the fully implicit constraint-preserving simulator. Figure 8 shows the convergence history plots of the nonlinear residual norms at the first time step. As introduced in Section 3, the price to pay for using implicit

Figure 4: Permeability field for Case-3. We use the logarithmic scale for the permeability, i.e., $\log(\kappa)$ where κ has a unit of md.

methods for the fully coupled problem is to solve a nonlinear system at each time step. When solving such nonlinear systems by the classical Newton method, however, we face the numerical challenges arising from the the predicted pressure and saturations can not frequently sit within physically meaningful range. As a result, for the classical Newton method, the value of the nonlinear residual function stagnates around 5×10^{-3} without any improvement after dozens of line search, even for tiny values of step length with $\alpha = 10^{-15}$. On the other hand, with the help of the variational inequality, it is clear that the proposed simulator converges much easier and can reach convergence with five nonlinear iterations for this test case.

Table 5: The historical values of the pressure and the saturations by using ASRS and INB at the position $(x, y, z) = (52.5, 52.5, 97.5)$.

Method	Iter	p	S_w	S_g	S_o
ASRS	0	1.065000e+1	1.000000e-4	1.000000e-1	8.999000e-1
	1	1.132668e+1	1.000415e-4	1.000000e-4	9.988000e-1
	3	1.143278e+1	1.000563e-4	2.133463e-2	9.785653e-1
	5	1.143230e+1	1.000568e-4	1.940666e-2	9.804933e-1
	5	1.143230e+1	1.000568e-4	1.940666e-2	9.804933e-1
INB	0	1.065000e+1	1.000000e-4	1.000000e-1	8.999000e-1
	1	1.132668e+1	1.000418e-4	-8.877602e-2	1.088676
	3	1.141187e+1	1.000743e-4	-5.933170e-2	1.059232
	5	1.140931e+1	1.000730e-4	-5.808857e-2	1.057988
	5	1.140931e+1	1.000730e-4	-5.808857e-2	1.057988

Finally, we test the proposed constraint-preserving simulator with a more difficult test case, in which the permeabilities of the porous medium are random. In this test case (Case-6), we consider a 3D domain with a $300 \text{ m} \times 150 \text{ m} \times 150 \text{ m}$ dimension and a random

(a) $p \geq p_\xi$ (b) $0 \leq S_w \leq 1$ (c) $0 \leq S_g \leq 1$ (d) $0 \leq S_o \leq 1$

Figure 5: Pressure and the saturation profiles by using ASRS for Case-3. The problem is solved on a 100×100 mesh, and the time step size is fixed to $\Delta t = 10^{-2}$ year.

Figure 6: Porosity and permeability for Case-4. We use the logarithmic scale for the permeability, i.e., $\log(\kappa)$ where κ has a unit of md.

distribution of permeability with the range $[0.0013, 161.6361]$, which are also generated by the open source code MRST. The injection well is located at the center of the domain and production wells are located at the four corners. The simulation is ended after 150 time steps on a $100 \times 50 \times 50$ mesh and with the time step size $\Delta t = 10^{-3}$ year. The plots of the solution profiles by using the variational inequality approach are shown in Figure 9. We can see that the proposed approach successfully resolves the rapid and abrupt evolution of the solution while keeping the pressure and saturations within the physically meaningful range.

4.2. Performance of the constraint-preserving simulator

In this section, we focus on the parallel performance of the proposed fully implicit constraint-preserving simulator by using Case-5. In the fully implicit solver, we set the tolerances as follows. For the nonlinear iteration, an absolute (relative) tolerance of 10^{-8} (10^{-6}) is utilized. For the linear iteration, the linear systems are solved by the Schwarz preconditioned GMRES method with absolute and relative tolerances of 10^{-6} and 10^{-4} , respectively.

4.2.1. Performance tuning

An important feature of the proposed method is the selection of linear preconditioner for the solution of linear system. We investigate the performance of the fully implicit solver when different types of the additive Schwarz preconditioners are employed and when different overlaps are taken. Here, we test the classical-AS (14), the left-RAS (15), and the right-RAS (16) preconditioners. For each numerical case, the overlapping size δ in additive Schwarz methods is gradually increased from 0 to 3. In the test, the mesh is $100 \times 50 \times 50$, the time step size is fixed to $\Delta t = 10^{-3}$ year. The simulation is stopped after 30 time steps. The numbers of nonlinear and linear iterations together with the execution time are listed in Table 6. To check the influence of the subdomain solvers, we limit the test to the right AS preconditioner

(a) $p \geq p_\xi$ (b) $0 \leq S_w \leq 1$ (c) $0 \leq S_g \leq 1$ (d) $0 \leq S_o \leq 1$

Figure 7: Pressure and the saturation profiles by using ASRS for Case-4. The problem is solved on a 220×60 mesh, and the time step size is fixed to $\Delta t = 10^{-3}$ year.

Figure 8: The convergence history plots of the nonlinear residual norms by using ASRS and INB at the first time step for Case-5.

and fix the overlapping size to $\delta = 1$. The ILU factorizations with different levels of fill-in and LU factorizations are considered. The numbers of nonlinear and linear iterations together with the execution time are provided in Table 7. We can obtain the observations from the results. The left-RAS and right-RAS preconditioner are superior to the classical-AS preconditioner. By increasing the overlapping size or by increasing the fill-in level, the number of linear iterations becomes smaller, but the computing time does not necessarily reduce because of the increased cost of communication or subdomain solver. In particular, when LU instead of ILU is used as the subdomain solver, the linear system converges with fewer iterations but the computing time is larger because LU is more expensive than ILU. We observe that the total number of nonlinear iterations is insensitive to the overlapping size and subdomain solver. It is clear that from Tables 6 and 7, we find that the optimal choice in terms of the execution time is ILU(1) subdomain solver with $\delta = 1$.

4.2.2. Parallel scalability

To study the parallel scalability, we again run Case-5 with different number of processor cores. The strong scalability *Speedup* and the parallel efficiency E_f , which are based on the computational time on the smallest processors number (N_{min}), are respectively defined as follows:

$$Speedup = \frac{T(N_{min})}{T(N_p)}, \quad E_f = \frac{T(N_{min}) \times N_{min}}{T(N_p) \times N_p},$$

where $T(N_p)$ denotes computational time on N_p processors. The simulation is stopped after five implicit time steps by using a fixed mesh $256 \times 128 \times 128$ and also a fixed time step size $\Delta t = 10^{-5}$ year. In the experiment, we use the right-RAS preconditioner with the overlapping size $\delta = 1$, and employ the sparse ILU(1) or LU factorization as the subdomain solver for the purpose of comparison. Table 8 shows the number of nonlinear and linear

Figure 9: Pressure and the saturation profiles by using ASRS for Case-6. The problem is solved on a $100 \times 50 \times 50$ mesh, and the time step size is fixed to $\Delta t = 10^{-3}$ year.

Table 6: Performance of ASRS with different types of the Schwarz preconditioners and different overlapping factors δ . The number of processors is $N_p = 144$.

Preconditioner	Overlapping size δ	0	1	2	3
Classical-AS	Average nonlinear iteration	3.3	3.3	3.3	3.3
	Average linear iteration	25.3	24.5	20.7	18.3
	Execution time (second)	209.8	171.1	291.7	454.3
Left-RAS	Average nonlinear iteration	3.3	3.3	3.3	3.3
	Average linear iteration	25.3	14.3	10.6	8.9
	Execution time (second)	209.8	160.4	270.8	432.4
Right-RAS	Average nonlinear iteration	3.3	3.3	3.3	3.3
	Average linear iteration	25.3	14.1	10.4	8.6
	Execution time (second)	209.8	159.4	271.1	427.3

Table 7: Performance of ASRS with different subdomain solvers. The number of processors is $N_p = 144$.

Subdomain solver	ILU(0)	ILU(1)	ILU(2)	ILU(3)	ILU(4)	LU
Average nonlinear iteration	3.3	3.3	3.3	3.3	3.3	3.3
Average linear iteration	26.9	19.4	16.7	15.3	14.6	14.1
Execution time (second)	59.9	57.6	63.3	78.9	110.1	159.4

iterations, the computing time, as well as the parallel scalability with respect to the number of processors. Below we list the observations made from the results.

- (1) We notice that the number of nonlinear iterations per time step is independent of the number of processors and subdomain solvers, and the the average number of linear iterations are almost unchanged during the increase of the number of processors, which implies that the number of nonlinear iterations and the average number of linear iterations are insensitive to the number of processors. In addition to that, the execution time decreases as the number of processor cores increases.
- (2) The effectiveness of the Schwarz preconditioner relies on its ability to mimic the spectrum of the linear operator and at the same time is relatively cheap to apply. We see that using the LU factorization in the construction of the Schwarz preconditioner provides less linear iteration counts and better parallel scalabilities. However, when compared with the Schwarz method with LU, the ILU approach is more attractive in the terms of the execution time, owing to its lower bandwidth and a less number of nonzeros in the sparse matrix. As a result, the tests are not carried out for the LU approach because of the lack of memory, when the number of processors is relatively small such as $N_p = 64$ and $N_p = 128$.

Table 8: Performance results with different number of processors N_p . The symbol “—” denotes that the tests are not carried out because of the lack of memory.

Subdomain solver	Number of processors	64	128	256	512	1024
ILU(1)	Average nonlinear iteration	2.2	2.2	2.2	2.2	2.2
	Average linear iteration	7.6	7.6	7.7	7.8	7.9
	Execution time (second)	213.2	109.2	61.1	40.3	29.7
	Speedup	1.00	1.95	3.49	5.32	7.18
	Efficiency	100.0%	97.5%	87.3%	66.5%	44.9%
LU	Average nonlinear iteration	—	—	2.2	2.2	2.2
	Average linear iteration	—	—	6.3	6.3	6.2
	Execution time (second)	—	—	694.9	229.5	104.7
	Speedup	—	—	1.00	3.03	6.64
	Efficiency	—	—	100.0%	151.3%	165.9%

5. Conclusions

In this paper, we present a fully implicit and constraint-preserving simulator for the solution of the black oil model. In the implementation, we adopt a variational inequality formulation to preserve the predicted solution to stay within the physical range. A fully implicit temporal scheme and a finite volume method in space, as a unconditionally stable discretization scheme of the black oil model, are introduced to couple the to-be-computed primary variables in a monolithic way. Additional challenging effects arising from nonlinear and linear fast solver technologies for the proposed parallel simulator are presented for the assurance of our simulator high efficiency. Several examples in one, two and three dimensions, consisting of highly heterogeneous permeabilities, and a well known benchmark problem, are provided in order to verify and demonstrate the performance of the algorithm.

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Declaration of interests

☒ The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

☐ The authors declare the following financial interests/personal relationships which may be considered as potential competing interests:

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