Prestack wavefield approximations

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SUMMARY

The double-square-root (DSR) relation offers a platform to perform prestack imaging using an extended single wavefield that honors the geometrical configuration between sources, receivers and the image point, or in other words, prestack wavefields. Extrapolating such wavefields, nevertheless, suffers from limitations chief among them is the singularity associated with horizontally propagating waves. I introduce approximations free of such singularities, and are highly accurate. Specifically, I use Pade expansions with denominators given by a power series that is an order lower than that of the numerator, and thus, introduce a free variable coefficient to balance the series order and normalize the singularity. For the higher order Pade approximation the errors are negligible. Additional simplifications, like recasting the DSR formula as a function of scattering angle, allow for a singularity free form that is useful for constant angle gather imaging. A dynamic form of this DSR formula can be supported by kinematic evaluations of the scattering angle to provide efficient prestack wavefield construction.

INTRODUCTION

The typical layout and design of seismic surveys allow for high redundancy in image representation conveniently described by the image point illumination, and this redundancy implies data dependency resulting in measurement connection that exists in the data space. Wave-extrapolation procedures with the survey-sinking or double-square-root (DSR) formulation of the wave equation (Claerbout, 1985; Popovici, 1996; de Hoop et al., 2003) provides a straightforward connection between the image and the data, free of the cross-correlation imaging step. A limitation of the DSR formulation is the one-way nature of wave extrapolation, which limits the imaging accuracy at large structural dips. An alternative is to extend the survey-sinking approach to extrapolation in time rather than depth in the full source and receiver wavefield (Alkhalifah and Fomel, 2010). Application of two-way extrapolators to modeling and migration follows the *exploding reflector* concept (Loewenthal et al., 1976; Claerbout, 1985) and allows for upgoing as well as downgoing wavefields. However, a major hinder in time extrapolating the DSR equation is an inherent singularity for horizontally traveling waves (Biondi, 2002; Duchkov and de Hoop, 2009; Alkhalifah and Fomel, 2010). Specifically, the singularities in this extended domain wave formulation prevents us from conventionally solving the dynamic form of the DSR equation in the space domain using, the usually more ef-

ficient for inhomogeneous media, finite difference methods.

In this abstract, I investigate this singularity and explore ways to circumvent it. I develop approximations of the DSR formula that are free of such singularities, and explore the prospect of solving these approximations in this extended domain by relaxing the accuracy requirements in the offset direction. This includes utilizing the geometrical description of waves (traveltimes and rays) to constrain the offset part of the solution.

THEORY

Consider a seismic survey $P(t, \mathbf{s}, \mathbf{r}, z)$ as a function of time t and source and receiver locations $\mathbf{s}$ and $\mathbf{r}$ at the surface at depth z . Our goal is to extrapolate the four-dimensional (six-dimensional in 3-D) wavefield $P(t, \mathbf{s}, \mathbf{r}, z)$ in time. Let $\mathbf{x}$ represent the space coordinates $\mathbf{x} = \{s, r, z\}$. The wave extrapolation operator, valid for small Δt , is

$$P(t + \Delta t, \mathbf{x}) + P(t - \Delta t, \mathbf{x}) \approx 2 \int \hat{P}(t, \mathbf{k}) \cos[\phi_0(\mathbf{x}, \mathbf{k}) \Delta t] e^{i \mathbf{k} \cdot \mathbf{x}} d\mathbf{k}, \quad (1)$$

where $\hat{P}(t, \mathbf{k})$ is the wavefield in the wavenumber domain given by $\mathbf{k} = \{k_s, k_r, k_z\}$. In the geometrical (high-frequency) approximation, the function $\phi_0(\mathbf{x}, \mathbf{k}, t)$ appearing in equation (1) should satisfy the appropriate eikonal equation, which is, in the case of prestack data, the double-square-root (DSR) equation. For isotropic media, it has the following form (Alkhalifah and Fomel, 2010)

$$\phi_0^2 = \frac{v_r^2 v_s^2 [(k_s^2 + k_r^2 + k_z^2)^2 - 4 k_r^2 k_s^2]}{v_+^2 k_z^2 - v_-^2 (k_s^2 - k_r^2) + 2D}, \quad (2)$$

where

$$D = \sqrt{k_z^2 [v_r^2 v_s^2 (k_s^2 + k_r^2 + k_z^2) - v_r^4 k_r^2 - v_s^4 k_s^2]}, \quad (3)$$

and we use the shortened notation $v_\xi = v(\xi, z)$, for $\xi = \{s, r\}$, $v_+^2 = v_s^2 + v_r^2$, $v_-^2 = v_s^2 - v_r^2$.

In the $v(z)$ case, $v_r = v_s = v$, and equation (2) reduces to

$$\phi_0^2 = \frac{v^2 [(k_s + k_r)^2 + k_z^2] [(k_s - k_r)^2 + k_z^2]}{4 k_z^2}, \quad (4)$$

which clearly includes the k_z singularity.

The Hamiltonian in equation (2) reduces to our familiar eikonal for waves emanating from a source when we set $k_s = k_r$ and $v = v_s = v_r$. Specifically, defining the half-offset slowness, $k_h = k_r - k_s$ and the midpoint slowness

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$k_x = k_r + k_s$, equation (2) reduces to $k_x^2 + k_z^2 = \frac{1}{v^2}$ when $k_h = 0$ and $v_- = 0$, which is free of singularities. Naturally, to construct a stable solution, we will need to perturb from this stable zero-offset solution.

THE $V(Z)$ ASSUMPTION

The laterally homogeneous based phase operator, equation 4, has only one singularity given by the $k_z = 0$, which corresponds to horizontally traveling waves. Despite that it is based on considering $v_s = v_r$, we can easily allow the velocity in the equation to vary laterally. This corresponds to an approximation in the velocity treatment for mainly the offset component. For many situations, this approximation is reasonable, especially for short offsets. Popovici (1996) demonstrated that such an approximation using a split-step implementation was good enough for the Marmousi data set. Thus, the $v(z)$ assumption considered here is only applied to the offset axis, as we consider the velocity for a particular offset is given by the midpoint velocity.

The $k_z = 0$ singularity is an artificial one caused by constraining the source and receiver wavefields to the same vertical wavenumber. The true solution for the phase operator for $k_z = 0$ is given by vk_h for $|k_h| > |k_x|$ and by vk_x for $|k_h| < |k_x|$. These represent the flanks of the pyramid in our offset-midpoint traveltime formulation, especially as the scatterer is near the surface ($k_z \rightarrow 0$). To enforce this solution, I first expand equation 4 as a function of k_h around its zero value using Pade approximation with a fourth-order numerator and second-order denominator, and obtain:

$$\phi_0 \approx \frac{v(8k_h^2k_z^2 + k_h^4 + 8k_z^4)k}{4k_z^2(k_h^2 + 2k_z^2)}, \quad (5)$$

where $k = \sqrt{k_x^2 + k_z^2}$. The uneven order of the Pade expansion allows us to add a k_h^4 term in the denominator with a coefficient that can be used to force the solution to approach the exact one for $k_z = 0$, or asymptotically with respect to the horizontal wavenumbers. Thus,

$$\phi_0 = \frac{v(8k_h^2k_z^2 + k_h^4 + 8k_z^4)k}{4k_z^2(k_h^2 + 2k_z^2) + Ak_h^4}. \quad (6)$$

If I set $k_z = 0$, then $A = 1$ yields the proper solution for horizontal reflectors for $k_h < k_x$ and $A = \frac{k_x}{k_h}$ for $k_h > k_x$.

A similar approximation can be obtained for a higher-order Pade expansion given by an eighth-order numerator and sixth-order denominator with adding Ak_h^8 term to the denominator to obtain

$$\phi_0 = \frac{v(32k_h^6k_z^2 + 160k_h^4k_z^4 + 256k_h^2k_z^6 + k_h^8 + 128k_z^8)k}{8k_z^2(10k_h^4k_z^2 + 24k_h^2k_z^4 + k_h^6 + 16k_z^6) + k_h^8}, \quad (7)$$

for $k_h < k_x$, and

$$\phi_0 = \frac{v(32k_h^6k_z^2 + 160k_h^4k_z^4 + 256k_h^2k_z^6 + k_h^8 + 128k_z^8)k}{8k_z^2(10k_h^4k_z^2 + 24k_h^2k_z^4 + k_h^6 + 16k_z^6) + k_h^7k_x}, \quad (8)$$

for $k_h > k_x$.

The difference between the two new phase extrapolation operators and the original $v(z)$ equation 4 are displayed in Figure 1. For a reasonable range of wavenumbers the difference is less than 1% for equation 6 and less than 0.01% for equations 7 and 8. Of course, for the large wavenumbers we are effectively approaching the singularity and the difference between the equations are due to the adjustment we added to normalize that artificial singularity.

Figure 1: The percent difference in the phase ϕ_0 between the original $v(z)$ equation 4 and the fourth-order Pade approximation 6 (top) and the 8th order Pade approximation 7 and 8 (bottom). The velocity is equal to 1 km/s and the vertical wavenumber, $k_z = 1 \text{ km}^{-1}$.

THE DSR-BASED PARTIAL DIFFERENTIAL EQUATION

To derive a partial differential equation capable of representing the behavior of waves in the extended source-receiver domain with kinematics described by the DSR equation, we cast the DSR relation (equation 2) in a polynomial form, set $\omega = \phi_0$ as follows,

$$\begin{aligned} & 2\omega^2 v_r^2 v_s^2 (k_r^2 (v_r - v_s) (v_r + v_s) - v_r^2 (k_s^2 + k_z^2) \\ & + v_s^2 (k_s - k_z) (k_s + k_z)) v_r^4 v_s^4 (2k_r^2 (k_z^2 - k_s^2) + k_r^4 + (k_s^2 + k_z^2)^2) \\ & + \omega^4 (v_r^2 - v_s^2)^2 = 0. \end{aligned} \quad (9)$$

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As a reminder, v_s and v_r are the source and receiver velocities, respectively, and can be represented as $v(s, z)$ and $v(r, z)$, since the velocity for the medium is a single function. Multiplying both sides of equation (9) by the wavefield in the Fourier domain, $F(k_x, k_y, k_z, \omega)$, as well as using inverse Fourier transform on k_x , k_y , and k_z ($k_x \rightarrow -i\frac{d}{dx}$, $k_y \rightarrow -i\frac{d}{dy}$, and $k_z \rightarrow -i\frac{d}{dz}$) as well as inverse Fourier transform on ω ($\omega \rightarrow i\frac{\partial}{\partial t}$), yields a wave equation in the space-time domain, given by

$$\begin{aligned} (v_r^2 - v_s^2) \frac{\partial^4 F}{\partial t^4} &= 2(v_r^4 v_s^2 - v_s^4 v_r^2) \left(\frac{\partial^4 F}{\partial t^2 \partial s^2} - \frac{\partial^4 F}{\partial t^2 \partial r^2} \right) \\ &+ 2(v_r^4 v_s^2 + v_s^4 v_r^2) \frac{\partial^4 F}{\partial t^2 \partial z^2} - v_r^4 v_s^4 \left(\frac{\partial^4 F}{\partial s^4} - 2 \frac{\partial^4 F}{\partial s^2 \partial r^2} \right. \\ &\quad \left. + \frac{\partial^4 F}{\partial r^4} + 2 \frac{\partial^4 F}{\partial s^2 \partial z^2} + 2 \frac{\partial^4 F}{\partial r^2 \partial z^2} + \frac{\partial^4 F}{\partial z^4} \right), \quad (10) \end{aligned}$$

where $v_s = v(s, z)$ and $v_r = v(r, z)$. Unlike in the conventional wave equation for 2-D media which is second order in time, equation (10) is fourth order in time, and thus can provide us up to four independent solutions. As we extrapolate the solution in time the equation has a regular singularity when the velocity for the source and receiver are equal as the equation reduces to second order in time.

SCATTERING ANGLE BASED FORMULATION

Considering that a convenient and useful representation of the offset axis is given by the scattering (reflection) angle, I transform the half-offset wavenumber in equation 9 to reflection angle using $k_h = \tan \theta k_z$. In addition, by substituting $v_s^2 = v_+^2 - v_-^2$ and $v_r^2 = v_+^2 + v_-^2$, we end up with the following

$$\begin{aligned} v_-^4 \phi_0^4 - 2(v_+^4 - v_-^4) \phi_0^2 [4v_-^2 k_x k_z \tan \theta + v_+^2 k_z^2] \\ + (v_+^4 - v_-^4)^2 [(2k_x^2 + 2k_z^2 \tan^2(\theta) + k_z^2)^2 \\ - 4(k_x^2 - k_z^2 \tan^2(\theta))] = 0. \quad (11) \end{aligned}$$

In this form, we are either considering θ to be constant, or we need to evaluate θ using other means. A constant θ along the wavefield might be valid for small offsets. A more appropriate representation is given by a constant k_h , which is valid for a vertically inhomogeneous medium approximation in the offset axis. Considering the relatively smaller scattering angles (limited by offset) that we deal with, compared with dips, this approximation is expected to work in many places. However, a representation in k_h , unlike that in θ , does not remove the k_z singularity in equation 9. Setting $v_- = 0$ in equation 11 yields the following easy form (Alkhalifah, 2012):

$$v_+^2 k_x^2 + v_+^2 k_z^2 = \cos(\theta)^2 \omega^2, \quad (12)$$

which is free of singularities. It introduces a $\cos \theta$ weighting function to the velocity, similar to what we observe in the NMO velocity for dipping reflectors (Levin, 1971).

It also provides a plane wave representation to the offset axis, which may be convenient. The dynamic form of this equation, obtained by introducing a wavefield in the Fourier domain followed by an inverse Fourier transform, is given by

$$\cos(\theta)^2 \frac{\partial^2 P}{\partial t^2} = v_+^2 \left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial z^2} \right). \quad (13)$$

Of course, this equation is a DSR formulation simplified by considering a $v(z)$ medium along the offset axis. Moreover, it includes a plane wave description of offset axis given by the scattering angle, θ . This equation can be used for prestack exploding reflector modeling (or imaging) applied to constant scattering angles. However, even in $v(z)$ media the scattering angle changes with depth (p_h is constant). Thus, we can utilize a $v(z)$ plane wave ray tracing along the offset axis to obtain an approximate $\theta(x, z)$ field per midpoint.

DIP ANGLE AS WELL

Likewise, the dip angle, ψ , satisfies $\tan \psi = \frac{p_x}{p_z}$ in the DSR formulation, and thus, equation 12 yields even an easier form (Alkhalifah, 2012):

$$v_+^2 k_z^2 = \cos(\theta)^2 \cos(\psi)^2 \omega^2. \quad (14)$$

Now, this additional $\cos \psi$ factor corresponds to dip. In homogeneous media, the normal moveout velocity is given by $\frac{v_+}{\cos(\psi)}$. A dynamic form of equation 14 is given by

$$\cos(\theta)^2 \cos(\psi)^2 \frac{\partial^2 P}{\partial t^2} = v_+^2 \frac{\partial^2 P}{\partial z^2}. \quad (15)$$

This one dimensional wave equation is valid for plane wave propagation in the dip and scattering angle. Layered media plane wave ray tracing can be used to obtain maps of $\theta(z, \psi)$ and $\psi(z, \theta)$ per location (image point). More exotic ray tracing can be employed to handle more complex media. The offset and midpoint information are embedded in the scattering and dip angles, respectively. Despite the ray tracing, we are still extrapolating prestack wavefields using equation 15, however, approximately.

THE IMPLEMENTATION

Marrying raytracing to wavefield extrapolation is always a challenge, however, the DSR formulation provides a suitable platform to do that as it explicitly provides the required geometrical information concerning the source-receiver-image point configuration. At first glance, the required ray tracing, per image point, seems exhaustive. However, phase escape based traveltime methods are made for such an implementation. They provide traveltime and angle information for all possible source location from every image point in one sweep so angle maps can be readily extracted from the solution.

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Figure 2: Snap shots of the prestack wavefield starting with time equal 0 (a) to time equal 4s (i) in an exploding reflector modeling application. The homogeneous medium has a velocity of 2km/s. The initial wavefield is given by a scattering angle decomposed image at zero time (the imaging condition). The solid lines cutting through each of the sections of the 3D represent the locations of the slices of the 3D volume.

Figure 3: A cube plot of the resulting prestack data at the surface using the proposed wave equation with snapshots shown in Figures 2(a)-2(d). The lines crossing each section of the cube represent the location of the corresponding slices.

I extrapolate prestack wavefields using equation 13 in a homogeneous medium to observe some the prestack wavefield features. Starting with an initial wavefield given by the image of the french model (French, 1974) in the angle domain (Figure 2(a)), I show snapshots of the wavefield at different times in Figures 2(b)-2(d). Clearly, the wavefields are moving upwards and downwards courtesy of the time extrapolation implementation. In larger angle slices the wave move faster as they experience higher effective velocity in the plane wave domain. The 2-D nature of our extrapolation operator given by equation 13 allows for separate angle section extrapolations, and a general reduction in the cost. The data recorded on the surface is given in Figure 3. Instead of the offset axis, the data is presented as a function of angle. A simple slant stack transform can take us to offset, which is the conventional product provided by our acquisition.

CONCLUSIONS

Combining raytracing and wavefield extrapolation is always a challenge, however, the DSR formulation provides a natural platform to do that as it explicitly provides geometrical information on the source-receiver-image point configuration. Thus, I develop DSR formulations that are capable of extrapolating wavefields in the prestack domain and are free of singularities. They are based on Pade approximations and reflection angle reformulations that are highly accurate and useful.

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EDITED REFERENCES

Note: This reference list is a copy-edited version of the reference list submitted by the author. Reference lists for the 2012 SEG Technical Program Expanded Abstracts have been copy edited so that references provided with the online metadata for each paper will achieve a high degree of linking to cited sources that appear on the Web.

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