A partial-low-rank method for solving acoustic wave equation

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SUMMARY

Numerical solutions of the acoustic wave equation, especially in anisotropic media, is crucial to seismic modeling, imaging and inversion as it provides efficient, practical, and stable approximate representation of the medium. However, a clean implementation (free of shear wave artifacts and dispersion) of wave propagation, especially in anisotropic media, requires an integral operator, the direct evaluation of which is extremely expensive. Recently, the low-rank method was proposed to provide a good approximation to the integral operator utilizing Fourier transforms. Thus, we propose to split the integral operator into two terms. The first term provides a differential operator that approximates that can be approximated with a standard finite-difference method. We, then, apply the low-rank approximation on the residual term of the finite-difference operator. We implement the two terms in two complementing steps, in which the spectral step corrects for any errors admitted by the finite difference step. Even though we utilize finite-difference approximations, the resulting algorithm admits spectral accuracy. Also, through the finite difference step, the method can deal approximately with the free surface and absorbing boundary conditions in a straight forward manner. Numerical examples show that the method is of high accuracy and efficiency.

INTRODUCTION

Wave extrapolation, especially in anisotropic media, is crucial to seismic modeling, imaging and inversion. The acoustic approximation is widely used in current industry imaging and inversion algorithms (Alkhalifah, 2000; Zhou et al., 2006; Fowler et al., 2010; Bakker and Duveneck, 2011; Zhang et al., 2011; Bube et al., 2012; Gholami et al., 2013; Alkhalifah and Plessix, 2014; Cheng et al., 2014b,a; Operto et al., 2015; da Silva et al., 2016).

The most popular method for solving acoustic wave equation is the finite-difference method. However, the finite-difference method often suffers from numerical dispersion and admits S -wave artifacts in the resulting wavefield. These artifacts are reduced when the source is located in an isotropic layer (Alkhalifah, 2000), but become unacceptable when the source is located in an anisotropic region or we have strong scattering acting as secondary sources (Wu and Alkhalifah, 2015; Alkhalifah and Wu, 2016; Wu and Alkhalifah, 2017). One family of high resolution algorithms for solving the anisotropic acoustic wave equation free of S -wave artifacts falls under the so-called spectral approach (Etgen and Brandsberg-Dahl, 2009; Du et al., 2010; Fomel et al., 2013; Fowler and Lapilli, 2012; Song et al., 2013; Song and Alkhalifah, 2013; Alkhalifah, 2014; Wu and Alkhalifah, 2014; Sun et al., 2016, 2015; Cheng et al., 2016; Sun et al., 2016). Numerically, they are reasonably accurate and admit no S -wave artifacts. Another group of efficient algorithms for computing pure quasi-P waves is the so-

called effective isotropic model approximation (Tariq Alkhalifah and Zuberi, 2013; Ibanez-Jacome et al., 2014; Waheed and Alkhalifah, 2015). Recently, and along the lines of the effective isotropic model approximation, Xu and Zhou (2014) proposed a new dynamic effective model formulation that decomposes the original pseudo-differential operator into two numerically solvable operators: a Laplacian operator and a scalar operator. To compensate for amplitude errors, Xu et al. (2015) proposed decomposing the original operator into elliptic anisotropic and anelliptic anisotropic components. Le et al. (2015) applied their method to full waveform inversion. To speed up the computation, Zhang et al. (2017) proposed to use the low rank approximation for the isotropic components of the resulting equation in Xu and Zhou (2014). To solve the anisotropic acoustic equation efficiently, Alkhalifah (2013) derived another formulation through perturbation theory. Meanwhile, Wu and Alkhalifah (2018) proposed a partially separable approximation to the anelliptic anisotropic component for the acoustic wave equation in VTI media described in the frequency domain.

In this abstract, we extend the idea in Wu and Alkhalifah (2018) to solve more general isotropic and anisotropic acoustic wave equations. Basically, we divide the integral operator into two steps. Firstly, we approximate the integral operator for anisotropic acoustic wave equation by a differential operator. The standard finite difference method can be utilized to approximate the relative differential operator. Secondly, we apply the low-rank method for the remaining components to correct the residual of the finite-difference operator. Through this two-steps procedure, the accuracy of the approximation can be much higher.

THEORY

The dispersion relation for a general anisotropic acoustic wave equation in the frequency-wavenumber domain can be represented as follows:

$$\omega^2 = W(\vec{x}, \vec{k}), \quad (1)$$

where ω is the angular frequency, and W is a given function of the space variable $\vec{x}$ and the wavenumber $\vec{k}$, as it describes the propagation of the seismic waves in the wavenumber domain.

For the dispersion relation shown in equation (1), the corresponding wave equation can be written as:

$$\frac{\partial^2 u}{\partial t^2}(\vec{x}) = - \int_{\vec{k}} W(\vec{x}, \vec{k}) \mathcal{F}(u)(\vec{k}) e^{i\vec{k} \cdot \vec{x}} d\vec{k}, \quad (2)$$

and $\mathcal{F}(u)(\vec{k})$ is the Fourier transform of u . If time discretization is taken into consideration, then we have the familiar formulation (Wards et al., 2008; Zhang and Zhang, 2009; Etgen and Brandsberg-Dahl, 2009; Soubaras and Zhang, 2008; Song and Alkhalifah, 2013; Sun et al., 2017):

$$\begin{aligned} & u(\vec{x}, t + \Delta t) + u(\vec{x}, t - \Delta t) - 2u(\vec{x}, t) \\ &= \int_{\vec{k}} \left(2 \cos(\Delta t \sqrt{W(\vec{x}, \vec{k})}) - 2 \right) \mathcal{F}(u)(\vec{k}) e^{i\vec{k} \cdot \vec{x}} d\vec{k}. \end{aligned} \quad (3)$$

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If we can find a separable approximation (Song, 2001; Chen and Liu, 2004) to the original mixed-domain operator, then we need one forward fast Fourier transform (FFT) and N inverse FFTs. So the complexity of the operator reduces to $O((N+1)N_x \log N_x)$. For an even more simplified case in which $W_e(\vec{x}, \vec{k}) = v_x^2 k_x^2 + v_y^2 k_y^2 + v_z^2 k_z^2$, we can represent the integral operator (2) with a partial differential equation given by:

$$\frac{\partial^2 u}{\partial t^2}(\vec{x}) = v_x(\vec{x})^2 \frac{\partial^2 u}{\partial x^2}(\vec{x}) + v_y(\vec{x})^2 \frac{\partial^2 u}{\partial y^2}(\vec{x}) + v_z(\vec{x})^2 \frac{\partial^2 u}{\partial z^2}(\vec{x}). \quad (4)$$

We call the above equation an elliptic-anisotropic-acoustic wave equation. For such an equation, we can solve it using standard finite-difference approximations with a complexity of $O(N_x)$. More precisely, we can approximate the original mixed domain operator using its Taylor expansion:

$$2 \cos(\Delta t \sqrt{W_e(\vec{x}, \vec{k})}) - 2 \approx -\Delta t^2 (v_x^2 k_x^2 + v_y^2 k_y^2 + v_z^2 k_z^2). \quad (5)$$

So the finite-difference approximate operator $W_{D,d}(\vec{x}, \vec{k}, \Delta t)$ is defined as:

$$\begin{aligned} & \Delta t^2 \left(\frac{a_0 + 2 \sum_{j=1}^{n_0} a_j \cos(\Delta x k_x)}{\Delta x^2} v_x^2 \right) \\ & + \Delta t^2 \left(\frac{a_0 + 2 \sum_{j=1}^{n_0} a_j \cos(\Delta y k_y)}{\Delta y^2} v_y^2 \right) \\ & + \Delta t^2 \left(\frac{a_0 + 2 \sum_{j=1}^{n_0} a_j \cos(\Delta z k_z)}{\Delta z^2} v_z^2 \right). \end{aligned} \quad (6)$$

For a general dispersion relation in anisotropic media $W(\vec{x}, \vec{k})$, we assume $W_D(\vec{x}, \vec{k})$ to be a polynomial of second order, which can be denoted as:

$$W_D(\vec{x}, \vec{k}) = \sum_{i_x, i_y, i_z} f_{i_x, i_y, i_z}(\vec{x}) k_x^{i_x} k_y^{i_y} k_z^{i_z}. \quad (7)$$

We assume that i_x , i_y , and i_z are integers and $|i_x| + |i_y| + |i_z| \leq 2$. We denote the relative finite-difference approximate operator as $W_{D,d}(\vec{x}, \vec{k}, \Delta t)$, similar to the formulation shown in equation (6). Then we can rewrite the original operator in the right side of equation (3) as:

$$2 \cos(\Delta t \sqrt{W(\vec{x}, \vec{k})}) - 2 = \left(1 + F(\vec{x}, \vec{k}, \Delta t) \right) W_{D,d}(\vec{x}, \vec{k}, \Delta t), \quad (8)$$

in which, $F(\vec{x}, \vec{k}, \Delta t)$ describes the components of the operator that can not be described by the finite-difference approximate operator. Since this forms, in some cases, a severe approximation, the accuracy of the approximation is limited. So the residual $F(\vec{x}, \vec{k}, \Delta t)$ is not small. And the residual operator $F(\vec{x}, \vec{k}, \Delta t)$ is still a mixed domain operator. Luckily, the low-rank approach (Engquist and Ying, 2009; Fomel et al., 2010) provides a relatively cheap alternative, as it provides an approximate separation given by

$$F(\vec{x}, \vec{k}, \Delta t) \approx \sum_{n=1}^N \left(\sum_{m=1}^M F(\vec{x}, \vec{k}_m, \Delta t) a_{mn} \right) F(\vec{x}_n, \vec{k}, \Delta t). \quad (9)$$

$F(\vec{x}, \vec{k}_m, \Delta t)$ is a submatrix of $F(\vec{x}, \vec{k}, \Delta t)$ that consists of a few columns associated with $\vec{k}_m$, $F(\vec{x}_n, \vec{k}, \Delta t)$ is another submatrix

that contains some rows associated with $\vec{x}_n$, and a_{mn} includes weighting coefficients. Then the relative approximate equation can be written as:

$$u(\vec{x}, t + \Delta t) + u(\vec{x}, t - \Delta t) - 2u(\vec{x}, t) = W_{D,d}u + \sum_{n=1}^N \left(\sum_{m=1}^M F(\vec{x}, \vec{k}_m) a_{mn} \right) \int_{\vec{k}} F(\vec{x}_n, \vec{k}) \mathcal{F}(W_{D,d}u)(\vec{k}) e^{i\vec{k} \cdot \vec{x}} d\vec{k}. \quad (10)$$

As defined before, $W_{D,d}$ is the finite-difference approximate operator. However, the accuracy of equation (10) is now not limited by the accuracy of the finite-difference method. Actually, the low-rank method will correct not only the phase error between the differential operator and the integration operator but also the dispersion error caused by the finite-difference method. So the accuracy is controlled only by the low-rank approximation and we can expect the spectral accuracy even though we have utilized finite-difference approximations in the algorithm.

EXAMPLES

Firstly, we will consider a smooth-isotropic model to show that the partial-low-rank method can efficiently reduce the dispersion error caused by the finite-difference method. In this case, there is no phase error between the integral operator and the differential operator. The analytic velocity satisfies:

$$v_h = 0.8 + \frac{(x_1 - 1)^2 + (x_2 - 1.2)^2}{10}. \quad (11)$$

A Ricker source wavelet with a peak frequency of 30 Hz is placed in the middle of the model. The space sample rate is 0.005 km and the time sample rate is 0.001 s. The solution obtained by finite-difference of 8th order at time $t = 1$ s is shown in Figure 1a. As shown in Figure 1a, the finite difference method of 8th order suffers from severe numerical dispersion due to the high frequency of the source wavelet. The solution obtained by partial-low-rank method at time $t = 1$ s shown in Figure 1b is reasonably accurate. Figure 1b indicates that the partial-low-rank method can reduce the numerical dispersion caused by finite-difference method.

Figure 1: (a) The snapshot at time $t = 1$ s of the finite-difference method of 8th order. (b) The snapshot at time $t = 1$ s of the partial-low-rank method.

Next, we will consider a simple two-dimensional-smooth VTI model. The horizontal velocity is the same as equation (11). The vertical velocity satisfies:

$$v_v = 0.7 + \frac{(x_2 - 1.2)^2}{10}. \quad (12)$$

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The anisotropic parameter $\eta = 0.3$. There are 256 grid points in both x- and z- directions. The space sample rate, time sample rate, source wavelet, and source location are the same as the previous example. The finite-difference solution at time $t = 1$ s with the elliptic-anisotropic equation (4) is shown in Figure 2a. The snapshot at time $t = 1$ s with the partial-low-rank method is shown in Figure 2b(b). In this case, the rank of $F(\vec{x}, \vec{k}, \Delta t)$ is only two, which means three FFTs are required for each time step. For comparison, we consider the solution obtained by the standard low-rank method (Fomel et al., 2010; Song and Alkhalifah, 2013) as the exact solution. The error of the finite-difference method using elliptic anisotropic equation is plotted in Figure 2c in the same scale. As shown in Figure 2c, the finite-difference method suffers from not only dispersion error but also phase error between the elliptic-anisotropic and VTI equations. The error of the partial-low-rank method is plotted in Figure 2d in the same scale. As shown in Figure 2d, the partial-low-rank method can correct not only the dispersion error, but also the phase error between elliptic-anisotropic equation and VTI equation.

Figure 2: (a) The snapshot at time $t = 1$ s of the background wavefield with elliptic-anisotropic equation. (b) The snapshot at time $t = 1$ s of the VTI equation using the partial-low-rank method with only 3 FFTs for each time steps. (c) The error of the snapshot at time $t = 1$ s shown in (a). (d) The error of the snapshot at time $t = 1$ s shown in (b).

Our next example is part of BP TTI model. The original model is a tilted TI model and we are considering only the VTI part of the model. The horizontal velocity and vertical velocity are shown in Figure 3a and 3b. To make the anisotropy stronger, we include a negative η , and thus, we set the actual $\eta = 4\eta_{ori} - 0.1$. The actual η is shown in Figure 3c. The space sampling is 0.0125 km and the time sampling is 0.001 s. In this case, the solution obtained by finite difference method at $t = 1.5$ s is shown in Figure 4a. After the correction with 3 FFTs per time step, we obtain the snapshot at $t = 1.5$ s using the partial-low-rank method shown in Figure 4b. As before, we consider the solution obtained by the standard low-rank method (Fomel et al., 2010; Song and Alkhalifah, 2013) as the exact solution. The error of the finite difference method is shown in Figure

4c. And the error of the partial-low-rank method is shown in Figure 4d. Comparing 4c and 4d, we can see that the low-rank correction is very important to improve the accuracy.

One advantage of the partial-low-rank method is that it can deal with the free surface boundary condition easily. The implementation of the free surface boundary condition and the absorbing boundary condition for standard finite-difference methods are well known and relatively easy. However, the conventional pseudo-spectral approach based on FFTs can not deal with the absorbing and free surface boundary condition easily because of the periodic nature of the Fourier transform. However, for the partial-low-rank method, we use finite difference for the calculation of $W_D u$. The implementation of the absorbing and free surface boundary condition is the same as the standard finite difference method. To demonstrate this advantage, we place a shot in the middle of the surface and apply a free surface boundary condition on the top boundary and an absorbing boundary condition on the other three boundaries. The seismogram is shown in Figure 5. It also demonstrates that the partial-low-rank method is stable for long time simulation.

Figure 3: The parameters for BP TTI model. (a) The horizontal velocity v_h . (b) The vertical velocity v_v . (c) The anisotropic parameter η .

At last, we will use an orthorhombic model. The relative material parameters, similar to the three dimensional VTI model, are shown in Figure 6. The source is located at the same place as the three dimensional VTI example. Figure 7a shows a snapshot at 0.9 s using the standard finite-difference method and Figure 7b shows that obtained by the partial-low-rank method with 3 extra FFTs per time step. The error of the finite-difference method and the partial-low-rank method are shown in Figure 7c and Figure 7d.

CONCLUSIONS

We developed a partial-low-rank method for solving the acoustic wave equation in isotropic and anisotropic medium. The partial-low-rank method approximates the general anisotropic acoustic wave equation through two steps. Firstly, the partial-low-rank method uses an approximate differential operator to represent the integral operator, and thus, utilizes a finite difference approximation. After that, the partial-low-rank method a low rank approximation to the residual terms. Through this, the partial-low-rank method admits the accuracy of the spectral method even though it has a finite-difference operator as part of the algorithm. Because of the combination of the finite-difference method and low-rank method, the partial-low-rank

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method can deal with a free surface and absorbing boundary conditions in a relatively easy way. The resulting approach can deal with different kinds of anisotropy in a uniform manner without admitting S-wave artifacts.

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Figure 4: The snapshot at $t = 1.5$ s. (a) The finite difference method of 8th order using with elliptic-anisotropic equation. (b) The partial-low-rank method with only 3 FFTs for each time steps. (c) The error of the finite difference method. (d) The error of the partial-low-rank method plotted in the same scale.

Figure 5: The observed data for a shot located in the middle of the surface.

Figure 6: The parameters for three dimensional SEAM model. (a) v_1 . (b) v_2 . (c) v_3 . (d) η_1 . (e) η_2 . (f) γ .

Figure 7: The snapshot for orthorhombic medium at 0.9 s using (a) the finite-difference method of 8th order and (b) the partial-low-rank method with 3 extra FFTs per time step. And the error of the finite-difference method (c) and the partial-low-rank method (d).

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