Two-To-One Internal Resonance of an Inclined Marine Riser under Harmonic Excitations

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Abstract

In this paper, we study the two-to-one internal resonance of an inclined marine riser under harmonic excitations. The riser is modeled as an Euler-Bernoulli beam accounting for mid-plane stretching, self-weight, and an applied axial top tension. Due to the inclination, the self-weight load causes a static deflection of the riser, which can tune the frequency ratio between the third and first natural frequencies near two. The multiple time scales method is applied to study the nonlinear equation accounting for the system nonlinearity. The solution is then compared to a Galerkin solution showing good agreement. A further investigation is carried out by plotting the frequency response curves, the force response curves, and the steady state response of the multiple scales solution, in addition to the dynamical solution obtained by Galerkin, as they vary with the detuning parameters. The results reveal that the riser vibrations can undergo multiple Hopf bifurcations and experience quasi-periodic motion that can lead to chaotic behavior. These phenomena lead to complex vibrations of the riser, which can accelerate its fatigue failure.

Keywords: *Harmonic excitation, internal resonance, marine riser, two-to-one.*

1 Introduction

Marine risers are pipe structures designed for transporting oil and gas in drilling and production platforms. The riser structure is subjected to various environmental loads, which include heave top motions, sea currents, and internal fluid flow. In addition, the riser structure inherits nonlinear geometric effects that can cause complex nonlinear dynamics of the beam, such as multi-modal interactions via internal resonance and parametric excitation, which can lead to

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chaotic behavior. The combined action of environmental loads and nonlinear geometric effects can result into vortex induced vibration (VIV) phenomena, which hinder the life of the riser by fatigue [1]. Therefore, it is necessary to study and examine the nonlinear response of the riser under these circumstances and understand the nonlinear interaction to prevent the failure of the riser structure.

The riser is often modeled as an Euler-Bernoulli beam with variable axial force due to self-weight. In addition, the presence of a self-weight distributed load causes a static deflection of the riser. This introduces quadratic nonlinearity, which in addition to the stretching nonlinearity, can lead to various internal resonance scenarios among the various vibration modes resulting into energy transfer among them. Examples of these internal resonances are 1:1 (one-to-one), 2:1 (two-to-one), and 3:1 (three-to-one) resonances. Internal resonances have been studied in different structures, such as beams [2-6], cables [7-12], and plates [13]. The variable axial load in the riser is often simplified by its average value [14] resulting into a beam equation subjected to a constant axial load.

This research work focuses on the case of two-to-one internal resonance. Related research in beams includes the work of Tien et al. [15] who theoretically examined the two-to-one internal resonance of a shallow arch beam under harmonic excitation using the averaging method. They showed that, using the Menlikov technique, the modulated system exhibits a Hopf bifurcation, which leads to chaotic motion via period doubling bifurcation. Chin et al. [5] studied the two-to-one internal resonance of a hinged-hinged buckled beam under parametric excitation considering the influence of the quadratic nonlinearity only and using the multiple time scales method (MTS). They demonstrated that when the critical forcing amplitude is exceeded the system undergoes Hopf bifurcation and series of period doubling, symmetry breaking, and cyclic folding bifurcations leading to

chaotic behavior. In addition, they reported that the two-to-one internal resonance cannot be excited in buckled beams for a fixed-fixed configuration. Öi and Pakdemirli [3] studied the two-to-one internal resonance of an arch beam resting on elastic foundation and found that the activation of the internal resonance depends on the type of assumed initial shape. Due to the symmetry between the initial shape and the mode shape, they found that the trigonometric initial solution cannot activate the internal resonance while the parabolic initial shape activates it. More recently, Xiong et al. [16] studied the influence of viscoelasticity on the two-to-one internal resonance of a post buckled beam. In their work, a two-term MTS expansion was applied on the discretized equation to analyze the effect of the viscoelastic damping. The result showed that viscoelasticity diminishes the response of the system and improves its stability. In another work, Zhuangpeng et al. [17] investigated the two-to-one internal resonance of an arch beam with constrained boundaries. The end constraints resulted into breaking the symmetry of the modes and activating the internal resonance. Furthermore, Ma et al. [18] studied the influence of soil-beam interaction of a pile immersed in the soil using the MTS method while considering the two-to-one internal resonance. The nonlinear interaction between the soil and the beam resulted into rich dynamics affecting the dynamic bifurcations of the beam. Recent works in this context include the work of Sahoo et al. [19] who studied the three-to-one internal resonance of a viscoelastic travelling beam under combined parametric excitations. In addition, Yu et al. [20] investigated the two-to-one internal resonance of an L-shaped beam under primary resonance. They utilized the energy-phase method to characterize the chaotic motion and phase jumps of the response. Furthermore, Zhu et al. [21] examined the three-to-one internal

resonance of an axially moving sandwiched beam subjected to parametric excitations.

On the subject of a riser structure, where self-weight axial force is considered, Neto and Mazzilli [22] developed an invariant manifold technique along with MTS to establish the nonlinear modes to analyze the riser dynamics under two-to-one internal resonance. Their work accounted for the quadratic nonlinearity where the influence of the internal resonance is present. Results showed good agreement when compared to long time integration. Srinil et al. [23-25] investigated the vortex-induced vibration of a catenary riser when the in-plane and out-of-plane frequencies are tuned to two-to-one. The results showed that the influence of the static solution causes the amplitude of the response to be higher when compared to the top tension risers in uniform and shear currents. Other works on this subject include parametric analysis of risers using the MTS method due to the effect of heave excitation [26,27] or internal flow [28].

Due to the lack of extensive research works addressing internal resonances in risers, this research investigates the two-to-one internal resonance of an inclined riser accounting for the variable axial load and mid-plane stretching. The model accounts for the distributed weight that causes a static deflection of the riser. The applied top tension and the riser's configuration are tuned such that the ratio of the third-to-first natural frequencies matches two. In addition, to reduce the computational challenges involved in resolving the fluid forces from a VIV problem, we assume the lift induced forces to be a simple harmonic excitation. Then, the nonlinear equation is treated using three-term expansion in the multiple time scales method to account for the influence of the cubic nonlinearity.

The structure of this paper is as follows: The mathematical formulation of the riser is discussed in Section 2. In Section 3, the perturbation solution of the

nonlinear equation is addressed. Next, validation of the results by comparing the multiple scales solution to long time integration is shown in Section 4 along with more parametric results and discussions. Section 5 presents concluding remarks.

2 Mathematical Formulation

We consider analyzing the interactions in the inclined riser shown in Fig. 1.

Figure 1. Schematic of the inclined riser.

The riser is modeled as an Euler-Bernoulli beam with variable axial tension and nonlinear mid-plane stretching. Then, the riser equation governing the deflection

$\hat{y}$ along the position $\hat{x}$ becomes [29]

$$m \frac{\partial^2 \hat{y}}{\partial \hat{t}^2} + EI \frac{\partial^4 \hat{y}}{\partial \hat{x}^4} - \left((T_e - We \sin(\theta))(L - \hat{x}) + \frac{EA_p}{2L} \int_0^L \left[\left(\frac{\partial \hat{y}}{\partial \hat{x}} \right)^2 \right] d\hat{x} \right) \left(\frac{\partial^2 \hat{y}}{\partial \hat{x}^2} \right) - We \sin(\theta) \left(\frac{\partial \hat{y}}{\partial \hat{x}} \right) + \bar{c} \frac{\partial \hat{y}_d}{\partial \hat{t}} = F_d - We \cos(\theta) \quad (1.1)$$

Subject to the boundary condition

$$\frac{\partial \hat{y}}{\partial \hat{x}} = 0 \text{ at } \hat{x} = 0 \text{ and } \hat{x} = L \quad (1.2)$$

$$\hat{y}(\hat{x}, \hat{t}) = 0 \text{ at } \hat{x} = 0 \text{ and } \hat{x} = L \quad (1.3)$$

where m is the total mass per unit length of the riser defined by $m = \rho_e A_e + \rho_p A_p$, ρ_e is the density of sea water, A_e is the area of the pipe based on the outside diameter, ρ_p is the pipe material density, A_p is the cross section area of the pipe, $\hat{t}$ is the time, EI is the flexural rigidity, T_e is the applied tension, We is the riser apparent weight defined by $We = (\rho_p A_p - \rho_e A_e)g$, g is the gravitational constant, θ is the inclination angle, L is the length of the pipe, $\bar{c}$ is the structural damping, and F_d is the dynamic applied load. The axial tension of the riser is assumed to vary linearly with the weight of the pipe. We assume that the sea current flow is orthogonal to the plane of curvature of the riser and, thereby, the influence of static drag is not considered in the analysis because it has minimal effect. In addition, the influence of the parametric (heave) excitation is neglected due to the application of a moderate static tension. Also, practically, a heave-compensation system is commonly installed on the riser-platform system to suppress the heave excitation. We assume the total riser deflection $\hat{y}_t(\hat{x}, \hat{t})$ solution composed of a static component $\hat{y}_s(\hat{x})$ superimposed to a dynamic component $\hat{y}_d(\hat{x}, \hat{t})$ as follows

$$\hat{y}_t(\hat{x}, \hat{t}) = \hat{y}_s(\hat{x}) + \hat{y}_d(\hat{x}, \hat{t}) \quad (2)$$

The static differential equation of the inclined riser is obtained by dropping the time dependent terms in Eq. (1.1), which results into

$$EI \frac{d^4 \hat{y}_s}{d\hat{x}^4} - \left((T_e - We \sin(\theta))(L - \hat{x}) + \frac{EA_p}{2L} \int_0^L \left[\left(\frac{d\hat{y}_s}{d\hat{x}} \right)^2 \right] d\hat{x} \right) \left(\frac{d^2 \hat{y}_s}{d\hat{x}^2} \right) - We \sin(\theta) \left(\frac{d\hat{y}_s}{d\hat{x}} \right) = -We \cos(\theta) \quad (3)$$

The static equation is solved using the boundary layer perturbation technique, the method of matched asymptotic expansion, assuming a tension-dominant riser and considering clamped boundary conditions. For a more comprehensive study, hinged boundary conditions can also be considered; however, due to the limitation imposed by the boundary-layer perturbation technique, we do not consider the

hinged boundaries [30]. The dynamic equation of motion of the inclined riser is obtained after dropping the static terms, Eq. (3). The dynamic equation is written after simplification as

$$\begin{aligned}
& m \frac{\partial^2 \hat{y}_d}{\partial \hat{t}^2} + EI \frac{\partial^4 \hat{y}_d}{\partial \hat{x}^4} - (T_e - We \sin(\theta)(L - \hat{x})) \left(\frac{\partial^2 \hat{y}_d}{\partial \hat{x}^2} \right) - We \sin(\theta) \left(\frac{\partial \hat{y}_d}{\partial \hat{x}} \right) \\
& - \left(\frac{EA_p}{2L} \left(\int_0^L \left(\frac{d\hat{y}_s}{d\hat{x}} \right)^2 d\hat{x} + \int_0^L \left(\frac{\partial \hat{y}_d}{\partial \hat{x}} \right)^2 d\hat{x} + \int_0^L \left[2 \frac{d\hat{y}_s}{d\hat{x}} \frac{\partial \hat{y}_d}{\partial \hat{x}} \right] d\hat{x} \right) \right) \left(\frac{\partial^2 \hat{y}_d}{\partial \hat{x}^2} \right) \\
& - \left(\frac{EA_p}{2L} \left(\int_0^L \left[2 \frac{d\hat{y}_s}{d\hat{x}} \frac{\partial \hat{y}_d}{\partial \hat{x}} \right] d\hat{x} + \int_0^L \left(\frac{\partial \hat{y}_d}{\partial \hat{x}} \right)^2 d\hat{x} \right) \right) \left(\frac{d^2 \hat{y}_s}{d\hat{x}^2} \right) + \bar{c} \frac{\partial \hat{y}_d}{\partial \hat{t}} = \bar{F}_d
\end{aligned} \quad (4)$$

For simplification, the following dimensionless variables are introduced

$$y_d = \frac{\hat{y}}{D} \quad x = \frac{\hat{x}}{L} \quad t = \sqrt{\frac{EI}{mL^4}} \hat{t} \quad (5)$$

where D is the diameter of riser pipe. The “hat” denotes the dimensional variables and the variables without the “hat” are dimensionless. Substituting Eq. (5) into (4) yields

$$\begin{aligned}
& \frac{\partial^2 y_d}{\partial t^2} + \frac{\partial^4 y_d}{\partial x^4} - \left(\bar{T} - \bar{\sigma}(1-x) - \bar{\eta} \left(\int_0^1 \left(\frac{dy_s}{dx} \right)^2 dx + \int_0^1 \left(\frac{\partial y_d}{\partial x} \right)^2 dx + \int_0^1 2 \frac{dy_s}{dx} \frac{\partial y_d}{\partial x} dx \right) \right) \left(\frac{\partial^2 y_d}{\partial x^2} \right) - \bar{\sigma} \left(\frac{\partial y_d}{\partial x} \right) \\
& - \left(\bar{\eta} \left(\int_0^1 2 \frac{dy_s}{dx} \frac{\partial y_d}{\partial x} dx + \int_0^1 \left(\frac{\partial y_d}{\partial x} \right)^2 dx \right) \right) \left(\frac{d^2 y_s}{dx^2} \right) + c \frac{\partial y_d}{\partial t} = \bar{F}_d
\end{aligned} \quad (6)$$

where

$$\bar{\sigma} = \frac{We \sin(\theta) L^3}{EI}; \quad \bar{\eta} = \frac{A_p D^2}{2I}; \quad c = \frac{cL^2}{\sqrt{mEI}}; \quad \bar{F}_d = \frac{L^4 F_d}{EID}; \quad \bar{T} = \frac{T_e L^2}{EI} \quad (7)$$

As noted in Eq. (6), the static deflection of the riser introduces a quadratic nonlinearity in addition to the existing cubic nonlinearity, which arises due to the stretch of the riser structure. The next step is to solve the linear eigenvalue problem in order to obtain the natural frequencies and mode shapes from Eq. (6) by dropping damping, forcing, and nonlinear terms yielding

$$\frac{\partial^2 y_d}{\partial t^2} + \frac{\partial^4 y_d}{\partial x^4} + \left(\bar{T}_s - \bar{\sigma}x - \bar{\eta} \int_0^1 \left(\frac{dy_s}{dx} \right)^2 dx \right) \frac{\partial^2 y_d}{\partial x^2} - \bar{\eta} \left(\int_0^1 2 \frac{dy_s}{dx} \frac{\partial y_d}{\partial x} dx \right) \left(\frac{d^2 y_s}{dx^2} \right) - \bar{\sigma} \frac{\partial y_d}{\partial x} = 0 \quad (8)$$

where $\bar{T}_s = \bar{\sigma} - \bar{T}$. The mode shapes and the natural frequencies of the structure are extracted by solving Eq. (8), using fifteen beam mode shapes, in a Galerkin method [31].

The riser structure is subjected to vortex-induced vibrations in the cross-flow direction. Several semi-empirical formulas have been developed [32,33] to reduce the level of fluid dynamics computations. However, in this work, these formulations increase the complexity of the analysis that may hinder in-depth study of the nonlinear interaction problems in the riser structure. Therefore, we choose the VIV excitation force to be of the form [1]

$$\bar{F}_d = f \cos(\Omega t) \quad (9)$$

where $f = \frac{L^4 C_L \rho_e U^2}{2EI}$, U is the external current velocity, and C_L is the average lift coefficient.

3 Two-To-One Internal Resonance (Perturbations Analysis)

In this section, we analyze the influence of the quadratic and cubic nonlinearities via the method of multiple scales where the expansion is applied directly on the integral partial differential equation of Eq. (6). Accordingly, we seek a three-order expansion of the form

$$y_d(x, t; \varepsilon) = \varepsilon y_1(x, T_0, T_1, T_2) + \varepsilon^2 y_2(x, T_0, T_1, T_2) + \varepsilon^3 y_3(x, T_0, T_1, T_2) \quad (10)$$

where ε is introduced as a bookkeeping parameter, $T_0 = \varepsilon t$, $T_1 = \varepsilon^2 t$, and $T_2 = \varepsilon^3 t$.

Due to the static deflection, a large difference is observed between the magnitudes of the quadratic and cubic nonlinearities. Thus, the scale chosen at Eq. (10) determines that the quadratic nonlinearity is at ε^2 and the cubic nonlinearity is at

ε^3 and the forcing and damping are ordered as $\varepsilon^2 f \cos(\Omega t)$ and εc , respectively.

We substitute Eq. (10) into Eq. (6) to obtain

Order ε

$$\ell(y_1) = D_0^2 y_1 + 2\sqrt{\beta\nu} D_0 y_1' + y_1^{iv} + \bar{T}_s y_1'' - x \bar{\sigma} y_1'' - \bar{\sigma} y_1' - 2\eta y_1'' \left(\int_0^1 y_s' y_1' dx \right) = 0 \quad (11)$$

Order ε^2

$$\ell(y_2) = -c D_0 y_1 - 2D_0 D_1 y_1 - 2\sqrt{\beta\nu} D_1 y_1' + \eta y_s'' \left(\int_0^1 (y_1')^2 dx \right) + 2\eta y_1'' \left(\int_0^1 y_s' y_1' dx \right) + \frac{f}{2} e^{i\Omega T_0} \quad (12)$$

Order ε^3

$$\begin{aligned} \ell(y_3) = & -D_1^2 y_1 - 2D_0 D_2 y_1 - 2D_0 D_1 y_2 - c D_1 y_1 - c D_0 y_2 + \eta y_1'' \int_0^1 (y_1')^2 dx \\ & + 2\eta \left(y_2'' \int_0^1 y_s' y_1' dx + y_s'' \int_0^1 y_1' y_2' dx + y_1'' \int_0^1 y_s' y_2' dx \right) \end{aligned} \quad (13)$$

where the derivatives are defined as $D_0^2 = \frac{\partial}{\partial T_0^2}$, $D_0 = \frac{\partial}{\partial T_0}$, $D_1^2 = \frac{\partial}{\partial T_1^2}$, $D_1 = \frac{\partial}{\partial T_1}$

, $D_2 = \frac{\partial}{\partial T_2}$ and $(\cdot)'$ denotes the derivative with respect to x and $\ell(\cdot)$ defines the

linear operator given by Eq. (11). It is observed that the first order Eq. (11) is equivalent to the eigenvalue problem defined by Eq. (8). Then, at higher orders, the coupling between the modes will occur due to the activation of the internal resonance. Therefore, the solution to Eq. (11) is written as the superposition of two different modes

$$\begin{aligned} y_1 = & A_n(T_1, T_2) \phi_n(x) e^{i\omega_n T_0} + A_m(T_1, T_2) \phi_m(x) e^{i\omega_m T_0} \\ & + \bar{A}_n(T_1, T_2) \phi_n(x) e^{-i\omega_n T_0} + \bar{A}_m(T_1, T_2) \phi_m(x) e^{-i\omega_m T_0} \end{aligned} \quad (14)$$

where $A_n(T_1, T_2)$ and $A_m(T_1, T_2)$ are the amplitudes associated with modes n and m , respectively, $\phi_n(x)$ and $\phi_m(x)$ are the mode shapes, and ω_n and ω_m are the natural frequencies. To determine the solvability condition at the second order, Eq. (14) is substituted into Eq. (12) to obtain

$$\begin{aligned}
\ell(y_2) = & -(\mathrm{i}\omega_m c\phi_m(x)A_m + 2\mathrm{i}\omega_m\phi_m(x)D_1A_m)e^{\mathrm{i}\omega_m T_0} - (\mathrm{i}\omega_n c\phi_n(x)A_n + 2\mathrm{i}\omega_n\phi_n(x)D_1A_n)e^{\mathrm{i}\omega_n T_0} \\
& + \left(\eta y_s'' \left(\int_0^1 (\phi_m')^2 dx \right) + 2\eta\phi_m'' \left(\int_0^1 y_s' \phi_m' dx \right) \right) A_m^2 e^{2\mathrm{i}T_0\omega_m} + 2\Lambda_{nm}A_n\bar{A}_m e^{\mathrm{i}T_0(\omega_n-\omega_m)} \\
& + \frac{f}{2}e^{\mathrm{i}\Omega T_0} + cc + NST
\end{aligned} \tag{15}$$

where cc is the complex conjugate of the expression, NST denotes the non-secular terms, and Λ_{nm} is defined as

$$\Lambda_{nm} = \left(\eta\phi_m'' \int_1^0 y_s' \phi_n' dx + \eta\phi_n'' \int_1^0 y_s' \phi_m' dx + \eta y_s'' \int_1^0 \phi_m' \phi_n' dx \right) \tag{16}$$

We consider the case when the excitation is around the first primary resonance defined by $\Omega = \omega_m + \varepsilon^2\alpha$, and the two-to-one internal resonance is given by $\omega_n = 2\omega_m + \delta\varepsilon^2$, where α and δ are detuning parameters. To obtain the solvability condition, the linear adjoint problem of Eq. (11) needs to be solved for $\phi_j(x)e^{\pm\mathrm{i}\omega_j T_0}$. The proof provided in Appendix A shows that the linear problem of Eq. (11) is self-adjoint. Then, we substitute $\omega_n = 2\omega_m + \delta\varepsilon$ and $\Omega = \omega_m + \varepsilon\alpha$ in Eq. (15) and demand that the right hand side of the equation is orthogonal to $\phi_m(x)e^{-\mathrm{i}\omega_m T_0}$ and $\phi_n(x)e^{-\mathrm{i}\omega_n T_0}$ while integrating to reduce to

$$2\mathrm{i}\omega_m D_1 A_m = -\mathrm{i}\mu_1 \omega_m A_m + R_1 A_n \bar{A}_m e^{\mathrm{i}\delta T_1} + F_m e^{\mathrm{i}\alpha T_1} \tag{17}$$

$$2\mathrm{i}\omega_n D_1 A_n = -\mu_2 \mathrm{i}\omega_n A_n + R_2 A_m^2 e^{-\mathrm{i}\delta T_1} \tag{18}$$

where the coefficients R_i, μ_1, μ_2, F_k are defined by

$$\begin{aligned}
R_1 &= \int_0^1 \phi_m \Lambda_{nm} dx, \quad \mu_1 = \int_0^1 c \phi_m^2 dx, \quad \mu_2 = \int_0^1 c \phi_n^2 dx, \quad F_k = \frac{1}{2} \int_0^1 f \phi_k dx, \\
R_2 &= \int_0^1 \phi_n \left(\eta y_s'' \left(\int_0^1 \phi_m'^2 dx \right) + 2\eta\phi_m'' \left(\int_0^1 y_s' \phi_m' dx \right) \right) dx
\end{aligned} \tag{19}$$

From Eq. (17) and Eq. (18), the necessary condition to activate the two-to-one internal resonance is given by non-zero coupling coefficients R_1 and R_2 . In the

absence of higher order nonlinearity, one can stop the analysis up to this order of ε^2 and analyze the modulation equations, Eq. (17) and Eq. (18). However, the mid-plane stretching has a cubic nonlinearity to be accounted for. For that reason, we extend the analysis to ε^3 and consider the particular solution of the non-secular terms in Eq. (15). The particular solution is written as

$$y_2 = \psi_1(x)A_m A_n e^{i(\omega_n + \omega_m)T_0} + \psi_2(x)A_n^2 e^{i2\omega_n T_0} + \psi_3(x)A_m \bar{A}_m + \psi_4(x)A_n \bar{A}_n + B_m \phi_m(x) e^{iT_0 \omega_m} + B_n \phi_n(x) e^{iT_0 \omega_n} + cc \quad (20)$$

where $\psi_1, \psi_2, \psi_3, \psi_4$ are given by solving the following boundary value problems:

$$\psi_1(x) : H[\omega_n + \omega_m] = 2\Lambda_{nm} \quad (21.1)$$

$$\psi_2(x) : H[2\omega_n] = \Lambda_n \quad (21.2)$$

$$\psi_3(x) : H[0] = \Lambda_m \quad (21.3)$$

$$\psi_4(x) : H[0] = \Lambda_n \quad (21.4)$$

and H is defined as

$$H[\omega_i] = H^{iv} - N H'' - 2\alpha_1 w_s'' \left(\int_0^1 w_s' H' dx \right) - \omega_i^2 H \quad (21.5)$$

where Λ_i is defined as

$$\Lambda_i = \eta y_s''(x) \left(\int_0^1 \phi_i'(x)^2 dx \right) + 2\eta \phi_i''(x) \left(\int_0^1 \phi_i'(x) y_s'(x) dx \right) \quad (21.6)$$

The boundary value problems in Eq. (21.1) – Eq. (21.4) are subjected to clamped-clamped boundary conditions defined by

$$\psi_i(0) = 0, \psi_i(1) = 0, \frac{d\psi_i(0)}{dx} = 0, \frac{d\psi_i(1)}{dx} = 0 \quad (22)$$

In addition, these equations are solved using Galerkin procedure prescribed in [31]. The expansion of the multiple scales solution up to ε^3 , in the two-to one internal resonance, yields secular terms that are not derivable from the time averaged Lagrangian. Therefore, $B_m(T_1, T_2)$, $B_n(T_1, T_2)$ are introduced as a correction to satisfy the Lagrangian formulation at the third order [34].

Substituting the solution from Eq. (14) and Eq. (20) into the third order Eq. (13) while considering $\omega_n = 2\omega_m + \varepsilon\delta$, $\Omega = \omega_m + \varepsilon\alpha$ and multiplying with the adjoint $\phi_m(x) e^{-i\omega_m T_0}$ and $\phi_n(x) e^{-i\omega_n T_0}$ yield the solvability condition at the third order given by

$$2i\omega_m D_2 A_m = -\mu_1 D_1 A_m - D_1^2 A_m + K_1 A_m A_n \bar{A}_n + K_2 A_m^2 \bar{A}_m - i\mu_1 \omega_m B_m - 2i\omega_m D_1 B_m + R_1 e^{i\delta T_1} B_n \bar{A}_m + R_1 e^{i\delta T_1} A_n \bar{B}_m \quad (23)$$

$$2i\omega_n D_2 A_n = -\mu_1 D_1 A_n - D_1^2 A_n + K_3 A_m A_n \bar{A}_m + K_4 A_n^2 \bar{A}_n + 2R_2 e^{-i\delta T_1} A_m B_m - i\mu_1 \omega_n B_n - 2i\omega_n D_1 B_n \quad (24)$$

where the coefficients K_1, K_2, K_3 and K_4 are defined in Appendix B. To ensure that the modulation equation satisfies the Lagrangian, we demand

$$D_1 [2i\omega_m B_m(T_1, T_2) + D_1 A_m(T_1, T_2)] = 0 \quad (25.1)$$

$$D_1 [2i\omega_n B_n(T_1, T_2) + D_1 A_n(T_1, T_2)] = 0 \quad (25.2)$$

Equations (25.1) and (25.2) yield

$$2i\omega_m B_m(T_1, T_2) + D_1 A_m(T_1, T_2) = c_1(T_2) \quad (26.1)$$

$$2i\omega_n B_n(T_1, T_2) + D_1 A_n(T_1, T_2) = c_2(T_2) \quad (26.2)$$

From Eq. (17) and Eq. (18), we note that $D_1 A_m(T_1, T_2)$ and $D_1 A_n(T_1, T_2)$ have no explicit dependence on T_2 . Then, $c_1(T_2) = c_2(T_2) = 0$, which yields

$$B_m = -\frac{i\mu_1}{4\omega_m} A_m + \frac{R_1}{4\omega_m^2} A_n \bar{A}_m e^{i\delta T_1} + \frac{F_m}{4\omega_m^2} e^{i\alpha T_1} \quad (27.1)$$

$$B_n = -\frac{i\mu_1}{4\omega_n} A_n + \frac{R_2}{4\omega_n^2} A_m^2 e^{-i\delta T_1} \quad (27.2)$$

The next step is to combine the solutions from Eq. (17), Eq. (18), Eq. (23), Eq. (24), Eq. (27.1), and Eq. (27.2) using the method of reconstitution [35] defined by

$$\frac{dA_i}{dt}(t, \varepsilon) = \varepsilon D_1 A_i + \varepsilon^2 D_2 A_i \quad (28)$$

Setting ε to unity and after further manipulation, the modulation equations are given by

$$\begin{aligned} \frac{dA_m}{dt} = & -\left(\frac{\mu_1}{2} + \frac{i\mu_1^2}{8\omega_m}\right)A_m - \left(\frac{iK_2}{2\omega_m} + \frac{iR_1R_2}{8\omega_m\omega_n^2}\right)A_m^2\bar{A}_m - \left(\frac{iK_1}{2\omega_m} + \frac{iR_1^2}{8\omega_m^3}\right)A_mA_n\bar{A}_n \\ & + \left(\frac{\mu_1R_1}{4\omega_m^2} - \frac{\mu_1R_1}{8\omega_m\omega_n} - \frac{iR_1}{2\omega_m}\right)A_n\bar{A}_me^{i\delta T_1} - \frac{iR_1F_m}{8\omega_m^3}A_ne^{-i(\alpha-\delta)T_1} + \left(\frac{\mu_1}{8\omega_m^2} - \frac{i}{2\omega_m}\right)e^{i\alpha T_1}F_m \end{aligned} \quad (29.1)$$

$$\begin{aligned} \frac{dA_n}{dt} = & -\left(\frac{\mu_1}{2} + \frac{i\mu_1^2}{8\omega_n}\right)A_n - \left(\frac{iK_3}{2\omega_n} + \frac{iR_1R_2}{4\omega_m^2\omega_n}\right)A_mA_n\bar{A}_m - \frac{iK_4}{2\omega_n}A_n^2\bar{A}_n - \frac{iR_2F_m}{4\omega_m^2\omega_n}A_me^{iT_1(\alpha-\delta)} \\ & + \left(\frac{\mu_1R_2}{8\omega_n^2} - \frac{2\mu_1R_2}{4\omega_m\omega_n} - \frac{2iR_2}{2\omega_n}\right)A_m^2e^{-i\delta T_1} \end{aligned} \quad (29.2)$$

Equation (29.1) and (29.2) resemble the solvability condition that captures the influence of the second and third orders respectively. Then, we substitute the polar transformation given by

$$A_m = \frac{1}{2}a_me^{i(\beta_m + \alpha t)} \quad (30.1)$$

$$A_n = \frac{1}{2}a_ne^{i(\beta_n + (2\alpha - \delta)t)} \quad (30.2)$$

Substituting Eq. (30.1) and Eq. (30.2) into Eq. (29.1) and Eq. (29.2) yields

$$\begin{aligned} a_m' = & -\frac{\mu_1}{2}a_m - \frac{R_1F_m}{8\omega_m^3}a_n\sin(\beta_m - \beta_n) + \frac{\mu_1F_m}{4\omega_m^2}\cos(\beta_m) - \frac{F_m}{\omega_m}\sin(\beta_m) \\ & - \frac{R_1}{4\omega_m}a_ma_n\sin(2\beta_m - \beta_n) + \left(\frac{\mu_1}{8\omega_m^2} - \frac{\mu_1}{16\omega_m\omega_n}\right)R_1a_ma_n\cos(2\beta_m - \beta_n) \end{aligned} \quad (31.1)$$

$$\begin{aligned} a_m\beta_m' = & -\left(\alpha + \frac{\mu_1^2}{8\omega_m}\right)a_m - \frac{\mu_1F_m}{4\omega_m^2}\sin(\beta_m) - \frac{F_m}{\omega_m}\cos(\beta_m) - \left(\frac{R_1R_2}{32\omega_m\omega_n^2} + \frac{K_2}{8\omega_m}\right)a_m^3 \\ & - \left(\frac{R_1^2}{32\omega_m^3} + \frac{K_1}{8\omega_m}\right)a_ma_n^2 - \frac{R_1F_m}{8\omega_m^3}a_n\cos(\beta_m - \beta_n) \\ & + \left(\frac{\mu_1R_1}{16\omega_m\omega_n} - \frac{\mu_1R_1}{8\omega_m^2}\right)a_ma_n\sin(2\beta_m - \beta_n) - \frac{R_1}{4\omega_m}a_ma_n\cos(2\beta_m - \beta_n) \end{aligned} \quad (31.2)$$

$$\begin{aligned} a_n' = & -\frac{\mu_1}{2}a_n + \frac{R_2F_m}{4\omega_m^2\omega_n}a_m\sin(\beta_m - \beta_n) + \left(\frac{R_2\mu_1}{16\omega_n^2} - \frac{R_2\mu_1}{8\omega_m\omega_n}\right)a_m^2\cos(2\beta_m - \beta_n) \\ & + \frac{R_2}{4\omega_n}a_m^2\sin(2\beta_m - \beta_n) \end{aligned} \quad (31.3)$$

$$\begin{aligned}
a_n \beta_n' = & \left(\delta - 2\alpha - \frac{\mu_1^2}{8\omega_n} \right) a_n - \frac{R_2 F_m}{4\omega_m^2 \omega_n} a_m \cos(\beta_m - \beta_n) - \left(\frac{K_3}{8\omega_n} + \frac{R_1 R_2}{16\omega_m^2 \omega_n} \right) a_m^2 a_n - \frac{K_4}{8\omega_n} a_n^3 \\
& - \frac{R_2}{4\omega_n} a_m^2 \cos(2\beta_m - \beta_n) + \left(\frac{R_2 \mu_1}{16\omega_n^2} - \frac{R_2 \mu_1}{8\omega_m \omega_n} \right) a_m^2 \sin(2\beta_m - \beta_n)
\end{aligned} \quad (31.4)$$

Equations (31) represent a system of equations that govern the phase and amplitude modulation. The fixed points (equilibrium solutions) of the system are obtained by setting the right-hand side of the equations to zero reducing them to a set of algebraic equations. The equations are solved using a pseudo-arc length continuation method [36]. The system response is then expressed as

$$\begin{aligned}
y = & y_s(x) + a_m \phi_m(x) \cos((\omega_m + \alpha)t + \beta_m) + \frac{\mu_1}{4\omega_m} a_m \phi_m(x) \sin((\omega_m + \alpha)t + \beta_m) \\
& + a_n \phi_n(x) \cos((\omega_n + 2\alpha - \delta)t + \beta_n) + \frac{\mu_2}{4\omega_n} a_n \phi_n(x) \sin((\omega_n + 2\alpha - \delta)t + \beta_n) \\
& + \frac{1}{4} a_m^2 \psi_3(x) + \frac{1}{4} a_n^2 \psi_4(x) + \frac{1}{2} a_m a_n \psi_1(x) \cos((\omega_m + \omega_n + 3\alpha - \delta)t + \beta_n + \beta_m) \\
& + \frac{1}{2} a_n^2 \psi_2(x) \cos(2(\omega_n + 2\alpha - \delta)t + 2\beta_n) + \frac{R_1}{8\omega_m^2} a_m a_n \phi_m(x) \cos((\omega_m + \alpha)t + \beta_n - \beta_m) \\
& + \frac{F_m}{2\omega_m^2} \phi_m(x) \cos((\omega_m + \alpha)t) + \frac{R_2}{8\omega_n^2} a_m^2 \phi_n(x) \cos((\omega_n + 2\alpha - \delta)t + 2\beta_m)
\end{aligned} \quad (32)$$

To analyze the stability of the system, we express the amplitudes using the complex Cartesian format to avoid numerical issues when the trivial solution is present [13]. This is done by letting

$$A_m = \frac{1}{2} (p_m - i q_m) e^{i\lambda_1 t} \quad (33.1)$$

$$A_n = \frac{1}{2} (p_n - i q_n) e^{i\lambda_2 t} \quad (33.2)$$

Substituting Eq. (33.1) and Eq. (33.2) into Eq. (29) yields

$$\begin{aligned}
p_m' = & -\frac{\mu_1}{2} p_m - \left(\alpha + \frac{\mu_1^2}{8\omega_m} \right) q_m + \frac{\mu_1 F_m}{4\omega_m^2} - \frac{R_1 F_m}{8\omega_m^3} q_n + \frac{R_1}{4\omega_m} (q_m p_n - p_m q_n) \\
& + \left(\frac{\mu_1 R_1}{8\omega_m^2} - \frac{\mu_1 R_1}{16\omega_m \omega_n} \right) (p_m p_n + q_m q_n) - q_m p_n^2 \left(\frac{K_1}{8\omega_m} + \frac{R_1^2}{32\omega_m^3} \right) - p_m^2 q_m \left(\frac{K_2}{8\omega_m} + \frac{R_1 R_2}{32\omega_m \omega_n^2} \right) \\
& - q_m^3 \left(\frac{K_2}{8\omega_m} + \frac{R_1 R_2}{32\omega_m \omega_n^2} \right) - q_m q_n^2 \left(\frac{K_1}{8\omega_m} + \frac{R_1^2}{32\omega_m^3} \right)
\end{aligned} \quad (34.1)$$

$$\begin{aligned}
q_m' = & -\frac{\mu_1}{2} q_m + \left(\frac{\mu_1^2}{8\omega_m} + \alpha \right) p_m + \frac{R_1 F_m}{8\omega_m^3} p_n + \frac{F_m}{\omega_m} - \left(\frac{\mu_1 R_1}{16\omega_m \omega_n} - \frac{\mu_1 R_1}{8\omega_m^2} \right) (p_m q_n + q_m p_n) \\
& + \frac{R_1}{4\omega_m} (q_m q_n + p_m p_n) + p_m q_n^2 \left(\frac{K_1}{8\omega_m} + \frac{R_1^2}{32\omega_m^3} \right) + p_m q_m^2 \left(\frac{K_2}{8\omega_m} + \frac{R_1 R_2}{32\omega_m \omega_n^2} \right) \\
& + p_m^3 \left(\frac{K_2}{8\omega_m} + \frac{R_1 R_2}{32\omega_m \omega_n^2} \right) + p_m p_n^2 \left(\frac{K_1}{8\omega_m} + \frac{R_1^2}{32\omega_m^3} \right)
\end{aligned} \tag{34.2}$$

$$\begin{aligned}
p_n' = & -\frac{\mu_1}{2} p_n - \left((2\alpha - \delta) + \frac{\mu_1^2}{8\omega_n} \right) q_n - \frac{R_2 F_m}{4\omega_m^2 \omega_n} q_m - \frac{R_2}{2\omega_n} p_m q_m + \left(\frac{\mu_1 R_2}{16\omega_n^2} - \frac{\mu_1 R_2}{8\omega_m \omega_n} \right) (p_m^2 - q_m^2) \\
& - p_m^2 q_n \left(\frac{K_3}{8\omega_n} + \frac{R_1 R_2}{16\omega_m^2 \omega_n} \right) - q_m^2 q_n \left(\frac{K_3}{8\omega_n} + \frac{R_1 R_2}{16\omega_m^2 \omega_n} \right) - \frac{K_4}{8\omega_n} p_n^2 q_n - \frac{K_4}{8\omega_n} q_n^3
\end{aligned} \tag{34.3}$$

$$\begin{aligned}
q_n' = & -\frac{\mu_1}{2} q_n + \left((2\alpha - \delta) + \frac{\mu_1^2}{4\omega_n} \right) p_n + \frac{R_2 F_m}{4\omega_m^2 \omega_n} p_m + \frac{R_2}{4\omega_n} (p_m^2 - q_m^2) - p_m q_m \left(\frac{\mu_1 R_2}{4\omega_m \omega_n} - \frac{\mu_1 R_2}{8\omega_n^2} \right) \\
& + p_m^2 p_n \left(\frac{K_3}{8\omega_n} + \frac{R_1 R_2}{16\omega_m^2 \omega_n} \right) + \frac{K_4}{8\omega_n} p_n q_n^2 + \frac{K_3}{8\omega_n} q_m^2 p_n + \frac{K_4}{8\omega_n} p_n^3 + \frac{R_1 R_2}{16\omega_m^2 \omega_n} q_m^2 p_n
\end{aligned} \tag{34.4}$$

where $\lambda_1 = \alpha$ and $\lambda_2 = 2\alpha - \delta$. Then, the stability of the system is obtained by examining the eigenvalues of the Jacobian of the system of Eqs. (34) after substituting the fixed points. To further investigate the stability of the response near Hopf bifurcations, the method of normal forms via second order multiple scales (MTS) is applied on Eq. (34.1) - Eq. (34.4) using the procedure prescribed in [37] to reduce to

$$r' = \kappa_1 (\alpha - \alpha_c) r + \kappa_2 r^3 \tag{35.1}$$

$$\mathcal{G}' = \kappa_3 (\alpha - \alpha_c) r + \kappa_4 r^2 \tag{35.2}$$

where r represents the amplitude of the limit cycle, α_c is the value at which Hopf bifurcation occurs, and $\mathcal{G}$ is the phase of the limit cycle. The limit cycle is stable when κ_2 is negative (supercritical Hopf bifurcation), and it is unstable when κ_2 is positive (subcritical Hopf bifurcation).

4 Results and Discussion

4.1 Natural Frequency Results

The solution from the eigenvalue problem from Eq. (8) is solved as a first step by varying the inclination angle θ and the applied tension. The parameters are tuned such that the ratio between the frequencies becomes 2. The data in Table 1 are used to interpret the results. The linear frequency results are depicted in Figure 2.

Table 1. Values of the geometric and material parameters used in the calculations.

Parameter	Value
Outside Diameter, D	0.26 m
Inside Diameter, D_i	0.2 m
Modulus of Elasticity, E	207 GPa
Density of Riser, ρ_p	7850 Kg/m ³
Density of Sea Water, ρ_e	1025 Kg/m ³
Depth of Sea	150 m

Figure 2. Natural frequency values at different inclination angles. (a) Lowest five natural frequencies at $\bar{T} = 2\bar{\sigma}$. (b) Normalized natural frequency values, with respect to the 1st natural frequency, at $\bar{T} = 2\bar{\sigma}$. (c) Lowest five natural frequencies at $\bar{T} = 5\bar{\sigma}$. (d) Normalized natural frequency at $\bar{T} = 5\bar{\sigma}$.

When the inclination angle is changed from lower values (close to horizontal configuration) to the higher angle values (close to vertical configuration), we notice that the frequency values tend to decrease due to the decrease in length and thereby the static deflection of the riser. This is apparent from Fig. 2a and Fig. 2c. Then, examining the frequency ratios with respect to the first natural frequency reveals the possibility of a two-to-one internal resonance between the first and third frequencies (in circle) at $\theta=32^\circ$ and $\theta=62^\circ$ for tension cases $\bar{T} = 2\bar{\sigma}$ and $\bar{T} = 5\bar{\sigma}$ at Fig. 2b and Fig. 2d, respectively. Since these results are of interest to this study, the configuration angles are fixed at these values and we sweep the applied tension to analyze the nonlinear interactions. The values of the frequencies are presented in Table 2.

Table 2. Frequency values for the two configurations under study.

$\bar{T}$	θ	ω_1	ω_3	δ
$2\bar{\sigma} = 414.04$	62°	186.11	372.63	0.41
$5\bar{\sigma} = 2873.67$	32°	443.59	887.79	0.60

In addition, we show the corresponding static and mode shapes plots in Fig. 3 and Fig. 4. Based on the static deflection in Fig. 3, the maximum deflection of the inclined riser is observed at $x \approx 0.494$ for $\theta = 32^\circ$ and at $x \approx 0.490$ for $\theta = 62^\circ$, which is close to the midpoint position of the riser. These results suggest that the influence of the self-weight is reduced at these values of applied tension.

Figure 3. (a) Static deflection profile along the position of the inclined riser for the configurations in Table 2. (b) Corresponding dimensional static deflection profiles with respect to the depth and offset parameters. The data in Table 1 are utilized to obtain the results.

The results in Fig. 4 show the lowest five mode shapes for $\theta = 62^\circ$ configuration confirming the fact that the maximum deflection is near the midpoint of the riser. Because the mode shape exhibits similarities to those observed in beams, the dynamic solutions from multiple scales and other method are calculated at the maximum deflection point, i.e. $x \approx 0.494$ for $\theta = 32^\circ$ and at $x \approx 0.490$ for $\theta = 62^\circ$.

Figure 4. The lowest five mode shapes of the inclined riser at a configuration angle of $\theta = 62^\circ$. (—) 1st mode shape, (-----) 2nd mode shape, (- · - · -) 3rd mode shape, (·····) 4th mode shape, (- - -) 5th mode shape.

4.2 Two-To-One Internal Resonance (Perturbation Results)

To ensure the validity of the multiple scale solution obtained by Eq. (31), the Galerkin method is applied to discretize Eq. (6), using five mode shapes from Eq. (8), into a set of ordinary differential equations in time, which are then integrated using fourth order Runge-Kutta technique in Mathematica [38] while updating initial conditions. The result of this comparison is shown in Fig. 5.

Figure 5. Frequency response curves showing the deflection y as a function of the detuning parameter α for $\theta = 62^\circ$, $\mu_1 = 0.08$, $\mu_2 = 0.08$, $F_m = 8$. (—) Stable fixed points, (---) unstable fixed points from the perturbation method. (□) Solutions by long time integration (forward sweep). (○) Solutions by long time integration (backward sweep).

A good agreement is observed between the multiple scales solution and the long time integration results. The multiple scales solution captures the main features of the interaction, i.e., the saddle node jumps at the hardening and the softening bands as well as the unstable bifurcated band. It is noteworthy that a computational challenge was experienced when capturing the solution at the upper softening branch by the Galerkin technique in the backward sweep. This may suggest that the basins of attraction of these solutions are very weak causing the solution to jump to the lower branch. Also, the estimated external current velocity corresponding to the forcing amplitude is $U_e = 0.09295 \text{ m/s}$, which suggests that the interaction between the modes can occur at very low velocities.

A further investigation is carried out by plotting the amplitudes corresponding to each mode in each configuration case. These results represent the fixed points obtained by solving the algebraic system of Eqs. (31). The results are shown in Fig. 6.

Figure 6. Frequency response curves of the first and third amplitudes as a function of the detuning parameter α . (—) Stable fixed points, (---) unstable fixed points. H denotes the point of a Hopf bifurcation. (a,b) $\theta = 62^\circ$, $\mu_1 = 0.08$, $\mu_2 = 0.08$, $F_m = 8$. (c,d) $\theta = 32^\circ$, $\mu_1 = 0.08$, $\mu_2 = 0.08$, $F_m = 23$.

As the frequency is swept from lower values, the response experiences a saddle node jump from the lower branch to the upper branch in the softening part of the response. Then, the system passes through forward and reverse subcritical Hopf bifurcations after which the system, due to internal resonance activation, experiences a hardening type response. This is apparent in both configuration cases. In addition, more forcing amplitude is required in case (b) to achieve a similar response observed in case (a) to compensate the influence of larger static deflection at lower configuration angles. In addition, the hardening type response in both cases is capped because the formulation accounts for the influence of cubic nonlinearities.

Furthermore, we examine the interaction by plotting the time histories and the corresponding phase spaces in Fig. 7 by substituting the amplitudes and corresponding phases in Eq. (32).

Figure 7. Response time history and their corresponding phase portraits for $\theta = 62^\circ$, $\mu_1 = 0.08$, $\mu_2 = 0.08$, $F_m = 8$. (a,b,c) $\alpha = -0.8$, (d,e,f) $\alpha = 0.45$.

In Fig. 7a, the time history of these results was obtained from the lower branch of the softening response type at $\alpha = -0.8$. Away from the interaction zone, the response is dominated by a single period and confirmed by a closed loop in the

phase plot Fig. 7b and the Fourier spectrum in Fig. 7c. In contrast, the plot in the interaction zone at $\alpha = 0.45$ (Fig. 7d) shows a two-period response confirmed by the two loops in Fig. 7e and the competing amplitudes in the Fourier spectrum corresponding to each mode in Fig. 7f.

Moreover, the force response curve is obtained for both amplitudes by plotting the amplitudes against the excitation amplitude for difference values of α .

The results are depicted in Fig. 8.

Figure 8. Force response curves of the first and third amplitudes as a function of forcing. $\theta = 62^\circ$, $\mu_1 = 0.08$, $\mu_2 = 0.08$. (—) Stable fixed points, (---) unstable fixed points. H denotes the point of a Hopf bifurcation. (a) $\alpha = 0.20$, (b) $\alpha = 0.45$, (c,d) $\alpha = 0$.

We plot the response in Fig. 8a and 8b for small forcing values and large forcing values in Fig. 8c, 8d. Since the excitation is around the first natural frequency, the trivial solution is not present and the interaction is always present. As the forcing increases in Fig. 8a, the exchange of energy starts at very low forcing values promoting the interaction between the two modes while increasing the forcing further results into a Hopf bifurcation. In addition, in Fig. 8b we observe the emergence of a saddle node bifurcation of the softening band followed by a Hopf bifurcation when the forcing is increased. Extending the forcing range further in Fig. 8c,d shows the occurrence of the saturation phenomena in the response of the third amplitude after Hopf bifurcation. These results suggest that care needs to be taken when designing a marine riser since the interaction can result into a bifurcation of the response even at very low external velocities.

To gain more insight on how the interaction varies with the applied tension, we vary the applied tension such that the third natural frequency is less than twice of the first natural frequency. Then, we increase the value of tension in a multistep process (solving for the static problem, linear eigenvalue problem, and then particular solutions) and solve the amplitude of each mode from Eq. (31). The detailed solutions of the first and third mode amplitudes corresponding to each tension value are available in the supplementary materials. The results at $\alpha = 0$ (softening regime) and $\alpha = 0.25$ (hardening regime) are shown in Fig. 9.

Figure 9. Frequency response curves showing the response of the first and third amplitude as a function of the detuning parameter δ . $\theta = 62^\circ$, $\mu_1 = 0.08$, $\mu_2 = 0.08$, $\alpha = 0$, $F_m = 8$. (—) Stable fixed solutions, (---) unstable fixed solution. H denotes the point of a Hopf bifurcation. (a,b) $\alpha = 0$, (c,d) $\alpha = 0.25$.

As noted from Fig. 9a, away from the interaction zone, at $\delta = -8$, the energy exchange due to the internal resonance activation is not present because the frequency ratio is not exactly two, $\omega_3 / \omega_1 \approx 1.95$. Increasing the tension further brings the ratio closer to two where we notice the first sign of the interaction in the peaks. As the tension is increased, the first mode amplitude decreases and the third natural frequency tends to decrease then increase as a result of the energy exchange. Then, increasing the tension further results in a ratio that is greater than two where the interaction decreases until it diminishes away. Similarly, in Fig. 9b, we notice that the interaction results into activation of the softening band at higher values of δ compared to Fig. 9a. This implies that early signs of interaction appear first at a lower frequency ratio in the hardening before the softening appears. In addition, a Hopf bifurcation is observed when increasing the tension further because of such interaction.

To further investigate the dynamics in between the two Hopf bifurcation points, we integrate in time the modulation equations, Eqs. (31). It is well known that a limit cycle solution is born whenever a Hopf bifurcation occurs [39] and this turns the fixed points of the system to unstable solution near the bifurcation points. Therefore, we plot the response of the riser and compare it with that

obtained by numerically integrating the reduced order model from the Galerkin solution. The results are depicted in Fig. 10 showing good agreement among the results.

Figure 10. Frequency response curves showing the response of the riser between the Hopf bifurcation points of Fig. 5. $\theta = 62^\circ$, $\mu_1 = 0.08$, $\mu_2 = 0.08$, $F_m = 8$. (—) Solutions obtained by integrating Eqs. (31), (---) unstable fixed point solutions, ($\square$, $\circ$) solutions obtained by numerically integrating the Galerkin solution. H denotes the point of a Hopf bifurcation. (a) Forward sweep, (b) backward sweep.

One can observe some differences between the forward and backward sweep results from MTS indicating multi coexisting states. This can be attributed to the presence of complex dynamics, potentially chaotic, which has been reported in the Hopf bifurcation regime in similar systems undergoing a two-one internal resonance [13].

Figure 11. Bifurcation diagram of the riser's deflection around the first primary resonance by plotting the Poincare displacements for each value of α . $\theta = 62^\circ$, $\mu_1 = 0.08$, $\mu_2 = 0.08$, $F_m = 8$.

A bifurcation diagram corresponding to the Galerkin solution is shown in Fig. 11. The solution after the Hopf bifurcation between $\alpha = 0.02$ and $\alpha = 0.1$ exhibits a periodic solution represented by discrete points in the Poincare diagram and confirmed by the time histories and phase portrait plots, as shown for example in Fig. 12a and 12b. As we sweep closer to $\alpha = 0.1$, a transition in the response occurs as observed from the shift of the harmonic peak in the time history Fig. 12c and the deformed phase portrait, Fig. 12d. Further analysis for the dynamic solutions near the Hopf bifurcation can be done based on continuation algorithms combined with the Floquet theory.

Figure 12. Time histories and their corresponding phase portraits at different values of α . $\theta = 62^\circ$, $\mu_1 = 0.08$, $\mu_2 = 0.08$, $F_m = 8$. (a,b) $\alpha = 0.05$, (c,d) $\alpha = 0.115$

To further analyze the type of the response after the transition regime, the time history, the Poincare diagram, and the power spectrum are plotted for $\alpha = 0.15$ in Fig. 13.

Figure 13. (a) Time history and the corresponding (b) Poincare section and (c) power spectra based on Galerkin method for $\alpha = 0.15$, $\theta = 62^\circ$, $\mu_1 = 0.08$, $\mu_2 = 0.08$, $F_m = 8$.

The results shown in Fig. 13 indicate that the solutions corresponding to $\alpha = 0.15$ show a quasi-periodic motion. This is confirmed by the presence of the closed loop in the Poincare section in Fig. 13b and the multiple harmonics that appear in the power spectral density of the response. The influence of two-to-one internal resonance on the response is realized by plotting the temporal modal coefficients corresponding to the first and third modes from the Galerkin solution in Fig. 14. The coefficients of other participating modes are neglected from the analysis because their amplitude is very low, as clarified in Appendix C.

Figure 14. (a) (b) Time histories of a_1 and a_3 using the Galerkin method. (c) (d) Poincaré sections (e) (f) Power spectra corresponding to a_1 and a_3 , respectively. $\theta = 62^\circ$, $\mu_1 = 0.08$, $\mu_2 = 0.08$, $\alpha = 0.15$, $\bar{F}_m = 8$.

It is noticeable from the time histories in Fig. 14a and Fig. 14b that the response is quasi-periodic. This is confirmed by the closed loop observed in the Poincare sections in Fig. 14c and Fig 14d. In addition, the power spectral density functions show that new frequencies start to appear apart from the first and third natural frequencies. This can lead to chaotic behavior and can be detrimental to the lifetime of the riser structure. Lastly, we compare the solution obtained by integrating the modulation equations with the previous results obtained by Galerkin and we plot the time history, the Poincare section, and power spectrum in Fig. 15.

Figure 15. (a) Time history and the corresponding (b) Poincare section and (c) power spectra based on MTS method for $\alpha = 0.15$, $\theta = 62^\circ$, $\mu_1 = 0.08$, $\mu_2 = 0.08$, $F_m = 8$.

The time history results from Fig. 15a show good agreement with that shown in Fig. 13a, based on the Galerkin solution. Similar agreement is observed in predicting the quasi-periodic closed-loop Poincare section in Fig. 15b when compared to that of Fig. 13b. Moreover, a difference is noted when examining the power spectrum of the MTS solution Fig. 15c compared to that of Fig. 13c. This is attributed to the influence of the other modes shapes not captured in the MTS and higher order corrections.

5 Conclusions

In this paper, the two-to-one internal resonance of an inclined riser has been studied. We used the method of multiple scales to investigate the interaction between the first and third natural frequencies when their frequency ratio gets close to 2. The results show good agreement between the multiple scales solution and the Galerkin solution when the internal resonance is activated. A further

investigation has been carried out by plotting the frequency response and force response curves showing the nature of the internal resonance interaction. Also, the system show bifurcation from a stable state to a limit cycle through Hopf bifurcation, which can lead to chaotic behavior that hinders the life time riser structure undergoing these oscillations. A future prospective from this work is realized in studying the full VIV interaction undergoing internal resonance interaction via a Van Der Pol oscillator using multiple scales theory and shooting method

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Conflict of interest

The authors declare that they have no conflict of interest.

Supplementary Material

Supplementary video 1 (avi 48.4 MB)

Supplementary video 2 (avi 48.4 MB)

Supplementary information 1 (pdf 165 kB)

Appendix A: Self-adjoint proof

In this appendix, we demonstrate by virtue of using integration by parts that the solution to the linear Eq. (11) is self-adjoint. Due to the internal resonance interaction, the solution to be utilized consists of two modes, namely, $\phi_m(x)e^{\pm i\omega_m T_0}$ and $\phi_n(x)e^{\pm i\omega_n T_0}$ corresponding to modes m and n , respectively. To verify that the problem is self-adjoint, we substitute the solutions from both modes in Eq. (11), and multiply the equation with $\phi_j(x)e^{\pm i\omega_j T_0}$, then integrate by parts from $x=0$ to $x=1$ to obtain

$$\begin{aligned}
& \left[\phi_j \phi_m''' - \phi_j' \phi_m'' + \phi_j'' \phi_m' - \phi_j''' \phi_m + \bar{T}_s \phi_j \phi_m' + \bar{T}_s \phi_j' \phi_m - \bar{\sigma} x \phi_j \phi_m' + \bar{\sigma} (x \phi_j)' \phi_m - \bar{\sigma} \phi_j \phi_m \right]_0^1 \\
& \left[\phi_j \phi_n''' - \phi_j' \phi_n'' + \phi_j'' \phi_n' - \phi_j''' \phi_n + \bar{T}_s \phi_j \phi_n' + \bar{T}_s \phi_j' \phi_n - \bar{\sigma} x \phi_j \phi_n' + \bar{\sigma} (x \phi_j)' \phi_n - \bar{\sigma} \phi_j \phi_n \right]_0^1 \\
& + 2\eta \left[\phi_j y_s'' \left(\int_0^1 y_s' \phi_m' dx \right) + \phi_m y_s'' \left(\int_0^1 y_s' \phi_j' dx \right) \right]_0^1 + 2\eta \left[\phi_j y_s'' \left(\int_0^1 y_s' \phi_n' dx \right) + \phi_n y_s' \left(\int_0^1 y_s' \phi_j' dx \right) \right]_0^1 \quad (\text{A.1}) \\
& \int_0^1 \phi_m \left[-\omega_m^2 \phi_j + \phi_j^{iv} + \bar{T}_s \phi_j'' - \bar{\sigma} (x \phi_j)'' + \bar{\sigma} \phi_j' - 2\eta y_s'' \left(\int_0^1 y_s' (x) \phi_j' dx \right) \right] dx \\
& \int_0^1 \phi_n \left[-\omega_n^2 \phi_j + \phi_j^{iv} + \bar{T}_s \phi_j'' - \bar{\sigma} (x \phi_j)'' + \bar{\sigma} \phi_j' - 2\eta y_s'' \left(\int_0^1 y_s' (x) \phi_j' dx \right) \right] dx
\end{aligned}$$

We observe from Eq. (A.1) that the solution $\phi_j(x)e^{\pm i\omega_j T_0}$ satisfies the eigenvalue problem Eq. (8) and it is self-adjoint.

Appendix B: Definition of third order solvability condition coupling coefficients

In this appendix, we provide the definition of the coupling coefficients K_1 ,

K_2 , K_3 and K_4 pertaining to Eq. (23) and Eq. (24) given by

$$\begin{aligned}
K_1 = & 2\eta \int_0^1 \left(\int_0^1 \phi_n' \psi_1' dx \right) \phi_m y_s'' dx + 2\eta \int_0^1 \left(\int_0^1 \phi_m' \psi_4' dx \right) \phi_m y_s'' dx + 2\eta \int_0^1 \left(\int_0^1 (\phi_n')^2 dx \right) \phi_m \phi_m'' dx \\
& + 2\eta \int_0^1 \left(\int_0^1 y_s' \psi_4' dx \right) \phi_m \phi_m'' dx + 4\eta \int_0^1 \left(\int_0^1 \phi_m' \phi_n' dx \right) \phi_m \phi_n'' dx + 2\eta \int_0^1 \left(\int_0^1 y_s' \psi_1' dx \right) \phi_m \phi_n'' dx \\
& + 2\eta \int_0^1 \left(\int_0^1 y_s' \phi_n' dx \right) \phi_m \psi_1'' dx + 2\eta \int_0^1 \left(\int_0^1 y_s' \phi_m' dx \right) \phi_m \psi_4'' dx
\end{aligned} \quad (\text{B.1})$$

$$\begin{aligned}
K_2 = & 2\eta \int_0^1 \left(\int_0^1 \phi_m' \psi_3' dx \right) \phi_m y_s'' dx + 3\eta \int_0^1 \left(\int_0^1 (\phi_m')^2 dx \right) \phi_m \phi_m'' dx + 2\eta \int_0^1 \left(\int_0^1 y_s' \psi_3' dx \right) \phi_m \phi_m'' dx \\
& + 2\eta \int_0^1 \left(\int_0^1 y_s' \phi_m' dx \right) \phi_m \psi_3'' dx
\end{aligned} \quad (\text{B.2})$$

$$\begin{aligned}
K_3 = & 2\eta \int_0^1 \left(\int_0^1 \phi_m' \psi_1' dx \right) \phi_n y_s'' dx + 2\eta \int_0^1 \left(\int_0^1 \phi_n' \psi_3' dx \right) \phi_n y_s'' dx + 4\eta \int_0^1 \left(\int_0^1 \phi_m' \phi_n' dx \right) \phi_n \phi_m'' dx \\
& + 2\eta \int_0^1 \left(\int_0^1 y_s' \psi_1' dx \right) \phi_n \phi_m'' dx + 2\eta \int_0^1 \left(\int_0^1 (\phi_m')^2 dx \right) \phi_n \phi_n'' dx + 2\eta \int_0^1 \left(\int_0^1 y_s' \psi_3' dx \right) \phi_n \phi_n'' dx \\
& + 2\eta \int_0^1 \left(\int_0^1 y_s' \phi_m' dx \right) \phi_n \psi_1'' dx + 2\eta \int_0^1 \left(\int_0^1 y_s' \phi_n' dx \right) \phi_n \psi_3'' dx
\end{aligned} \quad (\text{B.3})$$

$$\begin{aligned}
K_4 = & 2\eta \int_0^1 \left(\int_0^1 \phi_n' \psi_2' dx \right) \phi_n y_s'' dx + 2\eta \int_0^1 \left(\int_0^1 \phi_n' \psi_4' dx \right) \phi_n y_s'' dx + 3\eta \int_0^1 \left(\int_0^1 (\phi_n')^2 dx \right) \phi_n \phi_n'' dx \\
& + 2\eta \int_0^1 \left(\int_0^1 y_s' \psi_2' dx \right) \phi_n \phi_n'' dx + 2\eta \int_0^1 \left(\int_0^1 y_s' \psi_4' dx \right) \phi_n \phi_n'' dx + 2\eta \int_0^1 \left(\int_0^1 y_s' \phi_n' dx \right) \phi_n \psi_2'' dx \\
& + 2\eta \int_0^1 \left(\int_0^1 y_s' \phi_n' dx \right) \phi_n \psi_4'' dx
\end{aligned} \tag{B.4}$$

Appendix C: Time History of Modal coefficients pertaining to the Galerkin solution

In this appendix, we plot the modal coefficients corresponding to the riser solution in Fig. 13, which has weak or negligible contribution.

Figure C.1. Time histories of a_2 , a_4 and a_5 modal coefficients from the Galerkin solution.

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