Accepted Manuscript

A semi-analytic porosity evolution scheme for simulating wormhole propagation with the Darcy-Brinkman-Forchheimer model

Jisheng Kou, Shuyu Sun, Yuanqing Wu

PII: S0377-0427(18)30542-9
DOI: <https://doi.org/10.1016/j.cam.2018.08.055>
Reference: CAM 11892

To appear in: *Journal of Computational and Applied Mathematics*

Received date : 18 October 2017
Revised date : 4 June 2018

Please cite this article as: J. Kou, S. Sun, Y. Wu, A semi-analytic porosity evolution scheme for simulating wormhole propagation with the Darcy-Brinkman-Forchheimer model, *Journal of Computational and Applied Mathematics* (2018), <https://doi.org/10.1016/j.cam.2018.08.055>

This is a PDF file of an unedited manuscript that has been accepted for publication. As a service to our customers we are providing this early version of the manuscript. The manuscript will undergo copyediting, typesetting, and review of the resulting proof before it is published in its final form. Please note that during the production process errors may be discovered which could affect the content, and all legal disclaimers that apply to the journal pertain.

A SEMI-ANALYTIC POROSITY EVOLUTION SCHEME FOR SIMULATING WORMHOLE PROPAGATION WITH THE DARCY-BRINKMAN-FORCHHEIMER MODEL *

JISHENG KOU[†], SHUYU SUN[‡], AND YUANQING WU[§]

Abstract. In this paper, we consider numerical simulation of wormhole propagation with the Darcy-Brinkman-Forchheimer model. Matrix acidization in carbonate reservoirs is a widely practiced technique in the product enhancement of the oil and gas reservoir. A wormhole, i.e. a flow channel with high porosity, is generated during reactive dissolution of carbonates by the action of the injected acid. In the wormhole forming process, the porosity changes non-uniformly in space, and it even becomes close to unity in the central regions of a wormhole. The Darcy-Brinkman-Forchheimer model accounts for both the porous media and clear fluid area, so it can be used to model wormhole propagation perfectly. This model, however, strongly depends on porosity. Therefore, the time schemes for solving the evolutionary equation of porosity have a significant effect on accuracy and stability of the numerical simulation for wormhole propagation. We propose a semi-analytic time scheme, which solves the porosity equation analytically at each time step for given acid concentration. The proposed numerical method can improve the accuracy and stability of numerical simulation significantly. For theoretical analysis of the proposed time scheme for the wormhole simulation, we first reconstruct the analytical functions of porosity to analyze the time error of the porosity, and on the basis of error estimates of porosity, we employ a coupled analysis approach to achieve the estimates of pressure, velocity and solute concentration. The time error estimates for velocity, pressure, concentration and porosity are obtained in different norms. Finally, numerical results are provided to verify the effectiveness of the proposed scheme.

Key words. Wormhole; Darcy-Brinkman-Forchheimer model; Time scheme; Error estimate.

AMS subject classifications. 65M60; 65M12; 65M15

1. Introduction. The acid treatment of carbonate reservoirs plays an important role in enhancing oil production rate when muds and fines deposit at the perforated well pore pipe [4, 6, 9, 17, 19, 21, 23]. In the technique, acid is injected into matrix to dissolve the rocks and deposits around the well bore, and a channel with high porosity is formed. Such channel is called a wormhole. Oil and gas components in the reservoir can be pushed to the surface easily through the wormhole.

Numerical simulation based on the Darcy framework is applicable for the scenarios where the change of porosity is small, but it can be quite incorrect in cases where there is a large change of porosity (and consequently the permeability). Apparently, there are situations in which the porosity becomes close to unity, and as a result, there may exist an area of clear fluid within the porous medium. It is worth accounting for both the porous media and clear fluid area, and the Darcy-Brinkman-Forchheimer framework exactly deals with the problems [11, 24]. In [13], the Darcy-Forchheimer model is used to model the flow of high velocity in nonuniform porosity instead of the conventional Darcy framework. In this work, for the purpose of retrieving more

*This work is supported by National Natural Science Foundation of China (No. 11601345), Natural Science Foundation of SZU (No. 2017059), and Peacock Plan Foundation of Shenzhen (No. 000255).

[†]School of Mathematics and Statistics, Hubei Engineering University, Xiaogan 432000, Hubei, China.

[‡]Computational Transport Phenomena Laboratory, Division of Physical Science and Engineering, King Abdullah University of Science and Technology, Thuwal 23955-6900, Kingdom of Saudi Arabia. Email: shuyu.sun@kaust.edu.sa.

[§]Corresponding author. College of Mathematics and Statistics, Shenzhen University, Shenzhen, 518060, China. Email: wyq@szu.edu.cn.

reasonable results, numerical modeling is based on the Darcy-Brinkman-Forchheimer framework, which is more capable of handling with the multiple complication factors in wormhole propagation, including the boundary-layer development, macroscopic shear stress, microscopic shear stress and inertial force.

Numerical methods for simulating matrix acidization in carbonate reservoirs have been extensively studied in [4, 6, 9, 13–17, 19, 21, 23–25]. In [13], a mixed finite element scheme based on the Darcy-Forchheimer model has been suggested for the numerical simulation of wormholes, in which mixed finite element methods are used not only for the Darcy-Forchheimer flow equations but also for the solute transport equation by introducing an auxiliary flux variable to guarantee full mass conservation. The block-centered finite difference methods for simulating wormhole propagation have been developed and analyzed in [15, 16]. However, there are a few efforts on the numerical methods for the time discretization.

An essential feature of wormhole dynamics is that by the action of acids the porosity changes largely and irregularly with time. Therefore, the time schemes of porosity have a significant effect on accuracy and stability of the numerical simulation for wormhole propagation. A semi-implicit time scheme has been presented in [13], in which the porosity is implicitly treated in the equation of porosity. In this paper, we propose a semi-analytic time scheme, and its great virtue is that the equation of porosity is analytically solved at each time step for given acid concentration. So the proposed numerical method can improve significantly the accuracy and stability of the numerical simulation. For theoretical analysis of the proposed time scheme for wormhole simulation, the key issues are to estimate the error of porosity with time and to handle with the complication resulted from the fully coupling relation of multivariables, including porosity, pressure, velocity and solute concentration. Analytical functions of porosity are reconstructed to analyze the time error of the porosity. On the basis of error estimates of porosity, a coupled analysis approach is employed to achieve the estimates of pressure, velocity and solute concentration.

The rest of this paper is organized as follows. In Section 2, we describe the mathematical model for wormhole problems. In Section 3, a semi-analytic time method is proposed to simulate the wormhole problems. In Section 4, we analyze the time error estimates of the proposed method. In Section 5, we carry out numerical experiments to verify the proposed method. Finally, we make some conclusions in Section 6.

2. Mathematical model of the wormhole problem. A fundamental feature of wormhole propagation is that the advancement of chemical reaction front generally can not propagate uniformly along the acid injection direction. In fact, it usually advances in one certain direction more than the other directions. The heterogeneity of porosity and permeability in subsurface formations also promotes the chemical reaction front to be non-uniform. Because of acidization the porosity of a porous medium (denoted by ϕ) becomes very heterogeneous in space and varies with time in wormhole propagation, and as a result, the permeability K is also isotropic. The Darcy framework is usually applicable for the scenarios where the change of porosity is small, but it can be quite incorrect in cases where there is a large change of porosity (and consequently the permeability). Apparently, the porosity may become very large (but less than one) in the wormhole channels, and thus, the fluid velocity may be fast and close to the velocity of the clear fluid flow. Since the Brinkman model accounts for transitional flows between boundaries of porous media and clear fluids [3], and the Forchheimer model considers the inertial effect of high velocity, the Darcy-Brinkman-Forchheimer model that accounts for higher velocity and nonuni-

form porosity is strongly suggested to describe matrix acidization for the purpose of retrieving more reasonable results [24].

We denote the pressure by p and denote the velocity by $\mathbf{u}$. The flow model can be formulated by the continuity equation and the Darcy-Brinkman-Forchheimer flow equation as

$$\gamma \frac{\partial p}{\partial t} + \frac{\partial \phi}{\partial t} + \nabla \cdot \mathbf{u} = f, \quad (2.1)$$

$$\frac{\rho}{\phi} \frac{\partial \mathbf{u}}{\partial t} + \frac{\mu}{K} \mathbf{u} + \frac{\rho F}{\sqrt{K}} |\mathbf{u}| \mathbf{u} + \frac{\rho}{\phi^2} \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) - \frac{\mu}{\phi} \nabla^2 \mathbf{u} = -\nabla p + \rho \mathbf{g}, \quad (2.2)$$

where t is the time, $F = \frac{1.75}{\sqrt{150\phi^3}}$ is the Forchheimer number, μ is the fluid viscosity, ρ is the mass density and $\mathbf{g}$ is the gravity vector. In the continuity equation (2.1), γ is a pseudo-compressibility parameter [10] that accounts for a slight change of the fluid density. The second term at the left-hand side of (2.1) accounts for the effect of local volume changes during dissolution. f is the source term that is the sum of f_P and f_I , where f_P and f_I are production and injection rates respectively. The flow equation (2.2) is based on the general Navier-Stokes equation, and extended by the Darcy term, the Brinkman term and the Forchheimer term. In (2.2), $|\mathbf{u}|$ represents the 2-norm of $\mathbf{u}$, and the term $-\frac{\mu}{K} \mathbf{u} - \nabla p + \rho \mathbf{g}$ is the Darcy term. For the flow with very high velocity in porous media, inertial effects may become significant, therefore the inertial term $\frac{\rho F}{\sqrt{K}} |\mathbf{u}| \mathbf{u}$, which is called the Forchheimer term [7, 12, 20], is taken into account in (2.2). The term $\frac{\mu}{\phi} \nabla^2 \mathbf{u}$ is the Brinkman term [3].

The transport equation of acid in the fluid phase can be written as [19]

$$\frac{\partial(\phi c_f)}{\partial t} + \nabla \cdot (\mathbf{u} c_f) - \nabla \cdot (\phi \mathbf{D}_e \nabla c_f) = k_c a_v (c_s - c_f) + f_P c_f + f_I c_I, \quad (2.3)$$

where c_f is the cup-mixing concentration of the acid in the fluid phase, $\mathbf{D}_e$ stands for the effective dispersion tensor, k_c represents the local mass-transfer coefficient, a_v represents the interfacial area available for reaction per unit volume of the medium, and c_I stands for the injected concentration. The variable c_s is the concentration of the acid at the fluid-solid interface. The relationship between c_f and c_s is expressed as

$$c_s = \frac{c_f}{1 + k_s/k_c}, \quad (2.4)$$

where k_s is the surface reaction rate constant. In (2.3), the three terms at the left-hand side accounts for the accumulation, convection and dispersion of the acid respectively, and the first term at the right-hand side represents the transfer of acid species between the fluid phase and the fluid-solid interface [19]. Here, the effective dispersion tensor $\mathbf{D}_e$ is defined as

$$\mathbf{D}_e(\mathbf{u}) = d_m \mathbf{I} + |\mathbf{u}| \{ \alpha_l \mathbf{E}(\mathbf{u}) + \alpha_t (\mathbf{I} - \mathbf{E}(\mathbf{u})) \}, \quad (2.5)$$

where

$$(\mathbf{E}(\mathbf{u}))_{ij} = \frac{u_i u_j}{|\mathbf{u}|^2}, \quad 1 \leq i, j \leq d. \quad (2.6)$$

In (2.5), $d_m > 0$ represents the molecular diffusivity; α_l and α_t represent the longitudinal dispersivity and transverse dispersivity that are positive.

The evolution of porosity in the domain has the following form

$$\frac{\partial \phi}{\partial t} = \frac{\alpha k_c a_v (c_f - c_s)}{\rho_s}, \quad (2.7)$$

where α is the dissolving power of acid and ρ_s is the density of the solid phase.

The relationship between the permeability and the porosity is established by the Carman-Kozeny correlation [18]

$$\frac{K}{K_0} = \frac{\phi}{\phi_0} \left(\frac{\phi(1 - \phi_0)}{\phi_0(1 - \phi)} \right)^2, \quad (2.8)$$

where K_0 and ϕ_0 are the initial permeability and porosity respectively.

The interfacial area is calculated from porosity and permeability as

$$\frac{a_v}{a_0} = \frac{\phi}{\phi_0} \sqrt{\frac{K_0 \phi}{K \phi_0}}, \quad (2.9)$$

where a_0 is the initial interfacial area.

The mathematical model of wormhole propagation in porous media is formulated by a nonlinear system of equations including the mass conservation equation (Eq. (2.1)), the Darcy-Brinkman-Forchheimer equation (Eq. (2.2)), the solute reaction transport equation (Eq. (2.3)) and a series of semi-empirical equations that bridge the scale differences (Eqs. (2.4), (2.7), (2.8) and (2.9)).

We now describe the initial-boundary conditions of the wormhole problem. The initial pressure p_0 , initial velocity $\mathbf{u}_0$ and initial solute concentration c_0 are provided, and the initial porosity ϕ_0 , initial permeability K_0 and initial interfacial area a_0 are given as well. We assume $0 \leq c_0 \leq 1$. The following boundary conditions are used for the wormhole problem:

$$\mathbf{u} = 0, \quad \mathbf{x} \in \partial\Omega, \quad t \in (0, T],$$

$$(\mathbf{u}c_f - \phi \mathbf{D}_e \nabla c_f) \cdot \mathbf{n} = 0, \quad \mathbf{x} \in \partial\Omega, \quad t \in (0, T],$$

where $\Omega \subset \mathbf{R}^d (d = 2, 3)$ is a polygonal and bounded domain and $\mathbf{n}$ denotes the unit outward normal vector to the boundary $\partial\Omega$.

For the physical quantities, we assume that $0 < \phi_{0*} \leq \phi_0 \leq \phi_0^* < 1$, $0 < K_{0*} \leq K_0 \leq K_0^*$ and $0 \leq a_{0*} \leq a_0 \leq a_0^*$. We also assume that ρ, μ, k_c and k_s are all given positive constants. Because of the boundedness of a_0 , R_ϕ is also bounded as $0 \leq R_{\phi*} \leq R_\phi \leq R_\phi^*$.

3. A semi-analytic time method for simulating the wormhole problem.

We denote $R_\phi = \frac{\alpha a_0 k_c k_s}{\rho_s (k_c + k_s)}$. By the relations (2.4), (2.8) and (2.9), the evolution of porosity can be rewritten as

$$\frac{\partial \phi}{\partial t} = \frac{R_\phi (1 - \phi)}{1 - \phi_0} c_f. \quad (3.1)$$

With the relation (2.4), the transport equation (2.3) can be rewritten as

$$\frac{\partial(\phi c_f)}{\partial t} + \nabla \cdot (\mathbf{u}c_f) - \nabla \cdot (\phi \mathbf{D}_e \nabla c_f) = -\frac{k_c k_s}{k_c + k_s} a_v c_f + f_P c_f + f_I c_I. \quad (3.2)$$

Mixed finite element methods [2] are popularly applied to solve (3.2), so we reformulate it by a mixed form. To do this, we define an auxiliary flux variable $\mathbf{q}$ as

$$\mathbf{q} = \mathbf{u}c_f - \phi \mathbf{D}_e \nabla c_f. \quad (3.3)$$

Then (3.2) is reformulated as

$$\frac{\partial(\phi c_f)}{\partial t} + \nabla \cdot \mathbf{q} = -\frac{k_c k_s}{k_c + k_s} a_v c_f + f_P c_f + f_I c_I. \quad (3.4)$$

Furthermore, we define an auxiliary function of velocity as

$$\mathbf{M}_e(\mathbf{u}) = (\mathbf{D}_e(\mathbf{u}))^{-1} \mathbf{u}. \quad (3.5)$$

We define a cut-off operator $\mathcal{M}$ for the velocity as

$$\mathcal{M}(\mathbf{u}) := \begin{cases} \mathbf{u}, & \text{if } |\mathbf{u}| \leq M, \\ M\mathbf{u}/|\mathbf{u}|, & \text{if } |\mathbf{u}| > M, \end{cases} \quad (3.6)$$

where M is a given large positive constant.

Let the total time interval $J = (0, T]$, where $T > 0$. We divide J into N uniform subintervals $I_n = (t_n, t_{n+1}]$, where $t_0 = 0$ and $t_N = T$, and denote $\delta t = t_{n+1} - t_n$. For any scalar variable $v(t)$ or vector variable $\mathbf{u}(t)$, we denote by v^n or $\mathbf{u}^n$ its approximation at the time point t_n . A semi-implicit time discrete scheme is stated as

$$\gamma \frac{p^{n+1} - p^n}{\delta t} + \frac{\partial \phi^{n+1}}{\partial t} + \nabla \cdot \mathbf{u}^{n+1} = f(\mathbf{x}, t_{n+1}), \quad (3.7)$$

$$\begin{aligned} & \frac{\rho}{\phi^{n+1}} \frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\delta t} + \frac{\mu}{K(\phi^{n+1})} \mathbf{u}^{n+1} + \frac{\rho F(\phi^{n+1})}{\sqrt{K(\phi^{n+1})}} |\mathbf{u}_M^n| \mathbf{u}^{n+1} \\ & + \frac{\rho}{(\phi^{n+1})^2} \nabla \cdot (\mathbf{u}_M^n \otimes \mathbf{u}^{n+1}) - \frac{\mu}{\phi^{n+1}} \nabla^2 \mathbf{u}^{n+1} = -\nabla p^{n+1} + \rho \mathbf{g}, \end{aligned} \quad (3.8)$$

$$\begin{aligned} & \frac{\phi^{n+1} c_f^{n+1} - \phi^n c_f^n}{\delta t} + \nabla \cdot \mathbf{q}^{n+1} + \frac{k_c k_s}{k_c + k_s} a_v (\phi^{n+1}) c_f^{n+1} \\ & - f_P(\mathbf{x}, t^{n+1}) c_f^{n+1} = f_I(\mathbf{x}, t^{n+1}) c_I(\mathbf{x}, t^{n+1}), \end{aligned} \quad (3.9)$$

$$(\phi^{n+1} \mathbf{D}_e(\mathbf{u}_M^{n+1}))^{-1} \mathbf{q}^{n+1} = (\phi^{n+1})^{-1} \mathbf{M}_e(\mathbf{u}^{n+1}) c_f^{n+1} + \nabla c_f^{n+1}, \quad (3.10)$$

where $\mathbf{u}_M^n = \mathcal{M}(\mathbf{u}^n)$. The time derivative of porosity is approximated by

$$\frac{\partial \phi^{n+1}}{\partial t} = \frac{R_\phi (1 - \phi^{n+1})}{1 - \phi_0} \bar{c}_f^n, \quad t \in (t_n, t_{n+1}], \quad (3.11)$$

where $\bar{c}_f^n = \max(0, \min(c_f^n, 1))$.

We now focus on the discrete schemes of porosity. Based on the explicit treatment for the concentration and implicit treatment for the porosity in the porosity equation, a semi-implicit (SIM) scheme has been presented in [13]

$$\frac{\phi^{n+1} - \phi^n}{\delta t} = \frac{R_\phi (1 - \phi^{n+1})}{1 - \phi_0} \bar{c}_f^n. \quad (3.12)$$

We introduce the following notation

$$\theta_n = \delta t \frac{R_\phi}{1 - \phi_0} \bar{c}_f^n. \quad (3.13)$$

From (3.12), the porosity ϕ^{n+1} can be calculated as

$$\phi^{n+1} = G_{\text{sim}}(\phi^n, \theta_n), \quad G_{\text{sim}}(\phi, \theta) = \frac{\theta + \phi}{1 + \theta}. \quad (3.14)$$

In this work, we propose a semi-analytic (SAN) time scheme for the porosity by directly solving the following semi-discrete porosity equation

$$\frac{\partial \phi}{\partial t} = \frac{R_\phi (1 - \phi)}{1 - \phi_0} \bar{c}_f^n, \quad t \in (t_n, t_{n+1}]. \quad (3.15)$$

The solution of (3.15) gives us the following calculation formulation:

$$\phi^{n+1} = G_{\text{san}}(\phi^n, \theta_n), \quad G_{\text{san}}(\phi, \theta) = 1 - (1 - \phi) e^{-\theta}. \quad (3.16)$$

We note that the proposed method (3.16) can be reduced to (3.14) by substituting the rational approximation $e^{-\theta} \approx \frac{1}{1+\theta}$ into (3.14). We also note that the porosity ϕ^{n+1} computed by both methods satisfies the maximum principle for any time step size. Indeed, supposing that $0 < \phi^n < 1$, we deduce from (3.16) that $0 < \phi^n \leq \phi^{n+1} < 1$ holds for any time step size. By recurrence relation, we obtain $0 < \phi_0 \leq \phi^n < 1$ for $n \geq 1$. Apparently, the method (3.14) also has this property.

THEOREM 3.1. *The discrete values of porosity computed by two methods satisfy*

$$G_{\text{san}}(\phi^n, 0) = G_{\text{sim}}(\phi^n, 0) = \phi^n, \quad (3.17)$$

$$G_{\text{san}}(\phi^n, \theta_n) > G_{\text{sim}}(\phi^n, \theta_n), \quad \theta_n > 0. \quad (3.18)$$

Proof. The equation (3.17) is straightforward. The difference between two schemes is written as

$$\begin{aligned} G_{\text{san}}(\phi^n, \theta_n) - G_{\text{sim}}(\phi^n, \theta_n) &= \frac{1 - \phi^n}{1 + \theta_n} - (1 - \phi^n) e^{-\theta_n} \\ &= \frac{1 - \phi^n}{1 + \theta_n} (1 - (1 + \theta_n) e^{-\theta_n}). \end{aligned} \quad (3.19)$$

The inequality (3.18) is obtained from the monotonicity of the scalar function $(1 + x) e^{-x}$ on the interval $[0, +\infty)$. $\square$

4. Error estimates of semi-discrete time schemes.

4.1. Preliminary results. The traditional notations are used for the Lebesgue spaces $L^p(\Omega)$ and Sobolev spaces $W^{k,p}(\Omega)$ with $1 \leq p \leq \infty$ over the domain Ω . In particular, $W^{k,2}(\Omega)$ is denoted by $H^k(\Omega)$. The corresponding norms are denoted by $\|\cdot\|_{L^p(\Omega)}$, $\|\cdot\|_{W^{k,p}(\Omega)}$ and $\|\cdot\|_{H^k(\Omega)}$ respectively. The inner product of $L^p(\Omega)$ or $(L^p(\Omega))^d$ is denoted by $(\cdot, \cdot)$, and the inner product of the functions in the boundary Γ is denoted by $(\cdot, \cdot)_\Gamma$. We also use $|\cdot|$ to represent the 2-norm of a vector in $\mathbb{R}^d$

and a matrix in $\mathbb{R}^{d \times d}$. The norms for a vector-value function $\mathbf{v} \in (L^p(\Omega))^d$ and a matrix-value function $\mathbf{A} \in (L^p(\Omega))^{d \times d}$ are defined as

$$\|\mathbf{v}\|_{(L^p(\Omega))^d} := \|\mathbf{v}\|_{L^p(\Omega)}, \quad \|\mathbf{A}\|_{(L^p(\Omega))^{d \times d}} := \|\mathbf{A}\|_{L^p(\Omega)}.$$

For notation simplicity, the norms $\|\cdot\|_{L^2(\Omega)}$, $\|\cdot\|_{(L^2(\Omega))^d}$ and $\|\cdot\|_{(L^2(\Omega))^{d \times d}}$ are also written as $\|\cdot\|$ for scalar-value, vector-value, and matrix-value functions respectively. Let T be the final simulation time, then for a given normed space X and $1 \leq p \leq \infty$, we define

$$L^p(0, T; X) := \left\{ \varphi : \varphi(t) \in X, \quad \|\varphi\|_X \in L^p(0, T) \right\},$$

which is a normed linear space equipped with the norm given as

$$\|\varphi\|_{L^p(0, T; X)} := \|(\|\varphi\|_X)\|_{L^p(0, T)}.$$

We define two functional spaces as

$$\mathbf{V} = \{\mathbf{v} \in (H^1(\Omega))^d : \mathbf{v} = 0 \text{ on } \partial\Omega\},$$

$$\mathbf{H} = \{\mathbf{w} \in (L^2(\Omega))^d : \nabla \cdot \mathbf{w} \in L^2(\Omega), \quad \mathbf{w} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}.$$

Let $J = (0, T]$. We assume that the exact solutions are sufficiently smooth and bounded. We also assume that $f_P, f_I, c_I \in L^\infty(J; L^\infty(\Omega))$ and $f_P \leq 0, f_I \geq 0, 0 \leq c_I \leq 1$.

For the cut-off operator for velocity defined in (3.6), we assume that the constant M is chosen as $M > 2\|\mathbf{u}\|_{(L^\infty(J \times \Omega))^d}$. Under this assumption, using the similar approach in [13], we deduce that $|\mathbf{u}_M^n - \mathbf{u}(t_n)| \leq |\mathbf{u}^n - \mathbf{u}(t_n)|$ pointwise.

Let $\mathcal{W} = L^2(\Omega)$. Suppose that $p^n, c^n \in \mathcal{W}$ and $\mathbf{u}^n \in \mathbf{V}, \mathbf{q}^n \in \mathbf{H}$ ($0 \leq n \leq N-1$) are given or calculated, the semi-implicit mixed weak formulation is stated as below: find $p^{n+1}, c_f^{n+1} \in \mathcal{W}$ and $\mathbf{u}^{n+1} \in \mathbf{V}, \mathbf{q}^{n+1} \in \mathbf{H}$ such that

$$\gamma \left(\frac{p^{n+1} - p^n}{\delta t}, \varphi \right) + \left(\frac{\partial \phi^{n+1}}{\partial t}, \varphi \right) + (\nabla \cdot \mathbf{u}^{n+1}, \varphi) = (f(\mathbf{x}, t^{n+1}), \varphi), \quad \varphi \in \mathcal{W} \quad (4.1)$$

$$\begin{aligned} & \left(\frac{\rho}{\phi^{n+1}} \frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\delta t}, \mathbf{v} \right) + \left(\frac{\mu}{K(\phi^{n+1})} \mathbf{u}^{n+1}, \mathbf{v} \right) \\ & + \left(\frac{\rho F(\phi^{n+1})}{\sqrt{K(\phi^{n+1})}} |\mathbf{u}_M^n| \mathbf{u}^{n+1}, \mathbf{v} \right) - \left(\frac{\rho}{\phi^2} \mathbf{u}_M^n \otimes \mathbf{u}^{n+1}, \nabla \mathbf{v} \right) \\ & + \left(\frac{\mu}{\phi} \nabla \mathbf{u}^{n+1}, \nabla \mathbf{v} \right) = (p^{n+1}, \nabla \cdot \mathbf{v}) + (\rho \mathbf{g}, \mathbf{v}), \quad \mathbf{v} \in \mathbf{V}, \end{aligned} \quad (4.2)$$

$$\begin{aligned} & \left(\frac{\phi^{n+1} c_f^{n+1} - \phi^n c_f^n}{\delta t}, \psi \right) + (\nabla \cdot \mathbf{q}^{n+1}, \psi) + \frac{k_c k_s}{k_c + k_s} (a_v(\phi^{n+1}) c_f^{n+1}, \psi) \\ & - (f_P(\mathbf{x}, t^{n+1}) c_f^{n+1}, \psi) = (f_I(\mathbf{x}, t^{n+1}) c_I(\mathbf{x}, t^{n+1}), \psi), \quad \psi \in \mathcal{W}, \end{aligned} \quad (4.3)$$

$$\begin{aligned} ((\phi^{n+1} \mathbf{D}_e(\mathbf{u}_M^{n+1}))^{-1} \mathbf{q}^{n+1}, \mathbf{w}) &= \left((\phi^{n+1})^{-1} \mathbf{M}_e(\mathbf{u}^{n+1}) c_f^{n+1}, \mathbf{w} \right) \\ &\quad + \left(c_f^{n+1}, \nabla \cdot \mathbf{w} \right), \quad \mathbf{w} \in \mathbf{H}. \end{aligned} \quad (4.4)$$

We now estimate the errors of discrete time schemes with a uniform time step size δt . For a variable $v(\mathbf{x}, t)$, we use the notation $v^n = v(\mathbf{x}, t_n)$. The discrete time errors are defined as

$$\begin{aligned} E_\phi^n &= \phi(t_n) - \phi^n, & E_p^n &= p(t_n) - p^n, & E_{\mathbf{u}}^n &= \mathbf{u}(t_n) - \mathbf{u}^n, \\ E_c^n &= c_f(t_n) - c_f^n, & E_{\mathbf{q}}^n &= \mathbf{q}(t_n) - \mathbf{q}^n. \end{aligned}$$

In what follows, C is used to indicate a generic positive constant independent of δt , but probably having different values in different occurrences.

4.2. Error estimates for discrete porosity. We first consider the error bounds of the discrete porosity computed by (3.14) and (3.16).

LEMMA 4.1. *For $0 \leq m \leq N - 1$, the approximate error of the discrete porosity computed by (3.14) satisfies*

$$\begin{aligned} \|E_\phi^{m+1}\|^2 + \sum_{n=0}^m \|E_\phi^{n+1} - E_\phi^n\|^2 &\leq 2\delta t R_\phi^* \sum_{n=0}^m \left(\|E_\phi^{n+1}\|^2 + \|E_c^n\|^2 \right) \\ &\quad + C(\delta t)^2 \left(\int_{t_0}^{t_{m+1}} \left\| \frac{\partial c_f}{\partial t} \right\|^2 dt + \int_{t_0}^{t_{m+1}} \left\| \frac{\partial^2 \phi}{\partial t^2} \right\|^2 dt \right). \end{aligned} \quad (4.5)$$

Proof. Firstly, it is easy to know that

$$\frac{E_\phi^{n+1} - E_\phi^n}{\delta t} = -\frac{R_\phi E_\phi^{n+1}}{1 - \phi_0} c_f(t_{n+1}) + \frac{R_\phi (1 - \phi^{n+1})}{1 - \phi_0} (c_f(t_{n+1}) - \bar{c}_f^n) + \pi_\phi^{n+1}, \quad (4.6)$$

where

$$\pi_\phi^{n+1} = \frac{\phi(t_{n+1}) - \phi(t_n)}{\delta t} - \frac{\partial \phi}{\partial t}(t_{n+1}). \quad (4.7)$$

Multiplying (4.6) by E_ϕ^{n+1} , and then integrating it over Ω , we obtain that

$$\begin{aligned} \frac{1}{2\delta t} \left(\|E_\phi^{n+1}\|^2 - \|E_\phi^n\|^2 + \|E_\phi^{n+1} - E_\phi^n\|^2 \right) &= - \left(\frac{R_\phi}{1 - \phi_0} c_f(t_{n+1}) E_\phi^{n+1}, E_\phi^{n+1} \right) \\ &\quad + \left(\frac{R_\phi}{1 - \phi_0} (1 - \phi^{n+1}) (c_f(t_{n+1}) - \bar{c}_f^n), E_\phi^{n+1} \right) + \left(\pi_\phi^{n+1}, E_\phi^{n+1} \right) \\ &\leq \left(\frac{R_\phi}{1 - \phi_0} (1 - \phi^{n+1}) (c_f(t_{n+1}) - c_f(t_n) + c_f(t_n) - \bar{c}_f^n), E_\phi^{n+1} \right) + \left(\pi_\phi^{n+1}, E_\phi^{n+1} \right) \\ &\leq R_\phi^* \left(\|E_\phi^{n+1}\|^2 + \frac{1}{2} \|E_c^n\|^2 \right) + C \|\pi_\phi^{n+1}\|^2 + C \|c_f(t_{n+1}) - c_f(t_n)\|^2 \\ &\leq R_\phi^* \left(\|E_\phi^{n+1}\|^2 + \|E_c^n\|^2 \right) + C\delta t \int_{t_n}^{t_{n+1}} \left\| \frac{\partial^2 \phi}{\partial t^2} \right\|^2 dt + C\delta t \int_{t_n}^{t_{n+1}} \left\| \frac{\partial c_f}{\partial t} \right\|^2 dt, \end{aligned} \quad (4.8)$$

where two constants C depend R_ϕ and we have also used the following estimates

$$\|\pi_\phi^{n+1}\|^2 \leq \frac{1}{3} \delta t \int_{t_n}^{t_{n+1}} \left\| \frac{\partial^2 \phi}{\partial t^2} \right\|^2 dt, \quad (4.9)$$

$$\|c_f(t_{n+1}) - c_f(t_n)\|^2 \leq \delta t \int_{t_n}^{t_{n+1}} \left\| \frac{\partial c_f}{\partial t} \right\|^2 dt. \quad (4.10)$$

The inequality (4.5) is obtained by summing (4.8) from $n = 0$ to m and taking into account $\|E_\phi^0\| = 0$. $\square$

LEMMA 4.2. *For $0 \leq m \leq N - 1$, the approximate error of the discrete porosity computed by (3.16) satisfies*

$$\|E_\phi^{m+1}\|^2 \leq C\delta t \sum_{n=0}^m \|E_c^n\|^2 + C(\delta t)^2 \int_{t_0}^{t_{m+1}} \left\| \frac{\partial c_f}{\partial t} \right\|^2 dt, \quad (4.11)$$

$$\sum_{n=0}^m \|E_\phi^{n+1} - E_\phi^n\|^2 \leq C\delta t \sum_{n=0}^m (\|E_\phi^n\|^2 + \|E_c^n\|^2) + C(\delta t)^2 \int_{t_0}^{t_{m+1}} \left\| \frac{\partial c_f}{\partial t} \right\|^2 dt \quad (4.12)$$

Proof. The solution of the porosity evolution equation in the time subinterval $(t_n, t_{n+1}]$ can be expressed as

$$\phi(t) = 1 - (1 - \phi(t_n)) \exp \left(-\frac{R_\phi}{1 - \phi_0} \int_{t_n}^t c_f(s) ds \right). \quad (4.13)$$

It is deduced from (3.16) and (4.13) that

$$\begin{aligned} E_\phi^{n+1} &= (1 - \phi^n) \exp \left(-\frac{R_\phi \delta t}{1 - \phi_0} \bar{c}_f^n \right) - (1 - \phi(t_n)) \exp \left(-\frac{R_\phi}{1 - \phi_0} \int_{t_n}^{t_{n+1}} c_f(t) dt \right) \\ &= E_\phi^n \exp \left(-\frac{R_\phi \delta t}{1 - \phi_0} \bar{c}_f^n \right) \\ &\quad + (1 - \phi(t_n)) \left(\exp \left(-\frac{R_\phi \delta t \bar{c}_f^n}{1 - \phi_0} \right) - \exp \left(-\frac{R_\phi}{1 - \phi_0} \int_{t_n}^{t_{n+1}} c_f(t) dt \right) \right). \end{aligned} \quad (4.14)$$

Since $\exp \left(-\frac{R_\phi \delta t}{1 - \phi_0} \bar{c}_f^n \right) \leq 1$ and $0 < \phi < 1$, from (4.14), we derive that

$$\begin{aligned} \|E_\phi^{n+1}\| &\leq \|E_\phi^n\| + \left\| 1 - \exp \left(\frac{R_\phi}{1 - \phi_0} \int_{t_n}^{t_{n+1}} (\bar{c}_f^n - c_f(t)) dt \right) \right\| \\ &\leq \|E_\phi^n\| + C \left\| \int_{t_n}^{t_{n+1}} (\bar{c}_f^n - c_f(t)) dt \right\| \\ &\leq \|E_\phi^n\| + C\delta t \|E_c^n\| + C \left\| \int_{t_n}^{t_{n+1}} (c_f(t_n) - c_f(t)) dt \right\| \\ &\leq \|E_\phi^n\| + C\delta t \|E_c^n\| + C \int_{t_n}^{t_{n+1}} \|c_f(t_n) - c_f(t)\| dt \\ &\leq \|E_\phi^n\| + C\delta t \|E_c^n\| + C \int_{t_n}^{t_{n+1}} \left\| \int_{t_n}^t \frac{\partial c_f}{\partial s} ds \right\| dt \\ &\leq \|E_\phi^n\| + C\delta t \|E_c^n\| + C \int_{t_n}^{t_{n+1}} \int_{t_n}^t \left\| \frac{\partial c_f}{\partial s} \right\| ds dt \\ &\leq \|E_\phi^n\| + C\delta t \|E_c^n\| + C\delta t \int_{t_n}^{t_{n+1}} \left\| \frac{\partial c_f}{\partial t} \right\| dt, \end{aligned} \quad (4.15)$$

where the second inequality is obtained using the following bound

$$\frac{R_\phi}{1-\phi_0} \int_{t_n}^{t_{n+1}} (\bar{c}_f^n - c_f(t)) dt \leq \frac{R_\phi^* \delta t}{1-\phi_{0,*}} \leq \frac{R_\phi^* T}{1-\phi_{0,*}}.$$

Summing the inequality (4.15) from $n = 0$ to m leads to

$$\|E_\phi^{m+1}\| \leq C\delta t \sum_{n=0}^m \|E_c^n\| + C\delta t \int_{t_0}^{t_{m+1}} \left\| \frac{\partial c_f}{\partial t} \right\| dt. \quad (4.16)$$

Furthermore, we get

$$\begin{aligned} \|E_\phi^{m+1}\|^2 &\leq C(\delta t)^2 \left(\sum_{n=0}^m \|E_c^n\| \right)^2 + C(\delta t)^2 \left(\int_{t_0}^{t_{m+1}} \left\| \frac{\partial c_f}{\partial t} \right\| dt \right)^2 \\ &\leq C(\delta t)^2 (m+1) \sum_{n=0}^m \|E_c^n\|^2 + C(\delta t)^2 T \int_{t_0}^{t_{m+1}} \left\| \frac{\partial c_f}{\partial t} \right\|^2 dt. \end{aligned} \quad (4.17)$$

Taking into account $(m+1)\delta t \leq T$, we obtain (4.11) from (4.17).

We now turn to prove the estimate (4.12). Subtract E_ϕ^n from both sides of (4.14) as

$$\begin{aligned} E_\phi^{n+1} - E_\phi^n &= E_\phi^n \left(\exp \left(-\frac{R_\phi \delta t}{1-\phi_0} \bar{c}_f^n \right) - 1 \right) \\ &\quad + (1-\phi(t_n)) \left(\exp \left(-\frac{R_\phi \delta t}{1-\phi_0} \bar{c}_f^n \right) - \exp \left(-\frac{R_\phi}{1-\phi_0} \int_{t_n}^{t_{n+1}} c_f(t) dt \right) \right). \end{aligned} \quad (4.18)$$

Since $\left| \exp \left(-\frac{R_\phi \delta t}{1-\phi_0} \bar{c}_f^n \right) - 1 \right| \leq \frac{R_\phi}{1-\phi_0} \delta t$, by the similar routine used in (4.15), we obtain

$$\|E_\phi^{n+1} - E_\phi^n\| \leq C\delta t (\|E_\phi^n\| + \|E_c^n\|) + C\delta t \int_{t_n}^{t_{n+1}} \left\| \frac{\partial c_f}{\partial t} \right\| dt. \quad (4.19)$$

Furthermore, we derive that

$$\|E_\phi^{n+1} - E_\phi^n\|^2 \leq C(\delta t)^2 (\|E_\phi^n\|^2 + \|E_c^n\|^2) + C(\delta t)^3 \int_{t_n}^{t_{n+1}} \left\| \frac{\partial c_f}{\partial t} \right\|^2 dt. \quad (4.20)$$

The estimate (4.12) is a direct result of (4.20). $\square$

4.3. Error estimates. The error analysis for the discrete pressure, velocity, and concentration along with the auxiliary flux variable is carried out in the following lemmas. We note that Lemma 4.3 and Lemma 4.4 hold both for the proposed schemes with SIM scheme and SAN scheme for the porosity equation.

LEMMA 4.3. *For $0 \leq m \leq N-1$, the errors of discrete pressure and velocity satisfy*

$$\begin{aligned} &\|E_p^{m+1}\|^2 + \|E_u^{m+1}\|^2 + \delta t \sum_{n=0}^m (\|E_u^{n+1}\|^2 + \|\nabla E_u^{n+1}\|^2) \\ &\leq C\delta t \sum_{n=0}^{m+1} (\|E_u^n\|^2 + \|E_\phi^n\|^2 + \|E_p^n\|^2) + C\delta t \sum_{n=0}^m \|E_c^n\|^2 \end{aligned}$$

$$+ C(\delta t)^2 \int_{t_0}^{t_{m+1}} \left(\left\| \frac{\partial \mathbf{u}}{\partial t} \right\|^2 + \left\| \frac{\partial^2 \mathbf{u}}{\partial t^2} \right\|^2 + \left\| \frac{\partial c_f}{\partial t} \right\|^2 + \left\| \frac{\partial^2 p}{\partial t^2} \right\|^2 \right) dt. \quad (4.21)$$

Proof. The proof is listed in Appendix A. $\square$

LEMMA 4.4. For $0 \leq m \leq N-1$, the errors of discrete concentration and auxiliary flux variable satisfy

$$\begin{aligned} & \left\| (\phi^{n+1})^{1/2} E_c^{n+1} \right\|^2 + \delta t \sum_{n=0}^m \left\| (\phi^{n+1} \mathbf{D}_e(\mathbf{u}_M^{n+1}))^{-1/2} E_{\mathbf{q}}^{n+1} \right\|^2 \\ & \leq C \sum_{n=0}^m \left\| E_{\phi}^{n+1} - E_{\phi}^n \right\|^2 + C \delta t \sum_{n=0}^m \left(\left\| E_c^{n+1} \right\|^2 + \left\| E_{\mathbf{u}}^{n+1} \right\|^2 + \left\| E_{\phi}^{n+1} \right\|^2 \right) \\ & + C(\delta t)^2 \int_{t_0}^{t_{m+1}} \left\| \frac{\partial^2(\phi c_f)}{\partial t^2} \right\|^2 dt. \end{aligned} \quad (4.22)$$

Proof. The proof is listed in Appendix B. $\square$

Based on the above lemmas, we are now able to prove the main error estimates as below.

THEOREM 4.1. For $1 \leq m \leq N$, the discrete solutions satisfy the following error estimates

$$\begin{aligned} & \left\| E_{\phi}^m \right\|^2 + \left\| E_p^m \right\|^2 + \left\| E_{\mathbf{u}}^m \right\|^2 + \left\| E_c^m \right\|^2 \\ & + \delta t \sum_{n=1}^m \left(\left\| E_{\mathbf{q}}^n \right\|^2 + \left\| \nabla E_{\mathbf{u}}^n \right\|^2 \right) \leq C(\delta t)^2. \end{aligned} \quad (4.23)$$

Proof. It is followed from Lemmas 4.1-4.4 that for $0 \leq m \leq N-1$,

$$\begin{aligned} & \left\| E_{\phi}^{m+1} \right\|^2 + \left\| E_p^{m+1} \right\|^2 + \left\| E_{\mathbf{u}}^{m+1} \right\|^2 + \left\| E_c^{m+1} \right\|^2 \\ & + \delta t \sum_{n=0}^m \left\| E_{\mathbf{q}}^{n+1} \right\|^2 + \delta t \sum_{n=0}^m \left(\left\| E_{\mathbf{u}}^{n+1} \right\|^2 + \left\| \nabla E_{\mathbf{u}}^{n+1} \right\|^2 \right) \\ & \leq C \delta t \sum_{n=0}^m \left(\left\| E_p^{n+1} \right\|^2 + \left\| E_c^{n+1} \right\|^2 + \left\| E_{\mathbf{u}}^{n+1} \right\|^2 + \left\| E_{\phi}^{n+1} \right\|^2 \right) + C(\delta t)^2. \end{aligned} \quad (4.24)$$

Here, the values of different C depend on T and the exact solutions, which are a bit different for the two time schemes, but independent of δt . Thus, the estimate (4.23) is obtained by applying the discrete Gronwall's lemma to (4.24). $\square$

5. Numerical experiments. In this section, we carry out the numerical experiments to verify the theoretical analysis and effectiveness of the proposed scheme. In these examples, we use the finite volume method on the staggered grid, which is equivalent to a special mixed finite element method [1, 8]. The sparse direct solver UMFPACK [5] is used to solve the linear system.

5.1. Dissolution experiment. In the dissolution experiment, a 1-meter by 0.2-meter porous medium is considered. The initial porosity in the porous medium complies with the continuously uniform distribution and the range is from 0.05 to 0.35. The initial permeability of the porous medium also obeys the continuously uniform distribution and the average value is 10^{-8}m^2 . A kind of acid is injected

Fig. 5.1: The contours of concentration computed by the SIM (left column) and SAN (right column) schemes at the 10th(top row), the 30th(central row), and the 50th(bottom row) time step respectively. The color from blue to red means the range of the concentration from 0 to 10 mole/m².

from the left boundary of the porous medium and the velocity is 0.005m/s; that is, $q_l(x, y, t) = 0.005$ at the boundary $x = 0$ otherwise it equals zero. The acid goes out of the porous medium from the right boundary and the velocity is also 0.005m/s; that is, $q_r(x, y, t) = -0.005$ at the boundary $x = 1$ otherwise it equals zero. The up and down boundaries of the porous medium are closed. The initial pressure inside the porous medium is 10^5 Pa. The other properties of the acid and the porous medium can be seen in Table 5.1. The parameter γ is set to 0.01. The simulation period is 1.6×10^5 seconds and each time step is 3200 seconds. The configurations of the concentration and porosity are shown in Figure 5.1 and 5.2 respectively.

From the results, it can be seen that both methods can generate a major worm-

Fig. 5.2: The contours of porosity computed by the SIM (left column) and SAN (right column) schemes at the 10th(top row), the 30th(central row), and the 50th(bottom row) time step respectively. The color from blue to red means the range of porosity from 0 to 1.

hole, while the SAN's major wormhole propagates faster than the SIM's. This experiment demonstrates the correctness of the theoretical results proved in Theorem 3.1.

5.2. Convergence Experiment. In order to consider the convergence of both methods, we carry out the convergence experiment. In the experiment, the parameters are the same as the dissolution experiment, but the total number of time steps are increased from 50, 100 to 300. The configurations of the concentration and porosity are given in Figures 5.3 and 5.4 respectively. From the results, it can be observed that with the time step becoming finer, the results of both methods approach each other gradually, which indicates the correctness of the theoretical convergence analysis for

Table 5.1: Properties of acid and porous medium in the dissolution experiment

Property	Value
Viscosity of acid (μ)	$10^{-2}\text{Pa}\cdot\text{s}$
Density of acid (ρ)	$10^3\text{kg}/\text{m}^3$
Longitudinal dispersion coefficient	10^{-3}
Transverse dispersion coefficient	10^{-4}
Molecular diffusion coefficient	10^{-9}
Surface reaction rate constant (k_s)	$10\text{m}/\text{s}$
Local mass-transfer coefficient (k_c)	$1\text{m}/\text{s}$
Dissolving power of acid (α)	$0.1\text{kg}/\text{mole}$
Density of the rock (ρ_s)	$2500\text{kg}/\text{m}^3$
Initial interfacial area (a_0)	0.5m^{-1}
Injection concentration of acid ($c_I \times 10\text{mole}/\text{m}^3$)	$10\text{mole}/\text{m}^3$

both methods. We also observe that the SAN method converges more rapidly than the SIM method because the former has higher accuracy in the computation of the porosity equation.

5.3. 3D Experiment. Based on the new SAN method, we expand the former experiment to the 3D case. The experimental parameters are the same as shown in Table 5.1. The matrix dimension is of 0.2-m length in the x-direction, 1-m length in the y-direction and 0.1-m length in the z-direction. The acid goes into the matrix from one face perpendicular to the y-direction and out from another face perpendicular to the y-direction. The other faces of the matrix are closed. The 1.5 Iso-surface of the concentration, the 0.7 Iso-surface of the porosity and the streamlines at a moment can be seen in Figure 5.5 respectively. A main wormhole can be seen from the results.

6. Conclusions. A semi-analytic time scheme is proposed to simulate the wormhole propagation with the Darcy-Brinkman-Forchheimer model. The porosity equation is solved analytically at each time step for the given acid concentration, so the proposed numerical method can improve the accuracy and stability of the numerical simulation significantly. For the theoretical analysis, the analytical functions of porosity are reconstructed to analyze the time error of the porosity, and on the basis of error estimates of porosity, a coupled analysis approach is employed to achieve the estimates of pressure, velocity and solute concentration. The time error estimates for velocity, pressure, concentration and porosity are derived in different norms. Finally, numerical experiments verify the effectiveness of the proposed scheme.

Appendix A. The proof of Lemma 4.3.

We now consider the proof of Lemma 4.3. Firstly, from the discrete formulations of pressure and velocity, we obtain

$$\gamma \left(\frac{E_p^{n+1} - E_p^n}{\delta t}, \varphi \right) + (\nabla \cdot E_{\mathbf{u}}^{n+1}, \varphi) = (\pi_{\phi}^{n+1}, \varphi) + \gamma (\pi_p^{n+1}, \varphi), \quad (\text{A.1})$$

$$\begin{aligned} & \left(\frac{\rho}{\phi^{n+1}} \frac{E_{\mathbf{u}}^{n+1} - E_{\mathbf{u}}^n}{\delta t}, \mathbf{v} \right) + \left(\frac{\mu}{K(\phi^{n+1})} E_{\mathbf{u}}^{n+1}, \mathbf{v} \right) \\ & + \left(\frac{\rho F(\phi^{n+1})}{\sqrt{K(\phi^{n+1})}} (|\mathbf{u}(t_{n+1})| \mathbf{u}(t_{n+1}) - |\mathbf{u}_M^n| \mathbf{u}^{n+1}), \mathbf{v} \right) \end{aligned}$$

Fig. 5.3: The contours of concentration computed by the SIM (left column) and SAN (right column) schemes using different total number of time steps 50(top row), 100(central row), and 300(bottom row). The color from blue to red means the range of the concentration from 0 to 10 mole/m².

$$\begin{aligned}
 & + \left(\frac{\mu}{\phi^{n+1}} \nabla E_{\mathbf{u}}^{n+1}, \nabla \mathbf{v} \right) \\
 = & (E_p^{n+1}, \nabla \cdot \mathbf{v}) + \left(\frac{\rho E_{\phi}^{n+1}}{\phi^{n+1} \phi(t_{n+1})} \frac{\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n)}{\delta t}, \mathbf{v} \right) \\
 & + \left(\left(\frac{\mu}{K(\phi^{n+1})} - \frac{\mu}{K(\phi(t_{n+1}))} \right) \mathbf{u}(t_{n+1}), \mathbf{v} \right) \\
 & + \left(\left(\frac{\rho F(\phi^{n+1})}{\sqrt{K(\phi^{n+1})}} - \frac{\rho F(\phi(t_{n+1}))}{\sqrt{K(\phi(t_{n+1}))}} \right) |\mathbf{u}(t_{n+1})| \mathbf{u}(t_{n+1}), \mathbf{v} \right)
 \end{aligned}$$

Fig. 5.4: The contours of porosity computed by the SIM (left column) and SAN (right column) schemes using different total number of time steps 50(top row), 100(central row), and 300(bottom row) respectively. The color from blue to red means the range of the porosity from 0 to 1.

$$\begin{aligned}
 & + \left(\frac{\rho}{\phi(t_{n+1})} \pi_{\mathbf{u}}^{n+1}, \mathbf{v} \right) + \left(\left(\frac{\mu}{\phi^{n+1}} - \frac{\mu}{\phi(t_{n+1})} \right) \nabla \mathbf{u}(t_{n+1}), \nabla \mathbf{v} \right) \\
 & + \left(\frac{\rho}{(\phi(t_{n+1}))^2} \mathbf{u}(t_{n+1}) \otimes \mathbf{u}(t_{n+1}) - \frac{\rho}{(\phi^{n+1})^2} \mathbf{u}_M^n \otimes \mathbf{u}^{n+1}, \nabla \mathbf{v} \right), \quad (\text{A.2})
 \end{aligned}$$

where

$$\pi_{\phi}^{n+1} = \frac{\partial \phi^{n+1}}{\partial t} - \frac{\partial \phi(t_{n+1})}{\partial t},$$

(a)

(b)

(c)

Fig. 5.5: The 1.5 Iso-surface of concentration(a), the 0.7 Iso-surface of porosity(b), and the streamlines(c) in the 3D example computed by the SAN scheme.

$$\begin{aligned}\pi_p^{n+1} &= \frac{p(t_{n+1}) - p(t_n)}{\delta t} - \frac{\partial p(t_{n+1})}{\partial t}, \\ \pi_{\mathbf{u}}^{n+1} &= \frac{\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n)}{\delta t} - \frac{\partial \mathbf{u}(t_{n+1})}{\partial t}.\end{aligned}$$

Taking $\varphi = E_p^{n+1}$ in (A.1) and $\mathbf{v} = E_{\mathbf{u}}^{n+1}$ in (A.2), and then summing these two equations, we get

$$\begin{aligned}& \frac{\gamma}{2\delta t} \left(\|E_p^{n+1}\|^2 - \|E_p^n\|^2 + \|E_p^{n+1} - E_p^n\|^2 \right) + \frac{1}{\delta t} \left(\frac{\rho}{\phi^{n+1}} (E_{\mathbf{u}}^{n+1} - E_{\mathbf{u}}^n), E_{\mathbf{u}}^{n+1} \right) \\& + \left\| \frac{\sqrt{\mu}}{\sqrt{K(\phi^{n+1})}} E_{\mathbf{u}}^{n+1} \right\|^2 + \left\| \frac{\sqrt{\mu}}{\sqrt{\phi^{n+1}}} \nabla E_{\mathbf{u}}^{n+1} \right\|^2 \\& = \left(\frac{\rho F(\phi^{n+1})}{\sqrt{K(\phi^{n+1})}} (|\mathbf{u}_M^n| \mathbf{u}^{n+1} - |\mathbf{u}(t_{n+1})| \mathbf{u}(t_{n+1})), E_{\mathbf{u}}^{n+1} \right) \\& + \left(\frac{\rho E_{\phi}^{n+1}}{\phi^{n+1} \phi(t_{n+1})} \frac{\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n)}{\delta t}, E_{\mathbf{u}}^{n+1} \right) \\& + \left(\left(\frac{\mu}{K(\phi^{n+1})} - \frac{\mu}{K(\phi(t_{n+1}))} \right) \mathbf{u}(t_{n+1}), E_{\mathbf{u}}^{n+1} \right) \\& + \left(\left(\frac{\rho F(\phi^{n+1})}{\sqrt{K(\phi^{n+1})}} - \frac{\rho F(\phi(t_{n+1}))}{\sqrt{K(\phi(t_{n+1}))}} \right) |\mathbf{u}(t_{n+1})| \mathbf{u}(t_{n+1}), E_{\mathbf{u}}^{n+1} \right) \\& + (\pi_{\phi}^{n+1}, E_p^{n+1}) + \left(\frac{\rho}{\phi(t_{n+1})} \pi_{\mathbf{u}}^{n+1}, E_{\mathbf{u}}^{n+1} \right) + \gamma (\pi_p^{n+1}, E_p^{n+1}) \\& + \left(\left(\frac{\mu}{\phi^{n+1}} - \frac{\mu}{\phi(t_{n+1})} \right) \nabla \mathbf{u}(t_{n+1}), \nabla E_{\mathbf{u}}^{n+1} \right) \\& + \left(\frac{\rho}{(\phi(t_{n+1}))^2} \mathbf{u}(t_{n+1}) \otimes \mathbf{u}(t_{n+1}) - \frac{\rho}{(\phi^{n+1})^2} \mathbf{u}_M^n \otimes \mathbf{u}^{n+1}, \nabla E_{\mathbf{u}}^{n+1} \right) \\& := \sum_{i=1}^9 T_i.\end{aligned}\tag{A.3}$$

The second term on the left-hand side of (A.3) is bounded below as

$$\begin{aligned}\frac{1}{\delta t} \left(\frac{\rho}{\phi^{n+1}} (E_{\mathbf{u}}^{n+1} - E_{\mathbf{u}}^n), E_{\mathbf{u}}^{n+1} \right) &= \frac{1}{2\delta t} \left(\|\lambda^{n+1} E_{\mathbf{u}}^{n+1}\|^2 - \|\lambda^{n+1} E_{\mathbf{u}}^n\|^2 \right) \\& + \frac{1}{2\delta t} \|\lambda^{n+1} (E_{\mathbf{u}}^{n+1} - E_{\mathbf{u}}^n)\|^2 \\& \geq \frac{1}{2\delta t} \left(\|\lambda^{n+1} E_{\mathbf{u}}^{n+1}\|^2 - \|\lambda^n E_{\mathbf{u}}^n\|^2 \right),\end{aligned}\tag{A.4}$$

where $\lambda^n = \sqrt{\frac{\rho}{\phi^n}}$ and we have used the property of discrete porosity, i.e., $\phi^{n+1} \geq \phi^n$ almost everywhere.

We now turn to estimate the terms on the right-hand side of (A.3). The first term can be estimated as

$$T_1 = \left(\frac{\rho F(\phi^{n+1})}{\sqrt{K(\phi^{n+1})}} (|\mathbf{u}(t_n)| - |\mathbf{u}(t_{n+1})|) \mathbf{u}(t_{n+1}), E_{\mathbf{u}}^{n+1} \right)$$

$$\begin{aligned}
 & + \left(\frac{\rho F(\phi^{n+1})}{\sqrt{K(\phi^{n+1})}} (|\mathbf{u}_M^n| - |\mathbf{u}(t_n)|) \mathbf{u}(t_{n+1}), E_{\mathbf{u}}^{n+1} \right) \\
 & - \left(\frac{\rho F(\phi^{n+1})}{\sqrt{K(\phi^{n+1})}} |\mathbf{u}_M^n| E_{\mathbf{u}}^{n+1}, E_{\mathbf{u}}^{n+1} \right) \\
 & \leq C \|\mathbf{u}(t_n) - \mathbf{u}(t_{n+1})\|^2 + C \|E_{\mathbf{u}}^n\|^2 + \frac{1}{6} \mu \left\| K(\phi^{n+1})^{-1/2} E_{\mathbf{u}}^{n+1} \right\|^2 \\
 & \leq C \|E_{\mathbf{u}}^n\|^2 + C \delta t \int_{t_n}^{t_{n+1}} \left\| \frac{\partial \mathbf{u}}{\partial t} \right\|^2 dt + \frac{1}{6} \mu \left\| K(\phi^{n+1})^{-1/2} E_{\mathbf{u}}^{n+1} \right\|^2. \quad (\text{A.5})
 \end{aligned}$$

Using the boundedness of ϕ^n and the properties of functions F and K , we obtain the estimates

$$T_2 + T_3 + T_4 \leq \frac{1}{6} \mu \left\| K(\phi^{n+1})^{-1/2} E_{\mathbf{u}}^{n+1} \right\|^2 + C \|E_{\phi}^{n+1}\|^2, \quad (\text{A.6})$$

Taking into account the formulations of $\frac{\partial \phi^{n+1}}{\partial t}$ and $\frac{\partial \phi(t_{n+1})}{\partial t}$, we derive

$$\begin{aligned}
 T_5 & \leq \frac{1}{2} \left(\|\pi_{\phi}^{n+1}\|^2 + \|E_p^{n+1}\|^2 \right) \\
 & \leq \frac{1}{2} \|E_p^{n+1}\|^2 + \frac{1}{2} \left\| \frac{R_{\phi}(1 - \phi(t_{n+1}))}{1 - \phi_0} c_f(t_{n+1}) - \frac{R_{\phi}(1 - \phi^{n+1})}{1 - \phi_0} \bar{c}_f^n \right\|^2 \\
 & \leq \frac{1}{2} \|E_p^{n+1}\|^2 + C \left(\|c_f(t_{n+1}) - c_f(t_n)\|^2 + \|E_{\phi}^{n+1}\|^2 + \|E_c^n\|^2 \right) \\
 & \leq \frac{1}{2} \|E_p^{n+1}\|^2 + C \left(\|E_{\phi}^{n+1}\|^2 + \|E_c^n\|^2 \right) + C \delta t \int_{t_n}^{t_{n+1}} \left\| \frac{\partial c_f}{\partial t} \right\|^2 dt. \quad (\text{A.7})
 \end{aligned}$$

The approximate property of $\pi_{\mathbf{u}}^{n+1}$ and π_p^{n+1} , which is similar to (4.9), gives the estimates as

$$\begin{aligned}
 T_6 & \leq \frac{1}{6} \mu \left\| K(\phi^{n+1})^{-1/2} E_{\mathbf{u}}^{n+1} \right\|^2 + C \|\pi_{\mathbf{u}}^{n+1}\|^2 \\
 & \leq \frac{1}{6} \mu \left\| K(\phi^{n+1})^{-1/2} E_{\mathbf{u}}^{n+1} \right\|^2 + C \delta t \int_{t_n}^{t_{n+1}} \left\| \frac{\partial^2 \mathbf{u}}{\partial t^2} \right\|^2 dt, \quad (\text{A.8})
 \end{aligned}$$

$$\begin{aligned}
 T_7 & \leq \frac{\gamma}{2} \left(\|\pi_p^{n+1}\|^2 + \|E_p^{n+1}\|^2 \right) \\
 & \leq \frac{\gamma}{2} \|E_p^{n+1}\|^2 + C \delta t \int_{t_n}^{t_{n+1}} \left\| \frac{\partial^2 p}{\partial t^2} \right\|^2 dt, \quad (\text{A.9})
 \end{aligned}$$

$$T_8 \leq \frac{\mu}{4} \|\nabla E_{\mathbf{u}}^{n+1}\|^2 + C \|E_{\phi}^{n+1}\|^2. \quad (\text{A.10})$$

The last term is estimated as

$$\begin{aligned}
 T_9 & = \left(\left(\frac{\rho}{(\phi(t_{n+1}))^2} - \frac{\rho}{(\phi^{n+1})^2} \right) \mathbf{u}(t_{n+1}) \otimes \mathbf{u}(t_{n+1}), \nabla E_{\mathbf{u}}^{n+1} \right) \\
 & + \left(\frac{\rho}{(\phi^{n+1})^2} (\mathbf{u}(t_{n+1}) \otimes \mathbf{u}(t_{n+1}) - \mathbf{u}(t_n) \otimes \mathbf{u}(t_{n+1})), \nabla E_{\mathbf{u}}^{n+1} \right)
 \end{aligned}$$

$$\begin{aligned}
 & + \left(\frac{\rho}{(\phi^{n+1})^2} (\mathbf{u}(t_n) \otimes \mathbf{u}(t_{n+1}) - \mathbf{u}_M^n \otimes \mathbf{u}(t_{n+1})), \nabla E_{\mathbf{u}}^{n+1} \right) \\
 & + \left(\frac{\rho}{(\phi^{n+1})^2} (\mathbf{u}_M^n \otimes \mathbf{u}(t_{n+1}) - \mathbf{u}_M^n \otimes \mathbf{u}^{n+1}), \nabla E_{\mathbf{u}}^{n+1} \right) \\
 & \leq \frac{\mu}{4} \|\nabla E_{\mathbf{u}}^{n+1}\|^2 + C\|E_{\phi}^{n+1}\|^2 + C\|\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n)\|^2 \\
 & \quad + C\|\mathbf{u}(t_n) - \mathbf{u}_M^n\|^2 + C\|E_{\mathbf{u}}^{n+1}\|^2 \\
 & \leq \frac{\mu}{4} \|\nabla E_{\mathbf{u}}^{n+1}\|^2 + C\|E_{\phi}^{n+1}\|^2 + C\delta t \int_{t_n}^{t_{n+1}} \left\| \frac{\partial \mathbf{u}}{\partial t} \right\|^2 dt \\
 & \quad + C\|E_{\mathbf{u}}^n\|^2 + C\|E_{\mathbf{u}}^{n+1}\|^2. \tag{A.11}
 \end{aligned}$$

Finally, we combine the above estimates and obtain

$$\begin{aligned}
 & \frac{\gamma}{2\delta t} \left(\|E_p^{n+1}\|^2 - \|E_p^n\|^2 \right) + \frac{1}{2\delta t} \left(\|\lambda^{n+1} E_{\mathbf{u}}^{n+1}\|^2 - \|\lambda^n E_{\mathbf{u}}^n\|^2 \right) \\
 & + \frac{\mu}{2} \left\| K(\phi^{n+1})^{-1/2} E_{\mathbf{u}}^{n+1} \right\|^2 + \frac{\mu}{2} \|\nabla E_{\mathbf{u}}^{n+1}\|^2 \\
 & \leq C \left(\|E_{\mathbf{u}}^n\|^2 + \|E_{\mathbf{u}}^{n+1}\|^2 + \|E_{\phi}^{n+1}\|^2 + \|E_c^n\|^2 \right) + \frac{1+\gamma}{2} \|E_p^{n+1}\|^2 \\
 & \quad + C\delta t \int_{t_n}^{t_{n+1}} \left(\left\| \frac{\partial \mathbf{u}}{\partial t} \right\|^2 + \left\| \frac{\partial^2 \mathbf{u}}{\partial t^2} \right\|^2 + \left\| \frac{\partial c_f}{\partial t} \right\|^2 + \left\| \frac{\partial^2 p}{\partial t^2} \right\|^2 \right) dt. \tag{A.12}
 \end{aligned}$$

Summing (A.12) in time steps from 0 to m ($0 \leq m \leq N-1$), we reach the estimate (4.21).

Appendix B. The proof of Lemma 4.4.

The following results proved in [13, 22] show the properties of $\mathbf{D}_e$ and $\mathbf{M}_e$, which will be used in the theoretical analysis.

LEMMA B.1. *Let $\mathbf{D}_e(\mathbf{u})$ be defined as in (2.5) and $d_m > 0$. Suppose that α_l and α_t are non-negative functions of domain $\mathbf{x} \in \Omega$ and are uniformly bounded as $\alpha_l(\mathbf{x}) \leq \alpha_l^*$ and $\alpha_t(\mathbf{x}) \leq \alpha_t^*$. Then for $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$,*

$$|\mathbf{D}_e(\mathbf{u})^{-1} \mathbf{v}| \leq (d_m + \min(\alpha_l, \alpha_t) |\mathbf{u}|)^{-1} |\mathbf{v}| \leq d_m^{-1} |\mathbf{v}|, \tag{B.1}$$

$$|\mathbf{D}_e(\mathbf{u}) \mathbf{v}| \leq (d_m + \max(\alpha_l^*, \alpha_t^*) |\mathbf{u}|) |\mathbf{v}|, \tag{B.2}$$

$$|\mathbf{D}_e(\mathbf{u})^{-1} - \mathbf{D}_e(\mathbf{v})^{-1}| \leq d_m^{-2} (7\alpha_t^* + 6\alpha_l^*) d^{3/2} |\mathbf{u} - \mathbf{v}|, \tag{B.3}$$

where d is the dimension of domain Ω .

LEMMA B.2. *Let $\mathbf{M}_e(\mathbf{u})$ be defined as in (3.5) and $d_m > 0$. Suppose that α_l and α_t are positive functions of domain $\mathbf{x} \in \Omega$ and $0 < \alpha_{l*} \leq \alpha_l(\mathbf{x}) \leq \alpha_l^*$ and $0 < \alpha_{t*} \leq \alpha_t(\mathbf{x}) \leq \alpha_t^*$. Then for $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$,*

$$|\mathbf{M}_e(\mathbf{u})| \leq (\min(\alpha_{l*}, \alpha_{t*}))^{-1}, \tag{B.4}$$

$$|\mathbf{M}_e(\mathbf{u}) - \mathbf{M}_e(\mathbf{v})| \leq L_M |\mathbf{u} - \mathbf{v}|, \tag{B.5}$$

where $L_M = d_m^{-1} + (d_m \min(\alpha_{l*}, \alpha_{t*}))^{-1} (7\alpha_t^* + 6\alpha_l^*) d^{3/2}$ and d is the dimension of domain Ω .

We now prove Lemma 4.4. Noticing that

$$\begin{aligned} & \phi(t_{n+1})c_f(t_{n+1}) - \phi(t_n)c_f(t_n) - \phi^{n+1}c_f^{n+1} + \phi^n c_f^n \\ &= \phi^{n+1}E_c^{n+1} - \phi^n E_c^n + c_f(t_{n+1})E_\phi^{n+1} - c_f(t_n)E_\phi^n, \end{aligned} \quad (\text{B.6})$$

we obtain the transport equation satisfying

$$\begin{aligned} & \left(\frac{\phi^{n+1}E_c^{n+1} - \phi^n E_c^n}{\delta t}, \psi \right) + (\nabla \cdot E_\mathbf{q}^{n+1}, \psi) + \frac{k_c k_s}{k_c + k_s} (a_v(\phi^{n+1})E_c^{n+1}, \psi) \\ &= (\pi_c^{n+1}, \psi) - \left(\frac{c_f(t_{n+1})E_\phi^{n+1} - c_f(t_n)E_\phi^n}{\delta t}, \psi \right) \\ &+ \frac{k_c k_s}{k_c + k_s} ((a_v(\phi^{n+1}) - a_v(\phi(t_{n+1})))c_f(t_{n+1}), \psi) + (f_P E_c^{n+1}, \psi), \end{aligned} \quad (\text{B.7})$$

$$\begin{aligned} & ((\phi^{n+1}\mathbf{D}_e(\mathbf{u}_M^{n+1}))^{-1}E_\mathbf{q}^{n+1}, \mathbf{w}) = (E_c^{n+1}, \nabla \cdot \mathbf{w}) + ((\phi^{n+1})^{-1}\mathbf{M}_e(\mathbf{u}^{n+1})E_c^{n+1}, \mathbf{w}) \\ &+ (((\phi^{n+1}\mathbf{D}_e(\mathbf{u}_M^{n+1}))^{-1} - (\phi(t_{n+1})\mathbf{D}_e(\mathbf{u}(t_{n+1})))^{-1})\mathbf{q}(t_{n+1}), \mathbf{w}) \\ &+ ((\phi(t_{n+1})^{-1}\mathbf{M}_e(\mathbf{u}(t_{n+1})) - (\phi^{n+1})^{-1}\mathbf{M}_e(\mathbf{u}^{n+1}))c_f(t_{n+1}), \mathbf{w}), \end{aligned} \quad (\text{B.8})$$

where

$$\pi_c^{n+1} = \frac{\phi(t_{n+1})c_f(t_{n+1}) - (t_n)c_f(t_n)}{\delta t} - \frac{\partial(\phi(t_{n+1})c_f(t_{n+1}))}{\partial t}.$$

Taking $\psi = E_c^{n+1}$ in (B.7) and $\mathbf{w} = E_\mathbf{q}^{n+1}$ in (B.8) respectively, we sum these two equations

$$\begin{aligned} & \left(\frac{\phi^{n+1}E_c^{n+1} - \phi^n E_c^n}{\delta t}, E_c^{n+1} \right) + \left\| (\phi^{n+1}\mathbf{D}_e(\mathbf{u}_M^{n+1}))^{-1/2} E_\mathbf{q}^{n+1} \right\|^2 \\ &+ \frac{k_c k_s}{k_c + k_s} (a_v(\phi^{n+1})E_c^{n+1}, E_c^{n+1}) \\ &= (\pi_c^{n+1}, E_c^{n+1}) + \left(\frac{c_f(t_n)E_\phi^n - c_f(t_{n+1})E_\phi^{n+1}}{\delta t}, E_c^{n+1} \right) \\ &+ (f_P E_c^{n+1}, E_c^{n+1}) + \frac{k_c k_s}{k_c + k_s} ((a_v(\phi^{n+1}) - a_v(\phi(t_{n+1})))c_f(t_{n+1}), E_c^{n+1}) \\ &+ ((\phi^{n+1})^{-1}\mathbf{M}_e(\mathbf{u}^{n+1})E_c^{n+1}, E_\mathbf{q}^{n+1}) \\ &+ (((\phi^{n+1}\mathbf{D}_e(\mathbf{u}_M^{n+1}))^{-1} - (\phi(t_{n+1})\mathbf{D}_e(\mathbf{u}(t_{n+1})))^{-1})\mathbf{q}(t_{n+1}), E_\mathbf{q}^{n+1}) \\ &+ ((\phi(t_{n+1})^{-1}\mathbf{M}_e(\mathbf{u}(t_{n+1})) - (\phi^{n+1})^{-1}\mathbf{M}_e(\mathbf{u}^{n+1}))c_f(t_{n+1}), E_\mathbf{q}^{n+1}) \\ &:= \sum_{i=1}^7 T_i. \end{aligned} \quad (\text{B.9})$$

Because $\phi^{n+1} \geq \phi^n$, for the first term on the left-hand side of (B.9), we have

$$(\phi^{n+1}E_c^{n+1} - \phi^n E_c^n, E_c^{n+1}) = \left\| (\phi^{n+1})^{1/2} E_c^{n+1} \right\|^2 - (\phi^n E_c^n, E_c^{n+1})$$

$$\begin{aligned}
 &\geq \left\| (\phi^{n+1})^{1/2} E_c^{n+1} \right\|^2 \\
 &\quad - \frac{1}{2} \left(\left\| (\phi^n)^{1/2} E_c^n \right\|^2 + \left\| (\phi^n)^{1/2} E_c^{n+1} \right\|^2 \right) \\
 &\geq \frac{1}{2} \left(\left\| (\phi^{n+1})^{1/2} E_c^{n+1} \right\|^2 - \left\| (\phi^n)^{1/2} E_c^n \right\|^2 \right) \quad (\text{B.10})
 \end{aligned}$$

Since k_c, k_s and $a_v(\phi^{n+1})$ are all nonnegative, the third term on the left-hand side of (B.9) is also nonnegative.

The first term on the right-hand side of (B.9) is estimated as

$$\begin{aligned}
 T_1 &\leq \frac{1}{2} \left(\|E_c^{n+1}\|^2 + \|\pi_c^{n+1}\|^2 \right) \\
 &\leq \frac{1}{2} \|E_c^{n+1}\|^2 + \frac{1}{6} \delta t \int_{t_n}^{t_{n+1}} \left\| \frac{\partial^2(\phi c_f)}{\partial t^2} \right\|^2 dt. \quad (\text{B.11})
 \end{aligned}$$

The second term on the right-hand side of (B.9) is estimated as

$$\begin{aligned}
 T_2 &\leq \frac{1}{2} \|E_c^{n+1}\|^2 + \frac{1}{2\delta t} \left\| c_f(t_n) E_\phi^n - c_f(t_{n+1}) E_\phi^{n+1} \right\|^2 \\
 &\leq \frac{1}{2} \|E_c^{n+1}\|^2 + \frac{1}{\delta t} \left\| c_f(t_n) (E_\phi^n - E_\phi^{n+1}) \right\|^2 + \frac{1}{\delta t} \left\| (c_f(t_n) - c_f(t_{n+1})) E_\phi^{n+1} \right\|^2 \\
 &\leq \frac{1}{2} \|E_c^{n+1}\|^2 + \frac{1}{\delta t} \|E_\phi^{n+1} - E_\phi^n\|^2 + C \|E_\phi^{n+1}\|^2. \quad (\text{B.12})
 \end{aligned}$$

Since $f_P \leq 0$, the third term $(f_P E_c^{n+1}, E_c^{n+1}) \leq 0$. Noticing that $0 \leq a_v(\phi^{n+1}) \leq a_0$ and $|a_v(\phi^{n+1}) - a_v(\phi(t_{n+1}))| \leq \frac{a_0}{1-\phi_0} |\phi^{n+1}|$, we obtain the estimates

$$T_4 \leq \frac{1}{2} \|E_c^{n+1}\|^2 + \frac{1}{2} \frac{(k_c k_s a_0^*)^2}{(k_c + k_s)^2 (1 - \phi_{0*})^2} \|c_f\|_{L^\infty}^2 \|E_\phi^{n+1}\|^2. \quad (\text{B.13})$$

Using the boundedness of $\mathbf{M}_e(\mathbf{u}_h)$ and $\mathbf{D}_e(\mathbf{u}_M^n)$, we get

$$T_5 \leq \frac{1}{4} \left\| (\phi^{n+1} \mathbf{D}_e(\mathbf{u}_M^{n+1}))^{-1/2} E_{\mathbf{q}}^{n+1} \right\|^2 + C \|E_c^{n+1}\|^2. \quad (\text{B.14})$$

From the hypothesis regarding the choice of M , we have $|\mathbf{u}_M^n - \mathbf{u}| \leq |\mathbf{u}^n - \mathbf{u}|$ pointwise, so using the Lipschitz continuity of $\mathbf{M}_e(\cdot)$ and $\mathbf{D}_e(\cdot)$, we get

$$T_6 + T_7 \leq \frac{1}{4} \left\| (\phi^{n+1} \mathbf{D}_e(\mathbf{u}_M^{n+1}))^{-1/2} E_{\mathbf{q}}^{n+1} \right\|^2 + C \left(\|E_{\mathbf{u}}^{n+1}\|^2 + \|E_\phi^{n+1}\|^2 \right). \quad (\text{B.15})$$

We summarize the above estimates and get

$$\begin{aligned}
 &\frac{1}{2\delta t} \left(\left\| (\phi^{n+1})^{1/2} E_c^{n+1} \right\|^2 - \left\| (\phi^n)^{1/2} E_c^n \right\|^2 \right) + \frac{1}{2} \left\| (\phi^{n+1} \mathbf{D}_e(\mathbf{u}_M^{n+1}))^{-1/2} E_{\mathbf{q}}^{n+1} \right\|^2 \\
 &\leq \frac{1}{\delta t} \|E_\phi^{n+1} - E_\phi^n\|^2 + C \left(\|E_c^{n+1}\|^2 + \|E_{\mathbf{u}}^{n+1}\|^2 + \|E_\phi^{n+1}\|^2 \right) \\
 &\quad + \frac{1}{6} \delta t \int_{t_n}^{t_{n+1}} \left\| \frac{\partial^2(\phi c_f)}{\partial t^2} \right\|^2 dt. \quad (\text{B.16})
 \end{aligned}$$

Finally, the estimate (4.22) is obtained by summing (B.16) over time steps from 0 to m .

REFERENCES

- [1] T. Arbogast, M.F. Wheeler, and I. Yotov. Mixed finite elements for elliptic problems with tensor coefficients as cell-centered finite differences. *SIAM Journal on Numerical Analysis*, 34(2): 828–852, 1997.
- [2] F. Brezzi and M. Fortin. *Mixed and Hybrid Finite Element Methods*. Springer Verlag, 1991.
- [3] H. C. Brinkman. A calculation of the viscous force exerted by a flowing fluid on a dense swarm of particles. *Applied Scientific Research*, A1: 27–34, 1949.
- [4] G. Daccord, E. Touboul, R. Lenormand. Carbonate acidizing: toward a quantitative model of the wormholing phenomenon. *SPE Production Engineering*, 4: 63–68, 1989.
- [5] T. A. Davis. Algorithm 832: UMFPACK V4. 3—an unsymmetric-pattern multifrontal method. *ACM Transactions on Mathematical Software*, 30(2): 196–199, 2004.
- [6] C. N. Fredd and H. S. Fogler. Influence of transport and reaction on wormhole formation in porous media. *AIChE Journal*, 44(9):1933–1949, 1998.
- [7] V. Girault and M.F. Wheeler. Numerical discretization of a Darcy-Forchheimer model. *Numerische Mathematik*, 110:161–198, 2008.
- [8] V. Girault and H. Lopez. Finite element error estimates for the MAC scheme. *IMA J. Numer. Anal.*, 16: 347–379, 1996.
- [9] F. Golfier, C. Zarcone, B. Bazin, R. Lenormand, D. Lasseux, and M. Quintard. On the ability of a Darcy-scale model to capture wormhole formation during the dissolution of a porous medium. *Journal of Fluid Mechanics*, 457: 213–254, 2002.
- [10] T. T. Kieu. Analysis of expanded mixed finite element methods for the generalized Forchheimer flows of slightly compressible fluids. *Numerical Methods for Partial Differential Equations*, 32: 60–85, 2016.
- [11] N. Kladas, V. Prasad. Experimental verification of Darcy-Brinkman-Forchheimer flow model for natural convection in porous media. *Journal of Thermophysics and Heat Transfer*, 5: 560–76, 1991.
- [12] P. Knabner and J.E. Roberts. Mathematical analysis of a discrete fracture model coupling Darcy flow in the matrix with Darcy-Forchheimer flow in the fracture. *ESAIM: Mathematical Modelling and Numerical Analysis*, 48:1451–1472, 2014.
- [13] J. Kou, S. Sun and Y. Wu. Mixed finite element-based fully conservative methods for simulating wormhole propagation. *Computer Methods in Applied Mechanics and Engineering*, 298: 279–302, 2016.
- [14] H. P. Langtangen, K.-A. Mardal, and R. Winther. Numerical methods for incompressible viscous flow. *Advances in Water Resources*, 25:1125–1146, 2002.
- [15] X. Li, H. Rui. Characteristic block-centered finite difference method for simulating incompressible wormhole propagation. *Computers and Mathematics with Applications*, 73: 2171–2190, 2017.
- [16] X. Li, H. Rui. Block-Centered Finite Difference Method for Simulating Compressible Wormhole Propagation. *Journal of Scientific Computing*, 74: 1115–1145, 2018.
- [17] M. Liu, S. Zhang, J. Mou, and F. Zhou. Wormhole propagation behavior under reservoir condition in carbonate acidizing. *Transport in Porous Media*, 96: 203–220, 2013.
- [18] S. Mauran, L. Rigaud, and O. Coudeville. Application of the Carman-Kozeny correlation to a high-porosity and anisotropic consolidated medium: The compressed expanded natural graphite. *Transport in Porous Media*, 43: 355–376, 2001.
- [19] M.K.R. Panga, M. Ziauddin, and V. Balakotaiah. Two-scale continuum model for simulation of wormholes in carbonate acidization. *AIChE Journal*, 51(12): 3231–3248, 2005.
- [20] D. Ruth and H. Ma. On the derivation of the Forchheimer equation by means of the averaging theorem. *Transport in Porous Media*, 7: 255–264, 1992.
- [21] A.S. Smirnov, K.M. Fedorov, and A.P. Shevelev. Modeling the acidizing of a carbonate formation. *Fluid Dynamics*, 45(5): 779–786, 2010.
- [22] S. Sun and M.F. Wheeler. Discontinuous Galerkin methods for coupled flow and reactive transport problems. *Applied Numerical Mathematics*, 52(2): 273–298, 2005.
- [23] P. Szymczak and A.J.C. Ladd. Wormhole formation in dissolving fractures. *Journal of Geophysical Research*, 114, B06203, doi:10.1029/2008JB006122, 2009.
- [24] Y. Wu, A. Salama, and S. Sun. Parallel Simulation of Wormhole Propagation with the Darcy-Brinkman-Forchheimer Framework. *Computers and Geotechnics*, 69: 564–577, 2015.

- [25] C. Zhao, B.E. Hobbs, and A. Ord. Computational simulation of chemical dissolution-front instability in fluid-saturated porous media under non-isothermal conditions. *International Journal for Numerical Methods in Engineering*, 102: 135–156, 2015.