Adaptive Time-Splitting Scheme for Nanoparticles Transport with Two-phase Flow in Heterogeneous Porous Media

Mohamed F. El-Amin^{a,b,c,e}, Jisheng Kou^d, Shuyu Sun^c

^a*Effat University, Jeddah 21478, Kingdom of Saudi Arabia.*

^b*Mathematics Department, Faculty of Science, Aswan University, Aswan 81528, Egypt.*

^c*Computational Transport Phenomena Laboratory (CTPL), Division of Physical Sciences and Engineering (PSE), King Abdullah University of Science and Technology (KAUST), Thuwal, Kingdom of Saudi Arabia.*

^d*School of Mathematics and Statistics, Hubei Engineering University, Xiaogan 432000, Hubei, China.*

^e*momousa@effatuniversity.edu.sa*

Abstract

In this work, we introduce an efficient scheme using an adaptive time-splitting method to simulate the problem of nanoparticles transport with a two-phase flow in heterogeneous porous media. The pressure and saturation equations are coupled with the capillary pressure which is linearized in terms of saturation. An IMplicit Pressure Explicit Saturation-IMplicit Concentration (IMPES-IMC) scheme is used to solve the problem under consideration. The cell-centered finite difference (CCFD) method is used for the spatial discretization. The external time interval is divided into three levels, the first level is for the pressure, the second level of the saturation, while the third level is applied for the concentrations. This method can reduce the computational cost arisen from the implicit solution of the pressure equation and concentration equation as well as the rapid changes in saturation and concentration.

The time step-sizes for saturation and concentration equations are adaptive under computing and satisfying the Courant-Friedrichs-Lewy (CFL₁) condition, iteratively. These results show the good performance of the scheme. Moreover, a numerical example of a highly heterogeneous porous medium is introduced and adaptive time step-sizes are shown in graphs for different cases, in addition to saturation, nanoparticles concentration contours.

Keywords: time-splitting, IMPES, nanoparticles, two-phase flow, porous media, reservoir simulation

1. Introduction

The IMPLICIT Pressure Explicit Saturation (IMPES), is a conditionally stable approach, solves the pressure equation implicitly and updates the saturation explicitly. Hence it takes a very small time step size, in particular with heterogeneous porous media. The IMPES scheme has been improved in several versions (e.g. [1, 2, 3, 4, 5]). The temporal discretization scheme is considered an important factor that affects the efficiency of numerical reservoir simulators. The application of traditional single-scale temporal scheme is restricted by the rapid changes of the pressure and saturation with capillarity and concentrations if applicable. So, applying time splitting strategies has a significant improvement to treating the gap between the pressure and the saturation. Time splitting method has been considered in a number of publications such as [6]-[15]. For example, in Refs. [11, 12], an explicit sub-timing scheme is provided. On the other hand, an implicit time-stepping

scheme have been proposed by Bhallamudi et al. [9]. Similar methodology has been used to the problem of flow and transport simulations in fractured porous media [10]. VanderKwaak [16] have introduced a hybrid implicit-explicit scheme by treating implicitly some portions of the domain under certain stability conditions, while the rest of the domain is treated by the explicit time-stepping approach.

Recently, applications of nanoparticles have been considered in many branches of petroleum engineering, especially, enhanced oil recovery. Ju et al. [19, 20] have formulated a mathematical model of nanoparticles transport in two-phase flow in porous media based on the work in Refs. [21]. El-Amin et al. [22, 23] have introduced a model to simulate nanoparticles transport in two-phase flow in porous media. In [24], authors extended the model to include a negative capillary pressure and mixed relative permeabilities correlations to fit with the mixed-wet system. In Ref. [25], they have introduced numerical and dimensional analyses of nanoparticles transport with a two-phase flow in porous media. Salama et al. [26] have investigated the problem of transport of nanoparticles in anisotropic porous media numerically using the multipoint flux approximation. Recently, Chen et al. [27] presented a numerical simulation of drag reduction effects by hydrophobic nanoparticles adsorption method in water flooding processes. Chen et al. [28] have considered the dynamic update of anisotropic permeability field during the transport of nanoparticles in the subsurface. In [29] we have introduced nonlinear iterative IMPES-IMC (IMplicit Pressure Explicit Saturation-IMplicit Con-

centration) scheme to solve the flow equation of the model of nanoparticles transport in porous media.

In the current work, we introduce a time-stepping technique for the modeling and simulation of nanoparticles transport with two-phase flow. An IMPES-IMC scheme is used to solve the problem under consideration, while the CCFD method is used for the spatial discretization. The time-stepping scheme is applied under the CFL condition to adapting the time-steps sizes. Finally, numerical experiments are provided. The paper is organized as follows: The second section is devoted to the modeling and mathematical formulation. The third section is devoted to the time-stepping technique. The time-steps adaption is provided in the fourth section. The CCFD spatial discretization is introduced in Sec. 5. Then, Sec. 6 is devoted to numerical tests, while the conclusions are presented in Sec. 7.

2. Modeling and Mathematical Formulation

This paper considers the problem of nanoparticles transport with a two-phase immiscible incompressible flow in porous media, the system of governing equations consists of water saturation, Darcy's law, nanoparticles concentration, deposited nanoparticles concentration on the pore-wall, and entrapped nanoparticles concentration in the pore-throat. Also, the porosity and permeability vary due to the nanoparticles deposition/entrapment on/in the pores. In the following we introduce the governing equations briefly, (for details see Refs. [22, 23, 24, 25, 26, 28, 29]):

Momentum Conservation (Darcy's Law):

$$\mathbf{u}_\alpha = -\frac{k_\alpha}{\mu_\alpha} \nabla \Phi_\alpha, \quad \alpha = w, n, \quad (1)$$

where

$$k_\alpha = k_{r\alpha} \mathbf{K}, \quad \Phi_\alpha = p_\alpha + \rho_\alpha g \nabla z, \quad \alpha = w, n,$$

Mass Conservation (Saturation Equations):

$$\phi \frac{\partial s_\alpha}{\partial t} + \nabla \cdot \mathbf{u}_\alpha = q_\alpha, \quad \alpha = w, n, \quad (2)$$

where

$$s_w + s_n = 1.$$

where $\mathbf{K}$ is the permeability tensor $\mathbf{K} = k\mathbf{I}$, where $\mathbf{I}$ is the identity matrix and k is a positive real number. ϕ is the porosity, g is the gravitational acceleration, and z is the depth. $\mathbf{u}_\alpha, \Phi_\alpha, p_\alpha, \mu_\alpha, \rho_\alpha, k_\alpha, k_{r\alpha}, q_\alpha, s_\alpha$ are, respectively, the velocity, the pressure potential, the pressure, the viscosity, the density, the effective permeability, the relative permeability, the external mass flow rate, and the saturation of the phase α . w stands for the wetting phase (water), and n stands for the nonwetting phase (oil).

Providing the following definitions:

The capillary pressure: $p_c(s_w) = p_n - p_w$.

The total velocity: $\mathbf{u}_t = \mathbf{u}_w + \mathbf{u}_n$.

The flow fraction: $f_w = \lambda_w / \lambda_t$.

The phase mobility: $\lambda_\alpha = k_{r\alpha} / \mu_\alpha$.

The total mobility: λ_t .

The capillary pressure potential: $\Phi_c = p_c + (\rho_n - \rho_w) g \nabla z$.

The total source mass transfer: $q_t = q_w + q_n$.

After some mathematical manipulations and referring to Refs. [17], the pressure equation can be rewritten as,

$$\nabla \cdot \mathbf{u}_t = -\nabla \cdot \lambda_t \mathbf{K} \nabla \Phi_w - \nabla \cdot \lambda_n \mathbf{K} \nabla \Phi_c = q_t. \quad (3)$$

Because this equation contents the capillary pressure which is a function of saturation, it will be coupled with the following saturation equation to calculate the pressure,

$$\phi \frac{\partial s_w}{\partial t} - q_w = -\nabla \cdot \lambda_w \mathbf{K} \nabla \Phi_w. \quad (4)$$

However, the saturation is updated using the following form,

$$\phi \frac{\partial s_w}{\partial t} - q_w = -\nabla \cdot (f_w \mathbf{u}_a). \quad (5)$$

where $\mathbf{u}_w = f_w \mathbf{u}_a$ and $\mathbf{u}_a = -\lambda_t \mathbf{K} \nabla \Phi_w$.

Nanoparticles Transport Model:

Assuming that the nanoparticles exist only in the water phase of one size interval. So, The transport equation of the nanoparticles in the water phase

is given as ([18, 21, 19, 20, 22, 23, 24, 25, 26, 28, 29]),

$$\phi \frac{\partial (s_w c)}{\partial t} + \nabla \cdot (\mathbf{u}_w c - D \nabla c) = R + Q_c, \quad (6)$$

Nanoparticles Surface Deposition:

The surface deposition is expressed by,

$$\frac{\partial c_{s1}}{\partial t} = \begin{cases} \gamma_d |\mathbf{u}_w| c, & \mathbf{u}_w \leq u_r \\ \gamma_d |\mathbf{u}_w| c - \gamma_e |\mathbf{u}_w - u_r| c_{s1}, & \mathbf{u}_w > u_r \end{cases} \quad (7)$$

Nanoparticles Throat Entrapment:

The rate of entrapment of the nanoparticles is,

$$\frac{\partial c_{s2}}{\partial t} = \gamma_{pt} |\mathbf{u}_w| c, \quad (8)$$

where c is the nanoparticles concentrations. τ is the tortuosity parameter, of the flow of the water phase. Q_c is the rate of change of particle volume belonging to a source/sink term. c_{s1} is the deposited nanoparticles concentration on the pore surfaces of the porous medium. c_{s2} is the entrapped nanoparticles concentration in pore throats. γ_d is the rate coefficients for surface retention of the nanoparticles. γ_e is the rate coefficients for entrainment of the nanoparticles. u_r is the critical velocity of the water phase. where

γ_{pt} is the pore throat blocking constants. The diffusion-dispersion tensor is defined by,

$$D = \phi s_w \tau D_t, \quad D_t = D^{\text{Br}} + D^{\text{disp}} \quad (9)$$

where Br is the Brownian diffusion and D^{disp} is the dispersion coefficient which is defined by [2],

$$\phi s_w \tau D^{\text{disp}} = d_{t,w} |\mathbf{u}_w| \mathbf{I} + (d_{l,w} - d_{t,w}) \frac{\mathbf{u}_w \mathbf{u}_w^T}{|\mathbf{u}_w|} \quad (10)$$

Thus,

$$D = (\phi s_w \tau D^{\text{Br}} + d_{t,w} |\mathbf{u}_w|) \mathbf{I} + (d_{l,w} - d_{t,w}) \frac{\mathbf{u}_w \mathbf{u}_w^T}{|\mathbf{u}_w|} \quad (11)$$

where $d_{l,w}$ and $d_{t,w}$ are the longitudinal and transverse dispersion coefficients, respectively. R is the net rate of loss of nanoparticles which is defined by,

$$R = \frac{\partial c_{s1}}{\partial t} + \frac{\partial c_{s2}}{\partial t} \quad (12)$$

Initial and Boundary Conditions:

Consider the computational domain Ω with the boundary $\partial\Omega$ which is subjected to Dirichlet Γ_D and Neumann Γ_N boundaries, where $\partial\Omega = \Gamma_D \cup \Gamma_N$ and $\Gamma_D \cap \Gamma_N = \emptyset$. At the beginning of the injection process, we have,

$$s_w = s_w^0, \quad c = c_{s1} = c_{s2} = 0 \quad \text{in } \Omega \quad \text{at } t = 0, \quad (13)$$

The boundary conditions are given as,

$$p_w \text{ (or } p_n) = p^D \quad \text{on } \Gamma_D, \quad (14)$$

$$\mathbf{u}_t \cdot \mathbf{n} = q^N, \quad s_w = S^N, \quad c = c_0, \quad c_{s1} = c_{s2} = 0 \quad \text{on } \Gamma_N. \quad (15)$$

where $\mathbf{n}$ is the outward unit normal vector to $\partial\Omega$, p^D is the pressure on Γ_D and q^N the imposed inflow rate on Γ_N , respectively.

3. Multi-scale Time Splitting Method

The concept of time splitting method is to use different time step size for each equation has a time derivative. In the above-described method, the pressure is coupled with the saturation in each time-step. The time-step size for the pressure can be taken larger than the those of saturation and nanoparticle concentrations. So, for the pressure the total time interval $[0, T]$ is divided into N_p , time-steps as $0 = t_0 < t_1 < \dots < t_{N_p=T}$. Thus, the time-step length assigned for pressure is, $\Delta t^k = t^{k+1} - t^k$. Since the saturation varies more rapidly than the pressure, we use a smaller time-step size for the saturation equation. That is, each interval, $(t^k, t^{k+1}]$, will be divided into $N_{p,s}$ subintervals as $(t^k, t^{k+1}] = \cup_{l=0}^{N_{p,s}-1} (t^{k,l}, t^{k,l+1}]$. On the other hand, as the concentration varies more rapidly than the pressure (and may be saturation), we also use a smaller time-step size for the concentration equations. Thus, we partition each subinterval $(t^{k,l}, t^{k,l+1}]$ into $N_{p,s,c}$ subsubintervals as $(t^{k,l}, t^{k,l+1}] = \cup_{m=0}^{N_{p,s,c}-1} (t^{k,l,m}, t^{k,l,m+1}]$. Therefore, the system

of governing equations, (3), (4), (6), (7) and (8), is solved based on the adaptive time-splitting technique. The backward Euler time discretization is used for the equations of pressure, saturation, concentration and the two volume concentration. We linearized the capillary pressure function, Φ_c , in terms of saturation using this formula,

$$\Phi_c(s_w^*) \cong \Phi_c(s_w^k) + \Phi'_c(s_w^k) [s_w^{k+1} - s_w^k], \quad (16)$$

where Φ'_c is derivative of Φ_c . The quantity, $[s_w^{k+1} - s_w^k]$, can be calculated from the saturation equation,

$$s_w^{k+1} - s_w^k = \frac{\Delta t^k}{\phi(c_{s1}^k, c_{s2}^k)} [q_w^{k+1} - \nabla \cdot \lambda_t(s_w^k) \mathbf{K}(c_{s1}^k, c_{s2}^k) \nabla \Phi_w^{k+1}]. \quad (17)$$

In addition to the pressure equation,

$$-\nabla \cdot \lambda_t(s_w^k) \mathbf{K}(c_{s1}^k, c_{s2}^k) \nabla \Phi_w^{k+1} - \nabla \cdot \lambda_n(s_w^k) \mathbf{K}(c_{s1}^k, c_{s2}^k) \nabla \Phi_c(s_w^*) = q_t^{k+1}. \quad (18)$$

Then, the above coupled system (16), (17) and (18) is solved implicitly to obtain the pressure potential. Therefore, the saturation is updated explicitly with using the upwind scheme for the convection term as,

$$\phi(c_{s1}^k, c_{s2}^k) \frac{s_w^{k,l+1} - s_w^{k,l}}{\Delta t^l} + \nabla \cdot (f_w^k \mathbf{u}_a^{k+1}) = q_w^{k,l+1}. \quad (19)$$

Therefore, the nanoparticles concentration, deposited nanoparticles concentration on the pore-walls and entrapped nanoparticles concentration in the pore-throats are computed implicitly as follow,

$$\begin{aligned} \phi(c_{s1}^k, c_{s2}^k) \frac{s_w^{k+1} c^{k,l,m+1} - s_w^k c^{k,l,m}}{\Delta t^m} + \nabla \cdot \{ \mathbf{u}_w^{k+1} c^{k,l,m+1} - D(s_w^{k+1}, \mathbf{u}_w^{k+1}, c_{s1}^k, c_{s2}^k) \\ - R(\mathbf{u}_w^{k+1}, c_{s1}^k) \} \nabla c^{k,l,m+1} = Q_c^{k,l,m+1}, \end{aligned} \quad (20)$$

$$\begin{aligned} \frac{c_{s1}^{k,l,m+1} - c_{s1}^{k,l,m}}{\Delta t^m} = \begin{cases} \gamma_d |\mathbf{u}_w^{k+1}| c^{k+1}, & \mathbf{u}_w^{k+1} \leq u_r \\ \gamma_d |\mathbf{u}_w^{k+1}| c^{k+1} - \gamma_e |\mathbf{u}_w^{k+1} - u_r| c_{s1}^{k,l,m+1}, & \mathbf{u}_w^{k,l,m+1} > u_r \end{cases} \end{aligned} \quad (21)$$

and,

$$\frac{c_{s2}^{k,l,m+1} - c_{s2}^{k,l,m}}{\Delta t^m} = \gamma_{pt} |\mathbf{u}_w^{k+1}| c^{k,l,m+1} \quad (22)$$

Finally, the permeability, porosity, and other parameters such as $\lambda_w, \lambda_n, \lambda_t$ and f_w are updated. In the above system of equations both $\mathbf{K}$ and ϕ are functions in c_{s1} and c_{s2} , that are functions of u_w and c . On the other hand, $\mathbf{u}_w$ is a function of Φ_w and s_w , while D is functions of s_w , $\mathbf{u}_w$, c_{s1} and c_{s2} . D and R are functions of s_w and $\mathbf{u}_w$.

4. Adaptive Time-Stepping

In our algorithm, we have checked the Courant–Friedrichs–Lewy (CFL) condition to guarantee its satisfactory (i.e. $\text{CFL} < 1$). In order to achieve this idea, we have to define the following CFLs,

$$\text{CFL}_{s,x} = \frac{\mathbf{u}_x \Delta t^{k,l}}{\Delta x}, \quad (23)$$

$$\text{CFL}_{s,y} = \frac{\mathbf{u}_y \Delta t^{k,l}}{\Delta y}, \quad (24)$$

for saturation equation, and,

$$\text{CFL}_{c,x} = \frac{\mathbf{u}_x \Delta t^{k,l,m}}{\Delta x}, \quad (25)$$

and

$$\text{CFL}_{c,y} = \frac{\mathbf{u}_y \Delta t^{k,l,m}}{\Delta y}, \quad (26)$$

for concentration equations. It may be noted that the CFL depends on the ratio $\Delta t / \Delta x$ which can be fixed at larger time-steps and larger mesh-size. Thus, when we use a larger domain (larger mesh size) then we can use larger time step size. In the code, the initial time-step for the saturation equation is taken as the pressure time-step, i.e., $\Delta t^{k,0} = \Delta t^k$, and the initial time-step for the concentration equation is taken as the saturation time-step, i.e., $\Delta t^{k,l,0} = \Delta t^{k,l}$. Then, we check if $\text{CFL}_{s,x} > 1$ or $\text{CFL}_{s,y} > 1$, the saturation time-step will be divided by 2 and the $\text{CFL}_{s,x}$ and $\text{CFL}_{s,y}$ will be recalculated.

This procedure will be repeated until satisfying the condition $\text{CFL}_{s,x} < 1$ and $\text{CFL}_{s,y} < 1$, then the final adaptive saturation time-step will be obtained. Similarly, we check if the condition, $\text{CFL}_{c,x} > 1$ or $\text{CFL}_{c,y} > 1$ is satisfied, the concentration time-step will be divided by 2 and therefore, we recalculate both $\text{CFL}_{c,x}$ and $\text{CFL}_{c,y}$. We repeat this procedure to reach the condition $\text{CFL}_{c,x} < 1$ and $\text{CFL}_{c,y} < 1$, then we obtain the final adaptive concentration time-step.

5. Spatial Discretization

The cell-centered finite difference (CCFD) is a locally conservative method that is very useful in solving petroleum reservoir simulation problems. Arbogast et al. [31] have proved that the CCFD method is equivalent to the mixed finite element method in the case of rectangular elements. In this work, we apply the CCFD scheme to the system of equations (16)–(22). The corresponding algebraic coupled pressure and saturation equation are solved implicitly to obtain the pressure is given as,

$$\mathbf{A}_t(\mathbf{s}_w^k, \mathbf{c}_{s1}^k, \mathbf{c}_{s2}^k) \Phi_w^{k+1} = \mathbf{Q}_t(\mathbf{s}_w^k, \mathbf{c}_{s1}^k, \mathbf{c}_{s2}^k), \quad (27)$$

where

$$\mathbf{A}_t(\mathbf{s}_w^k, \mathbf{c}_{s1}^k, \mathbf{c}_{s2}^k) = \mathbf{A}_a(\mathbf{s}_w^k, \mathbf{c}_{s1}^k, \mathbf{c}_{s2}^k) - \Delta t^l \mathbf{A}_c(\mathbf{s}_w^k, \mathbf{c}_{s1}^k, \mathbf{c}_{s2}^k) \Phi'(\mathbf{s}_w^k) \mathbf{M}^{-1} \mathbf{A}_w(\mathbf{s}_w^k, \mathbf{c}_{s1}^k, \mathbf{c}_{s2}^k), \quad (28)$$

and,

$$\begin{aligned} \mathbf{Q}_t(\mathbf{s}_w^k, \mathbf{c}_{s1}^k, \mathbf{c}_{s2}^k) &= \mathbf{Q}_{ac}^{k+1} - \mathbf{A}_c(\mathbf{s}_w^k, \mathbf{c}_{s1}^k, \mathbf{c}_{s2}^k) [\Phi_c(\mathbf{s}_w^k) + \Phi'(\mathbf{s}_w^k)(\mathbf{s}_w^k - \mathbf{s}_w^{k+1})] \\ &\quad - \Delta t^l \mathbf{A}_c(\mathbf{s}_w^k, \mathbf{c}_{s1}^k, \mathbf{c}_{s2}^k) \Phi'(\mathbf{s}_w^k) \mathbf{M}^{-1} \mathbf{Q}_w^{k+1}. \end{aligned} \quad (29)$$

After calculating the pressure and the velocity, the CCFD scheme of (19) to update the saturation is the following algebraic equation,

$$\mathbf{M}(\mathbf{c}_{s1}^k, \mathbf{c}_{s2}^k) \frac{\mathbf{s}_w^{k,l+1} - \mathbf{s}_w^k}{\Delta t^l} + \mathbf{A}_s(\mathbf{u}_a^{k+1}) f_w(\mathbf{s}_w^k) = \mathbf{Q}_s^{k,l+1}. \quad (30)$$

Similarly, the CCFD discretization of the equations of concentration (20)–(22) can be obtained. The nanoparticles concentration CCFD equation is,

$$\mathbf{B}_t(\mathbf{s}_w^{k+1}, \mathbf{u}_w^{k+1}, \mathbf{c}_{s1}^k, \mathbf{c}_{s2}^k) \mathbf{c}^{k,l,m+1} = \mathbf{Q}_{ct}(\mathbf{s}_w^k, \mathbf{c}^k, \mathbf{c}_{s1}^k, \mathbf{c}_{s2}^k), \quad (31)$$

where

$$\begin{aligned} \mathbf{B}_t(\mathbf{s}_w^{k+1}, \mathbf{u}_w^{k+1}, \mathbf{c}_{s1}^k, \mathbf{c}_{s2}^k) &= \mathbf{B}_m(\mathbf{s}_w^{k+1}) + \Delta t^m \mathbf{M}(\mathbf{c}_{s1}^k, \mathbf{c}_{s2}^k)^{-1} \\ &\quad [\mathbf{B}_w(\mathbf{u}_w^{k+1}) - \mathbf{B}_D(\mathbf{s}_w^{k+1}, \mathbf{u}_w^{k+1}, \mathbf{c}_{s1}^k, \mathbf{c}_{s2}^k) - \mathbf{R}(\mathbf{u}_w^{k+1}, \mathbf{c}_{s1}^k)], \end{aligned}$$

and,

$$\mathbf{Q}_{ct}(\mathbf{s}_w^k, \mathbf{c}^k, \mathbf{c}_{s1}^k, \mathbf{c}_{s2}^k) = \Delta t^m \mathbf{M}(\mathbf{c}_{s1}^k, \mathbf{c}_{s2}^k)^{-1} \mathbf{Q}_c^{k,l,m+1} + \mathbf{B}_m(\mathbf{s}_w^k) \mathbf{c}^k.$$

The CCFD deposited nanoparticles concentration on the pore-walls is given by,

$$\mathbf{c}_{s1}^{k,l,m+1} = \begin{cases} \mathbf{R}_{1t}(\mathbf{u}_w^{k+1}, \mathbf{c}^{k+1}, \mathbf{c}_{s1}^k), & \mathbf{u}_w^{k+1} \leq u_r \\ [\mathbf{I} + \Delta t^m \mathbf{R}_2(\mathbf{u}_w^{k+1})]^{-1} \mathbf{R}_{1t}(\mathbf{u}_w^{k+1}, \mathbf{c}^{k+1}, \mathbf{c}_{s1}^k), & \mathbf{u}_w^{k+1} > u_r, \end{cases} \quad (32)$$

where

$$\mathbf{R}_{1t}(\mathbf{u}_w^{k+1}, \mathbf{c}^{k+1}, \mathbf{c}_{s1}^k) = \Delta t^m \mathbf{R}_1(\mathbf{u}_w^{k+1}, \mathbf{c}^{k+1}) + \mathbf{c}_{s1}^k,$$

$$\mathbf{R}_1(\mathbf{u}_w^{k+1}, \mathbf{c}^{k+1}) = \gamma_d |\mathbf{u}_w| \mathbf{c}^{k+1},$$

and,

$$\mathbf{R}_2(\mathbf{u}_w^{k+1}) = \gamma_e |\mathbf{u}_w - u_r|.$$

Finally, one may write the CCFD discretization of the $\mathbf{c}_{s2}$ equation as,

$$\mathbf{c}_{s2}^{k,l,m+1} = \Delta t^m \mathbf{R}_3(\mathbf{u}_w^{k+1}, \mathbf{c}^{k+1}) + \mathbf{c}_{s2}^k, \quad (33)$$

where

$$\mathbf{R}_3(\mathbf{u}_w^{k+1}, \mathbf{c}^{k+1}) = \gamma_{pt} |\mathbf{u}_w| \mathbf{c}^{k+1},$$

We can solve the above equations to find, $\mathbf{s}^{k+1} \equiv \mathbf{s}^{k,l+1}$, $\mathbf{c}^{k+1} \equiv \mathbf{c}^{k,l,m+1}$,

$\mathbf{c}_{s1}^{k+1} \equiv \mathbf{c}_{s1}^{k,l,m+1}$ and $\mathbf{c}_{s2}^{k+1} \equiv \mathbf{c}_{s2}^{k,l,m+1}$.

6. Numerical Tests

In order to examine the performance of the current scheme, we introduce some numerical examples in this section. Firstly, we introduce the required physical parameters used in the computations. Then, we study the performance of the scheme by introducing some results for the adaptive time steps based on values of the corresponding Courant–Friedrichs–Lewy (CFL). Then we present some results for the distributions of water saturation and nanoparticles concentrations. In this study, given the normalized wetting phase saturation,

$$S = \frac{s_w - s_{wr}}{1 - s_{nr} - s_{wr}}, \quad 0 \leq S \leq 1,$$

the capillary pressure is defined as,

$$p_c = -p_e \log S,$$

and the relative permeabilities are defined as,

$$k_{rw} = k_{rw}^0 S^2, \quad k_{rn} = k_{rn}^0 (1 - S)^2,$$

where

$$k_{rw}^0 = k_{rw}(S = 1), \quad k_{rn}^0 = k_{rn}(S = 1).$$

where p_e is the capillary pressure parameter, s_{wr} is the irreducible water saturation and s_{nr} is the residual oil saturation after water flooding. The

values and units of the physical parameters are inserted in Table 1.

Table 1: Values of the physical parameters .

Parameter	Value	Units
γ_d	16	m^{-1}
γ_{pt}	1.28	m^{-1}
γ_e	30	m^{-1}
$\mathbf{u}_r$	4.6×10^{-6}	m/s
c_0	0.1	—
S_{wr}	0.001	—
S_{nr}	0.001	—
ϕ_0	0.3	—
k_f	0.6	—
γ_f	0.01	—
μ_w	1	$\text{cP} = 1.0 \times 10^{-3} \text{ Pa.s}$
μ_n	0.45	cP
k_{rw0}	1	—
k_{ro0}	1	—
B_c	50	$\text{bar} = 1.0 \times 10^5 \text{ Pa}$
D	4.6×10^{-8}	m^2/s

In this example, we use a real permeability map of dimensions 120×50 , which is extracted from Ref. [30]. The permeabilities vary in a large scope and are they highly heterogeneous as shown in Fig. 1. We consider a domain of size $40\text{m} \times 16\text{m}$ which is discretized into 120×50 uniform rectangles grids. The injection rate was 0.01 Pore-Volume-Injection (PVI) and continued the

Figure 1: Real heterogeneous permeability map.

calculation until 0.5 PV. In this example, we choose the number of steps of the outer loop to be: (Case 1, $k=50$, Fig. 2; Case 1, $k=100$, Fig. 2; Case 1, $k=200$, Fig. 3). In these figures, the adaptive time-step sizes, Δt^l and Δt^m are plotted against the number of steps of the outer loop k and the number of the inner loops l and m . It can be seen from these figures (Fig.2 – Fig. 4) that the behavior of adaptive Δt^l and Δt^m are very similar. This may be because the velocity is large and dominate the CFL. Also, both Δt^l and Δt^m start with large values then they gradually become smaller and smaller as k increases. On the other hand, for the first two cases when $k=50, 100$, Δt^l and Δt^m are small when l and m are small, then they increase to reach a peak, then they are gradually decreasing. However, in the third case when $k=200$, both Δt^l and Δt^m start with large values then they gradually become smaller and smaller as l and m , respectively, increases.

Variations of saturation of the heterogeneous permeability are shown in Fig. 5. It is noteworthy that the distribution for water saturation is dis-

Figure 2: Adaptive time-step sizes, Δt^l and Δt^m against the number of steps of the outer loop k and the number of the inner loops l and m : Case 1 ($\Delta t^k=50$).

Figure 3: Adaptive time-step sizes, Δt^l and Δt^m against the number of steps of the outer loop k and the number of the inner loops l and m : Case 2 ($\Delta t^k=100$).

Figure 4: Adaptive time-step sizes, Δt^l and Δt^m against the number of steps of the outer loop k and the number of the inner loops l and m : Case 3 ($\Delta t^k=200$).

continuous due to the high heterogeneity. Thus, for example, we may note higher water saturation at higher permeability regions. Similarly, one may observe the discontinuity in the nanoparticles concentration as illustrated in Fig. 6. The contrast of the nanoparticles concentration distribution arisen from the heterogeneity of the medium can also be noted here. Moreover, the contours of the deposited nanoparticles concentration are shown in Fig. 7. The behavior of the entrapped nanoparticles concentration c_{s2} is very similar to the behavior of c_{s1} because of the similarity in their governing equation.

7. Conclusions

In this paper, we introduce an efficient time-stepping scheme with adaptive time-step sizes based on the CFL calculation. Hence, we have calculated the $\text{CFL}_{s,x}$, $\text{CFL}_{s,y}$, $\text{CFL}_{c,x}$ and $\text{CFL}_{c,y}$ at each substep and check if the

Figure 5: Variations of saturation of the real heterogenous permeability case.

Figure 6: Variations of nanoparticles concentration of the real heterogenous permeability case.

Figure 7: Variations of deposited nanoparticles concentration of the real heterogenous permeability case.

CFL condition is satisfied (i.e. $CFL < 1$). We applied this scheme with the IMES-IMC scheme to simulate the problem of nanoparticles transport in two-phase flow in porous media. The model consists of five differential equations, namely, pressure, saturation, nanoparticles concentration, deposited nanoparticles concentration on the pore-walls, and entrapped nanoparticles concentration in pore-throats. The capillary pressure is linearized and used to couple the pressure and the saturation equations. Then, the saturation equation is solved explicitly to update the saturation at each step. Therefore, the nanoparticles concentration equation is treated implicitly. Finally, the deposited nanoparticles concentration on the pore-walls equation, and entrapped nanoparticles concentration in pore-throats equation are solved implicitly. The CCFD method was used to discretize the governing equations spatially. In order to show the efficiency of the proposed scheme, we

presented some numerical experiments. The outer pressure time-step size was selected then the inner ones, namely, the saturation subtime-step and the concentration subtime-step were calculated and adaptive by the CFL condition. We presented three cases with different values of the outer pressure time-step. Moreover, distributions of water saturation, nanoparticles concentration, and deposited nanoparticles concentration on pore-wall are shown in graphs.

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