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Positivity-preserving CE/SE schemes for solving the compressible Euler and Navier-Stokes equations on hybrid unstructured meshes

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Abstract

We construct positivity-preserving space-time conservation element and solution element (CE/SE) schemes for solving the compressible Euler and Navier-Stokes equations on hybrid unstructured meshes consisting of triangular and rectangular elements. The schemes use an *a posteriori* limiter to prevent negative densities and pressures based on the premise of preserving optimal accuracy. The limiter enforces a constraint for spatial derivatives and does not change the conservative property of CE/SE schemes. Several numerical examples suggest that the proposed schemes preserve accuracy for smooth flows and strictly preserve positivity of densities and pressures for the problems involving near vacuum and very strong discontinuities.

Keywords: positivity-preserving, space-time conservation element and solution element scheme, the compressible Euler equations, the compressible Navier-Stokes equations, unstructured mesh

1. Introduction

In the gas dynamical case, the eigenvalues of the Jacobian matrices of inviscid fluxes contain the speed of sound, c , which is defined as $\sqrt{\gamma p/\rho}$ for an ideal gas. The entropy of an ideal gas is precisely defined as $S = \log p - \gamma \log \rho$, where γ , p and ρ respectively denote the ratio of specific heat, pressure and density. The dynamic viscosity, μ , of an ideal gas strongly depends on the absolute temperature, $T = p/(\rho R)$, and can be accurately correlated by the Sutherland's law [1] as $\mu = \mu_0 \left(\frac{T}{T_0}\right)^{3/2} \frac{T_0 + T_s}{T + T_s}$, where R is the ideal gas constant, T_0 is the reference temperature, μ_0 is the viscosity at T_0 and T_s is the Sutherland temperature. Given these facts, preserving the positivity of density and pressure

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is necessary for a numerical scheme to maintain the hyperbolic property of the compressible Euler equations and to derive a physically meaningful entropy and viscosity for the compressible Navier-Stokes equations. Many high-resolution numerical schemes suffer from numerical blow-ups due to the emergence of negative density or pressure when strong shock waves or strong expansion waves are encountered. When no alternative is possible, we have to force the negative density or pressure to be positive after each time step. Such brute treatment, however, leads to an increase of the total mass or total energy of the system. In other words, the numerical scheme is no longer conservative and therefore may converge to the wrong solution [2]. For accuracy and robustness, constructing conservative and positivity-preserving schemes for computational gas dynamics has attracted a lot of research interest.

It is well known that many first-order schemes, such as the Lax-Friedrichs scheme [3, 4], the Godunov scheme [5], the HLLC scheme with appropriate choice of the wavespeeds [6] and gas-kinetic schemes [7], fulfill the positivity-preserving property for the compressible Euler equations. Several high-order positivity-preserving schemes, such as the discontinuous Galerkin (DG) schemes [8, 9, 10] and the finite-volume and finite-difference weighted essentially non-oscillatory (WENO) schemes [11, 12, 13, 14], have recently been proposed for the compressible Euler equations. However, extending positivity-preserving schemes for the compressible Euler equations to the compressible Navier-Stokes equations is difficult, because positivity-preserving discretizations are required for the nonlinear diffusion operator. Zhang [15] proposed a local Lax-Friedrichs type positivity-preserving flux for the compressible Navier-Stokes equations. With this flux and a simple positivity-preserving limiter, Zhang constructed high-order positivity-preserving Runge-Kutta DG schemes without losing local conservation or high-order accuracy.

The space-time conservation element and solution element (CE/SE) method [16, 17] is a special finite-volume type algorithm for solving convection-diffusion equations. The original CE/SE scheme is a central scheme which was designed particularly for triangular meshes in two dimensions (2D) [18] and tetrahedral meshes in three dimensions (3D) [19]. Later, the scheme was extended to quadrilateral and hexahedral meshes [20, 21, 22]. Shen *et al.* [20] implemented the scheme on unstructured meshes consisting of hybrid elements and derived the corresponding high-order schemes. To improve the resolution of CE/SE schemes for contact discontinuities, Shen *et al.* [23, 24] proposed upwind CE/SE schemes by introducing characteristic-based fluxes into the framework of the CE/SE method. Then, a maximum-principle-satisfying CE/SE scheme [25] was constructed via the use of the simple and efficient limiter proposed by Zhang and Shu [26], which, in general, can be applied to both central and upwind CE/SE schemes. That scheme was successfully used in the simulation of multi-component flows to preserve the positivity of volume fractions. However, the positivity of the density and pressure was not completely guaranteed for the compressible Navier-Stokes equations under some extreme conditions. Hu and Khoo [27] first developed an effective slope limiter for the one-dimensional CE/SE Euler solver using a kinetic energy fix technique. In this study, we focus

on developing positivity-preserving CE/SE schemes for solving the compressible Euler and Navier-Stokes equations on unstructured hybrid meshes while maintaining the conservativeness and accuracy of baseline CE/SE schemes.

The remainder of this paper is organized as follows. In Section 2, we introduce the baseline CE/SE schemes for solving the compressible Navier-Stokes equations on hybrid unstructured meshes. In Section 3, we describe an *a posteriori* spatial limiter to prevent negative densities and pressures on the premise of preserving optimal accuracy. In Section 4, we present several numerical examples to demonstrate the accuracy and the positivity-preserving property of the proposed schemes. We offer concluding remarks in Section 5.

2. A compressible Navier-Stokes CE/SE solver on hybrid unstructured meshes

2.1. Governing equations

Consider the dimensionless 2D compressible Navier-Stokes equations

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} - \frac{\partial \mathbf{F}_v}{\partial x} + \frac{\partial \mathbf{G}}{\partial y} - \frac{\partial \mathbf{G}_v}{\partial y} = 0, \quad (1)$$

where $\mathbf{U} = [\rho, \rho u, \rho v, \rho e]$ is the vector of conservative variables. All the variables are non-dimensionalized with respect to free stream conditions. The inviscid fluxes, $\mathbf{F}$ and $\mathbf{G}$, and the viscous fluxes, $\mathbf{F}_v$ and $\mathbf{G}_v$, read

$$\mathbf{F} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ (\rho e + p)u \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ (\rho e + p)v \end{bmatrix}, \quad (2a)$$

$$\mathbf{F}_v = \begin{bmatrix} 0 \\ \tau_{xx} \\ \tau_{xy} \\ u\tau_{xx} + v\tau_{xy} - q_x \end{bmatrix} \quad \text{and} \quad \mathbf{G}_v = \begin{bmatrix} 0 \\ \tau_{yx} \\ \tau_{yy} \\ u\tau_{yx} + v\tau_{yy} - q_y \end{bmatrix}, \quad (2b)$$

where u , v and e denote the x -wise and y -wise velocities and specific total energy, respectively. Under Stokes' hypothesis of negligible bulk viscosity effects, the viscous stresses can be expressed as follows:

$$\tau_{xy} = \tau_{yx} = \frac{\mu}{Re} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right), \quad (3a)$$

$$\tau_{xx} = \frac{\mu}{Re} \left(\frac{4}{3} \frac{\partial u}{\partial x} - \frac{2}{3} \frac{\partial v}{\partial y} \right) \quad \text{and} \quad \tau_{yy} = \frac{\mu}{Re} \left(\frac{4}{3} \frac{\partial v}{\partial y} - \frac{2}{3} \frac{\partial u}{\partial x} \right), \quad (3b)$$

where Re is the Reynolds number. The heat fluxes appearing in the viscous fluxes are modeled according to Fourier's law as

$$q_x = -\frac{\mu}{Re(\gamma - 1)M_\infty^2 Pr} \frac{\partial T}{\partial x} \quad \text{and} \quad q_y = -\frac{\mu}{Re(\gamma - 1)M_\infty^2 Pr} \frac{\partial T}{\partial y}, \quad (4)$$

where M_∞ and Pr are the freestream Mach number and Prandtl number, respectively. The perfect gas assumption is used. Thus, the specific total energy and equation of state are given by

$$e = \frac{p}{(\gamma - 1)\rho} + \frac{1}{2}(u^2 + v^2) \quad (5)$$

and

$$p = \frac{\rho T}{\gamma M_\infty^2}. \quad (6)$$

2.2. Central CE/SE schemes implemented on hybrid meshes

Shen *et al.* [28] constructed a central CE/SE scheme on hybrid meshes for solving the compressible Euler equations. First, we present a straightforward extension of that scheme for solving the compressible Navier-Stokes equations. Second, we extend the upwind CE/SE scheme [23, 24] to hybrid unstructured meshes.

To simplify the mathematical notation, we define

$$\tilde{\mathbf{F}} = \mathbf{F} - \mathbf{F}_v \text{ and } \tilde{\mathbf{G}} = \mathbf{G} - \mathbf{G}_v. \quad (7)$$

Next, we let the spatial coordinates, x and y , together with the temporal coordinate, t , be the new coordinates of a three dimensional Euclidean space, $\mathcal{E}_3$. Then, Eq. (1) can be written as $\nabla \cdot \mathbb{H} = 0$ with $\mathbb{H} = (\tilde{\mathbf{F}}, \tilde{\mathbf{G}}, \mathbf{U})$. Utilizing the Gauss's divergence theorem, we obtain

$$\iiint_{\mathcal{V}} \nabla \cdot \mathbb{H} dV = \iint_{\partial\mathcal{V}} \mathbb{H} \cdot \mathbf{ds} = 0, \quad (8)$$

where $\partial\mathcal{V}$ is the boundary of an arbitrary closed space-time region, $\mathcal{V}$, in $\mathcal{E}_3$, and $\mathbf{ds} = d\sigma \mathbf{n}$, where $d\sigma$ and $\mathbf{n} = (n_x, n_y, n_t)$ are respectively the area and the unit outward normal vector of an infinitesimal surface element on $\partial\mathcal{V}$.

We consider a general hybrid unstructured mesh consisting of triangular and quadrilateral elements defined by the solid mesh lines, as shown in Fig. 1(a). An original element is divided into smaller sub-elements by the dashed lines that define the staggered mesh and elements. Hereafter, we adopt the following notation:

- V_i is the i th vertex (marked as $\bullet$);
- G_i is the geometrical center of the i th staggered element, $\mathcal{T}_i$ (marked as $\blacktriangle$);
- C_j is the geometrical center of the j th original element, $\mathcal{K}_j$ (marked as $\blacksquare$);
- E_k is the midpoint of the k th edge (marked as $\circ$);
- P^n is the point at time level t^n .

Figure 1: Schematic diagram of (a) a spatial hybrid mesh and (b) definition of conservation element (CE) and solution element (SE) for a general polygonal element.

We note that G_i does not coincide with V_i in a general unstructured mesh. Current CE/SE schemes alternatively update the solutions at C_j and G_i . That is to say that if we know the solutions at C_j^n , we try to solve the solutions at $G_i^{n+1/2}$. To this end, we define a solution element (SE) and a conservation element (CE) at each solution point. As shown in Fig. 1(b), there are a total of M elements associated with G_i , $C_{j[m]}^n$ is the center of the m th associated element and $E_{k[m]}^n$ is the center of the m th associated edge. $\text{CE}(G_i^{n+1/2})$ is defined by the polyhedron $C_{j[1]}^n E_{k[1]}^n \cdots C_{j[M]}^n E_{k[M]}^n C_{j[1]}^{n+1/2} E_{k[1]}^{n+1/2} \cdots C_{j[M]}^{n+1/2} E_{k[M]}^{n+1/2}$ which is a volume element, and $\text{SE}(G_i^{n+1/2})$ is a collection of $M+1$ surface elements of one dimension lower, including $C_{j[1]}^{n+1/2} E_{k[1]}^{n+1/2} \cdots C_{j[M]}^{n+1/2} E_{k[M]}^{n+1/2}$, $E_{k[1]}^n V_i^n V_i^{n+1} E_{k[1]}^{n+1}$, $\dots$ and $E_{k[M]}^n V_i^n V_i^{n+1} E_{k[M]}^{n+1}$, intersecting at $V_i^{n+1/2}$.

Imposing the space-time conservation law, Eq. (8), on $\text{CE}(G_i^{n+1/2})$ yields

$$\sum_{m=1}^M \iint_{\Omega_{m,0}^{n+1/2}} \mathbf{U} d\sigma = \sum_{m=1}^M \iint_{\Omega_{m,0}^n} \mathbf{U} d\sigma - \sum_{m=1}^M \iint_{\Omega_{m,1}^n + \Omega_{m,2}^n} (\tilde{\mathbf{F}} n_x + \tilde{\mathbf{G}} n_y) d\sigma = 0, \quad (9)$$

where $\Omega_{m,0}^n$, $\Omega_{m,1}^n$ and $\Omega_{m,2}^n$ represent quadrangles $C_{j[m]}^n E_{k[m]}^n V_i^n E_{k[m-1]}^n$, $C_{j[m]}^n E_{k[m]}^n E_{k[m]}^{n+1/2} C_{j[m]}^{n+1/2}$ and $E_{k[m-1]}^n C_{j[m]}^n C_{j[m]}^{n+1/2} E_{k[m-1]}^{n+1/2}$. (Here, we note that $E_{k[0]} = E_{k[M]}$). To solve the integral terms in Eq. (9), we assume all the space-time fluxes, $\mathbf{U}$, $\tilde{\mathbf{F}}$ and $\tilde{\mathbf{G}}$, in each SE to be continuous and approximated by first-order Taylor expansions of the associated solution point. This means that the fluxes are piece-wise linearly distributed on the boundary surfaces of

$\text{CE}(G_i^{n+1/2})$. Thus, the integrations of the fluxes on any specific surface are simply equal to the area multiplied by the corresponding values at the centroid, i.e.,

$$\sum_{m=1}^M \iint_{\Omega_{m,0}^{n+1/2}} \mathbf{U} d\sigma = \mathbf{U} (G_i^{n+1/2}) S_0, \quad (10a)$$

$$\sum_{m=1}^M \iint_{\Omega_{m,0}^n} \mathbf{U} d\sigma = \sum_{m=1}^M \mathbf{U} (\bar{G}_m^n) S_m, \quad (10b)$$

and

$$\begin{aligned} & \sum_{m=1}^M \iint_{\Omega_{m,1}^n + \Omega_{m,2}^n} (\tilde{\mathbf{F}} n_x + \tilde{\mathbf{G}} n_y) d\sigma \\ &= \sum_{m=1}^M \sum_{l=1}^2 S_{m,l} (n_{x,m,l} \bar{\mathbf{F}}_{m,l}^n + n_{y,m,l} \bar{\mathbf{G}}_{m,l}^n), \\ &= \Delta t \sum_{m=1}^M \sum_{l=1}^2 \Delta y_{m,l} \bar{\mathbf{F}}_{m,l}^n - \Delta x_{m,l} \bar{\mathbf{G}}_{m,l}^n, \end{aligned} \quad (10c)$$

where the unit outward normal of $\Omega_{m,l}^n$

$$(n_{x,m,l}, n_{y,m,l}) = \left(\frac{\Delta y_{m,l}}{\sqrt{\Delta x_{m,l}^2 + \Delta y_{m,l}^2}}, -\frac{\Delta x_{m,l}}{\sqrt{\Delta x_{m,l}^2 + \Delta y_{m,l}^2}} \right),$$

the area of $\Omega_{m,l}^n$

$$\begin{aligned} S_{m,l} &= \Delta t \sqrt{\Delta x_{m,l}^2 + \Delta y_{m,l}^2}, \\ \Delta x_{m,1} &= x(E_{k[m]}^n) - x(C_{j[m]}^n), \quad \Delta y_{m,1} = y(E_{k[m]}^n) - y(C_{j[m]}^n), \\ \Delta x_{m,2} &= x(C_{j[m]}^n) - x(E_{k[m-1]}^n), \quad \Delta y_{m,2} = y(C_{j[m]}^n) - y(E_{k[m-1]}^n), \\ \Delta t &= t(G_i^{n+1/2}) - t(G_i^n). \end{aligned}$$

and $\bar{G}_m^n$ and S_m are the centroid and area of $\Omega_{m,0}^n$, which is a part of $\text{SE}(C_{j[m]}^n)$ and $S_0 = \sum_{m=1}^M S_m$. In addition, $(\bar{\mathbf{F}}_{m,l}^n, \bar{\mathbf{G}}_{m,l}^n)$ represents the average flux through $\Omega_{m,l}^n$ ($l = 1$ or 2), which is also associated with $\text{SE}(C_{j[m]}^n)$. The Taylor expansion on $\text{SE}(C_{j[m]}^n)$ yields

$$\mathbf{U}(\bar{G}_m^n) = \mathbf{U}(C_{j[m]}^n) + \mathbf{U}_x(C_{j[m]}^n) \Delta x_{m,0} + \mathbf{U}_y(C_{j[m]}^n) \Delta y_{m,0}, \quad (11a)$$

$$\begin{aligned}\bar{\mathbf{F}}_{m,1}^n &= \tilde{\mathbf{F}}\left(C_{j[m]}^n\right) + \tilde{\mathbf{F}}_x\left(C_{j[m]}^n\right) \frac{\Delta x_{m,1}}{2} \\ &+ \tilde{\mathbf{F}}_y\left(C_{j[m]}^n\right) \frac{\Delta y_{m,1}}{2} + \tilde{\mathbf{F}}_t\left(C_{j[m]}^n\right) \frac{\Delta t}{2},\end{aligned}\quad (11b)$$

$$\begin{aligned}\bar{\mathbf{G}}_{m,1}^n &= \tilde{\mathbf{G}}\left(C_{j[m]}^n\right) + \tilde{\mathbf{G}}_x\left(C_{j[m]}^n\right) \frac{\Delta x_{m,1}}{2} \\ &+ \tilde{\mathbf{G}}_y\left(C_{j[m]}^n\right) \frac{\Delta y_{m,1}}{2} + \tilde{\mathbf{G}}_t\left(C_{j[m]}^n\right) \frac{\Delta t}{2},\end{aligned}\quad (11c)$$

$$\begin{aligned}\bar{\mathbf{F}}_{m,2}^n &= \tilde{\mathbf{F}}\left(C_{j[m]}^n\right) - \tilde{\mathbf{F}}_x\left(C_{j[m]}^n\right) \frac{\Delta x_{m,2}}{2} \\ &- \tilde{\mathbf{F}}_y\left(C_{j[m]}^n\right) \frac{\Delta y_{m,2}}{2} + \tilde{\mathbf{F}}_t\left(C_{j[m]}^n\right) \frac{\Delta t}{2},\end{aligned}\quad (11d)$$

$$\begin{aligned}\bar{\mathbf{G}}_{m,2}^n &= \tilde{\mathbf{G}}\left(C_{j[m]}^n\right) - \tilde{\mathbf{G}}_x\left(C_{j[m]}^n\right) \frac{\Delta x_{m,2}}{2} \\ &- \tilde{\mathbf{G}}_y\left(C_{j[m]}^n\right) \frac{\Delta y_{m,2}}{2} + \tilde{\mathbf{G}}_t\left(C_{j[m]}^n\right) \frac{\Delta t}{2},\end{aligned}\quad (11e)$$

where

$$\Delta x_{m,0} = x\left(\bar{G}_m^n\right) - x\left(C_{j[m]}^n\right) \text{ and } \Delta y_{m,0} = y\left(\bar{G}_m^n\right) - y\left(C_{j[m]}^n\right).$$

Finally, by plugging Eqs. (10a) - (10c) into Eq. (9), along with Eqs. (11a) - (11e), we obtain an explicit one-step time-marching scheme,

$$\mathbf{U}\left(G_i^{n+1/2}\right) = \frac{\sum_{m=1}^M \mathbf{U}\left(\bar{G}_m^n\right) S_m - \Delta t \sum_{m=1}^M \sum_{l=1}^2 \Delta y_{m,l} \bar{\mathbf{F}}_{m,l}^n - \Delta x_{m,l} \bar{\mathbf{G}}_{m,l}^n}{S_0} \quad (12)$$

for $\mathbf{U}\left(G_i^{n+1/2}\right)$.

The Taylor expansions, Eqs. (11a) - (11e), require knowledge of $\mathbf{U}$, $\tilde{\mathbf{F}}$, $\tilde{\mathbf{G}}$ and their first-order derivatives. However, we need only to update and store $\mathbf{U}$ and its spatial derivatives at each solution point, because $\tilde{\mathbf{F}}$, $\tilde{\mathbf{G}}$ and their derivatives can be calculated using $\mathbf{U}$ and the derivatives of $\mathbf{U}$. In addition, the temporal derivative of $\mathbf{U}$ can be derived using the Cauchy-Kovalewski procedure, which is based on repeated differentiation of the following governing partial differential equation,

$$\frac{\partial \mathbf{U}}{\partial t} = -\frac{\partial \tilde{\mathbf{F}}}{\partial x} - \frac{\partial \tilde{\mathbf{G}}}{\partial y}. \quad (13)$$

This approach can derive high-order temporal accuracy in only one time step and it has been employed in the original essential non-oscillation (ENO) [29] and arbitrary high order derivatives (ADER) schemes [30, 31, 32, 33].

In CE/SE schemes, the spatial derivatives are not reconstructed based on cell averages as in conventional finite-volume schemes. Instead, they are updated by independent schemes. Different CE/SE schemes share an identical time-marching scheme for $\mathbf{U}$, but they have different time-marching schemes for the spatial derivatives [23]. The scheme described in [28] is an implementation of the central ‘ $a - \alpha$ ’ scheme [17] on hybrid unstructured meshes that is sensitive to the local Courant–Friedrichs–Lewy (CFL) number. When solving the compressible Navier-Stokes equations, the mesh points are usually assembled in the boundary layer and the local CFL number is highly nonuniform. To overcome this shortcoming, Chang [34, 35] proposed a Courant number insensitive (CNI) CE/SE scheme that can tune the dissipation via the local CFL number. When the CFL number approaches 1, the CNI scheme approaches the dissipative central ‘ $a - \alpha$ ’ scheme, and when the CFL number decreases, the CNI scheme gradually approaches the non-dissipative ‘ a ’ scheme [17, 36] to avoid excessive dissipation caused by the staggered meshes.

In this study, we extend the CNI scheme to hybrid unstructured meshes. For the general hybrid unstructured mesh shown in Fig. 1, we define a new point, $Q_{j[m]}$, for which the coordinates are calculated by

$$x_i(Q_{j[m]}) = \frac{\nu}{\nu_0} x_i(C_{j[m]}) + \left(1 - \frac{\nu}{\nu_0}\right) x_i(\bar{G}_m), \quad (14)$$

where ν and ν_0 are the local and global Courant numbers, respectively. Using the Taylor expansions in $SE(G_i^{n+1/2})$, we have

$$\mathbf{U}(Q_{j[m]}^{n+1/2}) = \mathbf{U}(G_i^{n+1/2}) + \mathbf{U}_x(G_i^{n+1/2}) \delta x_m + \mathbf{U}_y(G_i^{n+1/2}) \delta y_m, \quad (15)$$

where $\delta x_m = x(Q_{j[m]}) - x(G_i)$, $\delta y_m = y(Q_{j[m]}) - y(G_i)$ and $\mathbf{U}(Q_{j[m]}^{n+1/2})$ can be explicitly calculated using the Taylor expansion at $C_{j[m]}^n$. The same relation can be derived for $m+1$. And then the two unknown derivatives, $\mathbf{U}_x(G_i^{n+1/2})$ and $\mathbf{U}_y(G_i^{n+1/2})$, can be solved from the two equations for m and $m+1$. We can derive a total of M sets of derivatives for each element. A weighted average function then calculates optimal derivatives to capture shocks and to preserve accuracy in smooth regions [20, 28]. We note that when $\nu = \nu_0$, $Q_{j[m]}$ and $C_{j[m]}$ coincide and the present CNI scheme is equivalent to the central ‘ $a - \alpha$ ’ scheme in [28]. When $\nu \rightarrow 0$, $Q_{j[m]} \rightarrow \bar{G}_m$ and the scheme approaches a non-dissipative scheme.

2.3. Upwind CE/SE schemes implemented on hybrid meshes

Recently, Shen et al. [23, 24, 25] proposed an upwind CE/SE scheme for rectangular meshes that improves the scheme’s ability to capture contact discontinuities. Here, for the first time, we present a straightforward extension of that scheme to hybrid unstructured meshes.

In the upwind CE/SE scheme, $\text{CE}(G_i^{n+1/2})$ remains the same and is divided into M parts. The m -th part, $\text{CE}(G_{i,m}^{n+1/2})$, is defined by the polyhedron $C_{j[m]}^n E_{k[m]}^n V_i^n E_{k[m-1]}^n C_{j[m]}^{n+1/2} E_{k[m]}^{n+1/2} V_i^{n+1/2} E_{k[m-1]}^{n+1/2}$. However, the definition of SE slightly changes and $\text{SE}(G_i^{n+1/2})$ is defined by the polyhedron $C_{j[1]}^{n+1/2} E_{k[1]}^{n+1/2} \dots C_{j[M]}^{n+1/2} E_{k[M]}^{n+1/2} C_{j[1]}^{n+1} E_{k[1]}^{n+1} \dots C_{j[M]}^{n+1} E_{k[M]}^{n+1}$.

Integrating Eq. (8) over $\text{CE}(G_{i,m}^{n+1/2})$, we obtain

$$\begin{aligned} \mathbf{U}(\bar{G}_m^{n+1/2}) &= \mathbf{U}(\bar{G}_m^n) - \frac{\Delta t}{S_m} \sum_{l=1}^2 \Delta y_{m,l} \bar{\mathbf{F}}_{m,l}^n - \Delta x_{m,l} \bar{\mathbf{G}}_{m,l}^n \\ &\quad + \frac{\Delta t}{S_m} (L_{m-1} \hat{\mathbf{F}}_{m-1} - L_m \hat{\mathbf{F}}_m), \end{aligned} \quad (16)$$

where L_m is the length of $\overline{V_i E_{k[m]}}$ and $\hat{\mathbf{F}}_m$ is the average flux in the normal direction through the inner boundary, $V_i^n E_{k[m]}^n E_{k[m]}^{n+1/2} V_i^{n+1/2}$, of $\text{CE}(G_i^{n+1/2})$. In the modified definition of SE, $\Omega_{m,l}^n$ ($l = 0 - 2$) remain affiliated with $\text{SE}(C_{j[m]}^n)$, and hence $\mathbf{U}(\bar{G}_m^n)$ and $(\bar{\mathbf{F}}_{m,l}^n, \bar{\mathbf{G}}_{m,l}^n)$ ($l = 1 \text{ or } 2$) are continuous and can also be calculated by the Taylor expansions in $\text{SE}(C_{j[m]}^n)$ (Eqs. (11a)-(11e)). However, since $V_i^n E_{k[m]}^n E_{k[m]}^{n+1/2} V_i^{n+1/2}$ is located between $\text{SE}(C_{j[m]}^n)$ and $\text{SE}(C_{j[m+1]}^n)$, $\hat{\mathbf{F}}_m$ can be discontinuous. The inviscid part of $\hat{\mathbf{F}}_m$ is calculated by solving a local Riemann problem in the normal direction,

$$\hat{\mathbf{F}}_{inv,m} = \mathfrak{R}(\mathbf{U}_L, \mathbf{U}_R), \quad (17)$$

where $\mathbf{U}_L$ and $\mathbf{U}_R$ are the conservative values at the center of $V_i^n E_{k[m]}^n E_{k[m]}^{n+1/2} V_i^{n+1/2}$ reconstructed from $\text{SE}(C_{j[m]}^n)$ and $\text{SE}(C_{j[m+1]}^n)$, respectively. The derivatives are reconstructed by the WBAP limiter [37, 23] as

$$\mathbf{U}_{\xi,L}^* = \mathbf{U}_{\xi,L} \text{WBAP}(1, \theta_L) \text{ and } \mathbf{U}_{\xi,R}^* = \mathbf{U}_{\xi,R} \text{WBAP}(1, \theta_R), \quad (18)$$

where ξ represents x or y , $\theta_L = \frac{\mathbf{U}_{\xi,R}}{\mathbf{U}_{\xi,L}}$ and $\theta_R = \frac{\mathbf{U}_{\xi,L}}{\mathbf{U}_{\xi,R}}$. In Shen *et al.* [23, 24], Harten, Lax and van Leer (HLL) [38], contact restoring Harten, Lax and van Leer (HLLC) [39] and Roe [40] Riemann solvers are coupled with the upwind CE/SE scheme. In this study, we use the local Lax-Friedrichs (LLF) flux,

$$\hat{\mathbf{F}}_{inv,m}^{LF} = \frac{1}{2} [\hat{\mathbf{F}}_{inv,L} + \hat{\mathbf{F}}_{inv,R} - |v_n + c|_{max} (\mathbf{U}_R - \mathbf{U}_L)], \quad (19)$$

where v_n is the normal velocity. Hereafter, we refer to the upwind CE/SE scheme with LLF flux as the 'LLF CE/SE' scheme. As we can see, the LLF flux requires only information of the local maximum wave speed, which is necessary

for computing the time steps. The viscous flux is calculated by an arithmetic average,

$$\hat{\mathbf{F}}_{v,m} = \frac{1}{2} \left(\hat{\mathbf{F}}_{v,L} + \hat{\mathbf{F}}_{v,R} \right). \quad (20)$$

From the area-weighted average of $\mathbf{U} \left(\bar{G}_m^{n+1/2} \right)$ in Eq. (16), all the inner characteristic-based fluxes are canceled and exactly the same time-marching scheme as the central CE/SE scheme, Eq. (12), can be derived for $\mathbf{U} \left(G_i^{n+1/2} \right)$. Replacing $Q_{j[m]}^{n+1/2}$ with $\bar{G}_m^{n+1/2}$ in Eq. (15), since $\mathbf{U} \left(\bar{G}_m^{n+1/2} \right)$ is already solved by Eq. (16), we can calculate $\mathbf{U}_x \left(G_i^{n+1/2} \right)$ and $\mathbf{U}_y \left(G_i^{n+1/2} \right)$ using the same procedure as that of the central CE/SE scheme. We note that, in the upwind CE/SE scheme, we always use $\bar{G}_m^{n+1/2}$ to calculate the spatial derivatives. The scheme can thus avoid excessive dissipation when the local CFL number is too small without additional parameters [23, 24].

3. Positivity-preserving CE/SE schemes for Navier-Stokes equations

3.1. Preliminaries

Since $\rho(\mathbf{U}) = u_1$, where u_i is the i -th component of the conservative vector $\mathbf{U}$, we have

$$\rho \left(\sum \theta_i \mathbf{U}_i \right) = \sum \theta_i \rho(\mathbf{U}_i) \text{ for } \theta_i \in \mathbb{R}. \quad (21)$$

According to Eq. (5), the pressure, p , can be calculated by

$$p(\mathbf{U}) = (\gamma - 1) \left[u_4 - \frac{u_2^2 + u_3^2}{2u_1} \right]. \quad (22)$$

If $\rho(\mathbf{U}) \geq 0$, it is easy to verify that $\frac{\partial^2 p}{\partial^2 u_i} \leq 0$, and hence p is a concave function of $\mathbf{U}$. Jensen's inequality implies that

$$p \left(\frac{\sum \theta_i \mathbf{U}_i}{\sum \theta_i} \right) \geq \frac{\sum \theta_i p(\mathbf{U}_i)}{\sum \theta_i} \text{ for } \theta_i \geq 0. \quad (23)$$

3.2. The positivity-preserving slope limiter

Neither the central CE/SE scheme nor the upwind CE/SE scheme can preserve the positivity of densities and pressures. We are interested in constructing a CE/SE scheme which only admits positive densities and pressures without loss of accuracy and conservativeness.

Theorem 1. *If the spatial derivatives of $\mathbf{U}$ are equal to $\mathbf{0}$, then the CE/SE time-marching scheme expressed by Eq. (12) satisfies the positivity-preserving property.*

Proof. Define $\tilde{\mathbf{U}}_{j[m]}^{high} = \mathbf{U}(\bar{G}_m^n) - \frac{\Delta t}{S_m} \sum_{l=1}^2 \Delta y_{m,l} \bar{\mathbf{F}}_{m,l}^n - \Delta x_{m,l} \bar{\mathbf{G}}_{m,l}^n$. Then, Eq. (12) can be written as

$$\mathbf{U}(G_i^{n+1/2}) = \frac{\sum_{m=1}^M \tilde{\mathbf{U}}_{j[m]}^{high} S_m}{\sum_{m=1}^M S_m}. \quad (24)$$

When the spatial derivatives of $\mathbf{U}$ are equal to $\mathbf{0}$, the vectors $\mathbf{U}_t$, $\mathbf{F}_v$, $\mathbf{G}_v$, $\tilde{\mathbf{F}}_x$, $\tilde{\mathbf{F}}_y$, $\tilde{\mathbf{F}}_t$, $\tilde{\mathbf{G}}_x$, $\tilde{\mathbf{G}}_y$ and $\tilde{\mathbf{G}}_t$ are all equal to $\mathbf{0}$. As a result, Eq. (12) reduces to

$$\mathbf{U}(G_i^{n+1/2}) = \frac{\sum_{m=1}^M \tilde{\mathbf{U}}_{j[m]}^{low} S_m}{\sum_{m=1}^M S_m}, \quad (25)$$

where $\tilde{\mathbf{U}}_{j[m]}^{low} = \mathbf{U}(C_{j[m]}^n) - \frac{\Delta t}{S_m} [\Delta y_m \mathbf{F}(C_{j[m]}^n) - \Delta x_m \mathbf{G}(C_{j[m]}^n)]$ with $\Delta x_m = \Delta x_{m,1} + \Delta x_{m,2}$ and $\Delta y_m = \Delta y_{m,1} + \Delta y_{m,2}$. Define

$$\sigma_x = \frac{|\Delta x_m|(|v| + c)}{|\Delta x_m|(|v| + c) + |\Delta y_m|(|u| + c)}$$

and

$$\sigma_y = \frac{|\Delta y_m|(|u| + c)}{|\Delta x_m|(|v| + c) + |\Delta y_m|(|u| + c)}.$$

Then, we can split $\tilde{\mathbf{U}}_{j[m]}^{low}$ into two parts as

$$\tilde{\mathbf{U}}_{j[m]}^{low} = \tilde{\mathbf{U}}_{j[m],1}^{low} + \tilde{\mathbf{U}}_{j[m],2}^{low}, \quad (26)$$

where

$$\tilde{\mathbf{U}}_{j[m],1}^{low} = \sigma_y \mathbf{U}(C_{j[m]}^n) - \frac{\Delta t \Delta y_m}{S_m} \mathbf{F}(C_{j[m]}^n) \quad (27)$$

and

$$\tilde{\mathbf{U}}_{j[m],2}^{low} = \sigma_x \mathbf{U}(C_{j[m]}^n) + \frac{\Delta t \Delta x_m}{S_m} \mathbf{G}(C_{j[m]}^n). \quad (28)$$

To simplify the derivation, we drop the notation $C_{j[m]}^n$. Thus, we have

$$\tilde{\mathbf{U}}_{j[m],1}^{low} = \sigma_y \mathbf{U} - \frac{\Delta t \Delta y_m}{S_m} \mathbb{A} \cdot \mathbf{U}, \quad (29)$$

where the Jacobian

$$\mathbb{A} = \frac{\partial \mathbf{F}}{\partial \mathbf{U}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{\gamma-3}{2}u^2 + \frac{\gamma-1}{2}v^2 & (3-\gamma)u & (1-\gamma)v & \gamma-1 \\ -uv & v & u & 0 \\ -\gamma eu + (\gamma-1)Ku & \gamma e - \frac{\gamma-1}{2}(3u^2 + v^2) & (1-\gamma)uv & \gamma u \end{bmatrix}$$

with $K = u^2 + v^2$. The eigenvalues and corresponding eigenvectors of $\mathbb{A}$ are

$$\lambda_1 = u - c, \lambda_2 = u, \lambda_3 = u, \lambda_4 = u + c, \quad (30)$$

$$\mathbf{r}_1 = \begin{bmatrix} 1 \\ u - c \\ v \\ h - uc \end{bmatrix}, \mathbf{r}_2 = \begin{bmatrix} 0 \\ 0 \\ v \\ v^2 \end{bmatrix}, \mathbf{r}_3 = \begin{bmatrix} 1 \\ u \\ v \\ \frac{K}{2} \end{bmatrix} \text{ and } \mathbf{r}_4 = \begin{bmatrix} 1 \\ u + c \\ v \\ h + uc \end{bmatrix}, \quad (31)$$

where the specific total enthalpy is defined as $h = e + \frac{p}{\rho}$. The conservative vector $\mathbf{U}$ can be written as a convex combination of the eigenvectors as

$$\mathbf{U} = \frac{\rho}{2\gamma} \mathbf{r}_1 + \frac{(\gamma - 1)\rho}{\gamma} \mathbf{r}_3 + \frac{\rho}{2\gamma} \mathbf{r}_4. \quad (32)$$

Plugging Eq. (32) into Eq. (29) yields

$$\begin{aligned} \tilde{\mathbf{U}}_{j[m],1}^{low} &= \frac{\rho}{2\gamma} \left[\sigma_y - \frac{\Delta t \Delta y_m \lambda_1}{S_m} \right] \mathbf{r}_1 + \frac{(\gamma - 1)\rho}{\gamma} \left[\sigma_y - \frac{\Delta t \Delta y_m \lambda_3}{S_m} \right] \mathbf{r}_3 \\ &+ \frac{\rho}{2\gamma} \left[\sigma_y - \frac{\Delta t \Delta y_m \lambda_4}{S_m} \right] \mathbf{r}_4. \end{aligned} \quad (33)$$

If the following CFL condition,

$$\Delta t < \text{MIN} \left\{ \frac{S_m}{|\Delta x_m| (|v| + c) + |\Delta y_m| (|u| + c)} \right\}, \quad (34)$$

holds, then $\tilde{\mathbf{U}}_{j[m],1}^{low}$ is obviously a convex combination of $\mathbf{r}_1$, $\mathbf{r}_3$ and $\mathbf{r}_4$. In addition, it is trivial to verify that

$$\rho(\mathbf{r}_1) = \rho(\mathbf{r}_3) = \rho(\mathbf{r}_4) = 1, \quad (35)$$

$$p(\mathbf{r}_1) = p(\mathbf{r}_4) = \frac{\gamma(3 - \gamma)p}{2\rho} > 0 \text{ and } p(\mathbf{r}_3) = 0. \quad (36)$$

According to Eqs. (21) and (23), if the solution at the previous time step is positive, we have $\rho(\tilde{\mathbf{U}}_{j[m],1}^{low}) > 0$ and $p(\tilde{\mathbf{U}}_{j[m],1}^{low}) > 0$. Using the same derivation, we can prove that $\rho(\tilde{\mathbf{U}}_{j[m],2}^{low}) > 0$ and $p(\tilde{\mathbf{U}}_{j[m],2}^{low}) > 0$. Again, utilizing Eqs. (21) and (23), we have $\rho(\tilde{\mathbf{U}}_{j[m]}^{low}) > 0$ and $p(\tilde{\mathbf{U}}_{j[m]}^{low}) > 0$ under the CFL condition in Eq. (34). ■

Next, we show that when we use a limiter for the spatial derivatives, CE/SE schemes guarantee the positivity of densities and pressures for solving the Navier-Stokes equations on hybrid unstructured meshes. When solving $\mathbf{U}(G_i^{n+1/2})$,

we limit the derivatives at the original cell centers, C_j^n . We first enforce the densities to be positive using Algorithm 1. Then we enforce the pressures to be positive using Algorithm 2. (We note that, in Algorithm 1 and Algorithm 2, the index m means C_j is the m th element center connected with G_i , while the index k means G_i is the k th vertex connected with C_j .) Since $\rho(\tilde{\mathbf{U}}_{j[m]}^{low}) > 0$

Algorithm 1 The limiter for preserving positive densities

```

1: for  $k = 1 \rightarrow K$  (the number of vertexes,  $G_i^{n+1/2}$ , connected with  $C_j^n$ ) do
2:   if  $\rho(\tilde{\mathbf{U}}_{j[m]}^{high}) \leq 0$  then
3:     Set  $\varepsilon_\rho = \text{MIN}(10^{-13}, \rho(\tilde{\mathbf{U}}_{j[m]}^{low}))$ ;
4:      $\theta_{\rho,j[k]} = \frac{\varepsilon_\rho - \rho(\tilde{\mathbf{U}}_{j[m]}^{low})}{\rho(\tilde{\mathbf{U}}_{j[m]}^{high}) - \rho(\tilde{\mathbf{U}}_{j[m]}^{low})}$ ;
5:   else
6:      $\theta_{\rho,j[k]} = 1$ ;
7:   end if
8: end for
9:  $\theta_{\rho,j} = \text{MIN}(\theta_{\rho,j[1]}, \dots, \theta_{\rho,j[K]})$ ;
10:

```

$$\tilde{\mathbf{U}}_{j[m]}^* = \theta_{\rho,j} \tilde{\mathbf{U}}_{j[m]}^{high} + (1 - \theta_{\rho,j}) \tilde{\mathbf{U}}_{j[m]}^{low}. \quad (37)$$

Algorithm 2 The limiter for preserving positive pressures

```

1: for  $k = 1 \rightarrow K$  (the number of vertexes,  $G_i^{n+1/2}$ , surrounding  $C_j^n$ ) do
2:   if  $p(\tilde{\mathbf{U}}_{j[m]}^*) \leq 0$  then
3:      $\varepsilon_p = \text{MIN}(10^{-13}, p(\tilde{\mathbf{U}}_{j[m]}^{low}))$ ;
4:      $\theta_{p,j[k]} = \frac{\varepsilon_p - p(\tilde{\mathbf{U}}_{j[m]}^{low})}{p(\tilde{\mathbf{U}}_{j[m]}^*) - p(\tilde{\mathbf{U}}_{j[m]}^{low})}$ ;
5:   else
6:      $\theta_{p,j[k]} = 1$ ;
7:   end if
8: end for
9:  $\theta_{p,j} = \text{MIN}(\theta_{p,j[1]}, \dots, \theta_{p,j[K]})$ ;
10:

```

$$\tilde{\mathbf{U}}_{j[m]}^{**} = \theta_{p,j} \tilde{\mathbf{U}}_{j[m]}^* + (1 - \theta_{p,j}) \tilde{\mathbf{U}}_{j[m]}^{low}. \quad (38)$$

and $p(\tilde{\mathbf{U}}_{j[m]}^{low}) > 0$ according to Theorem 1, therefore we have $0 \leq \theta_{\rho,j} \leq 1$ and $0 \leq \theta_{p,j} \leq 1$ in Algorithm 1 and Algorithm 2. Utilizing Eqs. (21) and (23), it is easy to verify that $\rho(\tilde{\mathbf{U}}_{j[m]}^*) > 0$, $\rho(\tilde{\mathbf{U}}_{j[m]}^{**}) > 0$ and $p(\tilde{\mathbf{U}}_{j[m]}^{**}) > 0$ via simple algebra. Finally, we replace $\tilde{\mathbf{U}}_{j[m]}^{high}$ in Eq. (24) with $\tilde{\mathbf{U}}_{j[m]}^{**}$ derived by Eq. (38).

Obviously, the time-marching scheme, Eq. (24), is positivity-preserving after implementing this limiter process, since $\mathbf{U} \left(G_i^{n+1/2} \right)$ is a convex combination of $\tilde{\mathbf{U}}_{j[m]}^{high}$. Combining Eqs. (37) and (38), we have

$$\tilde{\mathbf{U}}_{j[m]}^{**} = \tilde{\mathbf{U}}_{j[m]}^{low} + \theta_{\rho,j} \theta_{p,j} \left(\tilde{\mathbf{U}}_{j[m]}^{high} - \tilde{\mathbf{U}}_{j[m]}^{low} \right). \quad (39)$$

In $\left(\tilde{\mathbf{U}}_{j[m]}^{high} \right)$, the viscous terms are directly calculated from $\mathbf{U} \left(C_j^n \right)$, $\mathbf{U}_x \left(C_j^n \right)$ and $\mathbf{U}_y \left(C_j^n \right)$, e.g.,

$$\frac{\partial u}{\partial x} = \frac{1}{u_1} \frac{\partial u_2}{\partial x} - \frac{u_2}{u_1^2} \frac{\partial u_1}{\partial x}. \quad (40)$$

Neglecting the derivatives of viscous flux, which consists of second-order terms, we have

$$\frac{\partial \tilde{\mathbf{F}}}{\partial \xi} = \frac{\partial \mathbf{F}}{\partial \mathbf{U}} \cdot \frac{\partial \mathbf{U}}{\partial \xi} \text{ and } \frac{\partial \tilde{\mathbf{G}}}{\partial \xi} = \frac{\partial \mathbf{G}}{\partial \mathbf{U}} \cdot \frac{\partial \mathbf{U}}{\partial \xi}. \quad (41)$$

Therefore, the term $\left(\tilde{\mathbf{U}}_{j[m]}^{high} - \tilde{\mathbf{U}}_{j[m]}^{low} \right)$ in Eq. (39) is a linear combination of $\mathbf{U}_x \left(C_j^n \right)$ and $\mathbf{U}_y \left(C_j^n \right)$ and the limiting process is equivalent to applying a limiter, $\theta_{\rho,j} \theta_{p,j}$, to the spatial derivatives. The limiter does not affect the cell averages, and the conservation law is thus preserved. Furthermore, similar to the multi-dimensional optimal order detection (MOOD) method [41, 42, 10], the limiter is based on an *a posteriori* detection of problematic cells in which negative densities or pressures appear. This will therefore not disrupt the overall accuracy of the original scheme. We note that this limiter can be considered a variation of Hu *et al.*'s [12] positivity-preserving flux limiter for finite-volume/difference schemes solving the Euler equations. A rigorous proof of the accuracy of the limiter can be found therein. Since central and upwind CE/SE schemes share the same time-marching scheme for $\mathbf{U}$, the limiter can be used with both central and upwind CE/SE schemes.

4. Numerical examples

4.1. Isentropic vortex evolution problem

We use this problem to test the effect of the positivity-preserving limiter on the accuracy of the CE/SE schemes for smooth flow regions. The computational domain is $[-5, 5] \times [-5, 5]$ and periodic boundary conditions are applied for all boundaries. Initially, the status of the mean flow is $\rho = 1, p = 1$ and $(u, v) = (1, 1)$. An isentropic vortex (i.e. perturbations in (u, v) and the temperature $T = p/\rho$, and no perturbation in the entropy, $S = p/\rho^\gamma$) is added to the mean flow. The resulting perturbations can be expressed as

$$(\delta u, \delta v) = \frac{\psi}{2\pi} e^{0.5(1-r^2)} (-y, x), \delta T = -\frac{(\gamma-1)\psi^2}{8\gamma\pi^2} e^{(1-r^2)} \text{ and } \delta S = 0,$$

Figure 2: (a) The uniform rectangular mesh and the irregular triangular mesh for isentropic vortex evolution problem, and the mesh size $h=1/2$.

h	CNI CE/SE				LLF CE/SE			
	L_1 error	Order	L_∞ error	Order	L_1 error	Order	L_∞ error	Order
1/2	4.51E-3	-	9.06E-2	-	2.31E-3	-	3.70E-2	-
1/4	9.24E-4	2.29	1.91E-2	2.25	4.50E-4	2.36	7.30E-3	2.34
1/8	1.66E-4	2.48	3.75E-3	2.35	8.56E-5	2.39	1.22E-3	2.58
1/16	2.91E-5	2.51	6.77E-4	2.47	1.98E-5	2.11	2.54E-4	2.26
1/32	6.00E-6	2.28	1.30E-4	2.38	5.01E-6	1.98	6.80E-5	1.90

Table 1: Accuracy of the positivity-preserving CNI CE/SE and LLF CE/SE schemes for isentropic vortex evolution problem on uniform rectangular meshes.

where $r^2 = x^2 + y^2$ and ψ is the vortex strength which is set as 5 in current tests. In order to compare with the exact solution, the Reynolds number is set to ∞ .

The exact solution of the problem is the passive convection of the vortex at the mean velocity. We compute this problem to $t = 2.0$ using uniform rectangular meshes and irregular triangular meshes, as shown in Fig. 2. The average errors, L_0 , and the maximum errors, L_∞ , of the densities calculated by the positivity-preserving CNI CE/SE and LLF CE/SE schemes on uniform rectangular meshes and irregular triangular meshes are listed in Tables 1 and 2, respectively. The results show that the positivity-preserving slope limiter does not affect the convergence rates of the CE/SE schemes on both meshes.

4.2. Sedov's point blast wave problem

The computational domain is a circle with $r = 1.2$. There are 4,752 triangular elements within $r < 0.5$ and 19,200 quadrilateral elements within $0.5 <$

h	CNI CE/SE				LLF CE/SE			
	L_1 error	Order	L_∞ error	Order	L_1 error	Order	L_∞ error	Order
1/2	3.53E-3	-	6.88E-2	-	1.50E-3	-	2.02E-2	-
1/4	6.85E-4	2.37	1.39E-2	2.31	2.77E-4	2.44	4.75E-3	2.09
1/8	1.14E-4	2.59	2.07E-3	2.75	6.08E-5	2.19	1.00E-3	2.25
1/16	1.94E-5	2.56	4.41E-4	2.23	1.50E-5	2.02	2.95E-4	1.76
1/32	4.03E-6	2.27	1.06E-4	2.06	3.91E-6	1.94	8.20E-5	1.85

Table 2: Accuracy of the positivity-preserving CNI CE/SE and LLF CE/SE schemes for isentropic vortex evolution problem on irregular triangular meshes.

Figure 3: Zoom-in view of the mesh for Sedov's point blast wave problem.

Figure 4: (a, b) Density contours at $t = 1$, (c, d) density profiles along the horizontal radius, (e, f) histories of minimum density and minimum pressure and (g, h) errors of the mass and total energy for Sedov's point blast wave problem with $Re = 10^5$. Left column: results by CNI CE/SE, right column: results by LLF CE/SE.

$r < 1.2$, as shown in Fig. 3. The initial condition is

$$(\rho, u, v, p) = \begin{cases} (1, 0, 0, 0.979264(\gamma - 1)/A_0), & x = 0, y = 0, \\ (1, 0, 0, 10^{-8}), & \text{otherwise,} \end{cases} \quad (42)$$

where $\gamma = 1.4$ and A_0 is the area of the element located at the origin of the axes. We set $Re = 10^5$ with constant viscosity. The extremely large pressure ratio between the two initial states in Eq. (42) yields the formation of a strong blast wave that is accompanied by a sudden strong expansion fan. This feature leads to the appearance of a region very close to the vacuum state. Therefore, negative densities and pressures very likely appear in the numerical simulations. The baseline CE/SE schemes without the positivity-preserving limiter would lead to negative density and the simulation would rapidly blow up. Fig. 4 shows the density contours at $t = 1$, the density profile along the horizontal radius, the history of the minimum density and minimum pressure, and the errors in the mass and total energy computed using the CNI CE/SE and LLF CE/SE schemes. The error in the generic quantity, X , is measured by $(X - X_0)/X_0$, where X_0 is the initial value of X . Good agreement between Sedov's inviscid theory and the numerical results is achieved as shown in Figs. 4(c) and 4(d). Figs. 4(e) and 4(f) clearly show that the minimum density and minimum pressure remain positive during the whole simulations. Meanwhile, the mass and the total energy of the system are well conserved as observed in Figs. 4(g) and 4(h).

4.3. The two-dimensional Noh problem

The Noh problem [43] depicts an shock waves traveling outward at infinite large Mach number. This test case has been widely used to test the numerical schemes' ability to simulate zero pressure flows involving strong shock waves [12, 44, 45]. The computational domain is a circle with $R = 0.25$ divided by a similar mesh with the Sedov's point blast wave problem. There are 1,810 irregular triangular elements within $r < 0.1$ and 100×100 quadrilateral elements within $0.1 < r < 0.25$. The initial condition is $(\rho, u, v, p) = (1, -x/r, -y/r, 10^{-8})$, where $r = \sqrt{x^2 + y^2}$. The ratio of specific heat, γ , is equal to $5/3$. We set $Re = 10^5$ with constant viscosity. According to the literature [43], the radial velocity of the shock is $1/3$ and the analytical inviscid solution of the density for the 2D Noh problem is

$$\rho = \begin{cases} 16, & r < t/3, \\ 1 + t/r, & r \geq t/3. \end{cases} \quad (43)$$

On the boundary, the velocity and pressure are set to the initial value, and the density is set according to the analytical solution. We terminate the computations at $t = 0.6$. Fig. 5 shows the numerical results comparing with the analytical solution. The numerical density near the origin of coordinates is lower than the analytical solution, because the origin of coordinates is a singular point. Similar results are observed by other schemes [12, 44, 45].

Figure 5: (a, b) Density contours and (c, d) density profiles along the horizontal radius at $t = 0.6$ for the 2D Noh problem with $Re = 10^5$. Left column: results computed by CNI CE/SE; right column: results computed by LLF CE/SE.

Figure 6: Zoom-in view of the mesh for extremely high Mach number jet problem.

Figure 7: (a, c) Density contours and (b, d) pressure contours of the logarithmic scale at $t = 10^{-3}$ for the extremely high Mach number jet problem with $Re = 10^5$. Upper: results computed by CNI CE/SE; lower: results computed by LLF CE/SE.

Figure 8: Zoom-in view of the mesh for the viscous shock tube problem.

4.4. The extremely high Mach number jet problem

The computational domain is a $[0, 1] \times [-0.25, 0.25]$ rectangle. The lower half of the domain is divided into 800×200 uniform rectangular elements and the upper half of the domain is divided into 365,278 nonuniform triangular elements. A zoom-in view of the mesh near the central line is shown in Fig. 6. Initially, the computational domain is full of ambient gas with $(\rho, u, v, p) = (0.5, 0, 0, 0.4127)$. The inflow condition,

$$(\rho, u, v, p) = \begin{cases} (0.5, 0, 0, 0.4127), & \text{if } |y| > 0.05, \\ (5, 800, 0, 0.4127), & \text{otherwise,} \end{cases}$$

is applied to the left boundary. All the other boundaries are nonreflective. We set $\gamma = 5/3$. The speed of jet is 800, which is approximately Mach 2157 with respect to the speed of sound in the jet's gas. This case has been used to test positivity-preserving Euler solvers in [8, 11, 12, 14]. To validate our positivity-preserving Navier-Stokes CE/SE solver, we set $Re = 10^5$ with constant viscosity. Fig. 7 shows the density contours and pressure contours of the logarithmic scale at $t = 10^{-3}$ computed by CNI CE/SE and LLF CE/SE schemes. We observe that the triangular mesh and the rectangular mesh give similar solutions. The results computed by LLF CE/SE are better than those computed by CNI CE/SE and are comparable with the inviscid solutions obtained by high-order WENO schemes [11, 14] using the same grid resolution on uniform rectangular meshes.

4.5. The viscous shock tube problem

The viscous shock tube problem [46, 47, 48] is a classical test of the accuracy of compressible Navier-Stokes solvers in complex viscous flows in shock tubes.

(a)

(b)

Figure 9: Density contour lines computed by (a) CNI CE/SE and (b) LLF CE/SE at $t = 1$ for the viscous shock tube problem with $Re = 200$.

-	CNI CE/SE		LLF CE/SE	
	lower vortex	upper vortex	lower vortex	upper vortex
Height	0.147	0.152	0.163	0.168

Table 3: The sizes of the primary vortex in the viscous shock tube problem computed by the CNI CE/SE and LLF CE/SE schemes.

Although positivity does not pose stability issues in the original solvers for this test case, we use it to verify the accuracy of the positivity-preserving CE/SE schemes for the viscosity dominated complex flows. The computational domain is a unit square. The domain is first divided into 500×500 uniform rectangular elements, and then the upper half is diagonalized into $500 \times 250 \times 2$ uniform triangular elements. A zoom-in view of the mesh near the central line is shown in Fig. 8. Initially, the gas is divided into two parts by a vertical diaphragm at the center, $x = 0$. The state of the gas is given by

$$(\rho, u, v, p) = \begin{cases} (120, 0, 0, 120/\gamma), & \text{if } x < 0, \\ (1.2, 0, 0, 1.2/\gamma), & \text{otherwise,} \end{cases}$$

where $\gamma = 1.4$. The Reynolds number is 200 with constant viscosity. All the boundaries are non-slip adiabatic walls.

The simulations are terminated at $t = 1$. Fig. 9 shows the density contour lines computed by the CNI CE/SE and LLF CE/SE schemes. Compared with the results in literature, the CNI CE/SE result is similar to that computed by the MUSCL-TVD scheme in [46] and [47], while the LLF CE/SE result exhibits less dissipation and is comparable with that computed by the high-order WENO, ACM [47] and AUSMPW+ [48] schemes using the same grid resolution. The height of the primary vortex, which is sensitive to numerical dissipation, converged to 0.168 with the high-order AUSMPW+ scheme [48]. Table 3 lists the sizes of the primary vortex computed by the CNI CE/SE and LLF CE/SE schemes. The size of the upper vortex is larger than the lower one because the size of the triangular mesh is smaller than the size of the rectangular mesh. Moreover, the size of the vortex computed by the LLF CE/SE scheme is larger than that computed by the CNI CE/SE scheme, once again demonstrating that LLF CE/SE scheme is less dissipative than CNI CE/SE scheme.

5. Conclusions

We propose positive-preserving CE/SE schemes for solving the Euler and Navier-Stoke equations on hybrid unstructured meshes. The schemes require an *a posteriori* limiter that can be applied to both central and upwind CE/SE schemes. The proposed approach is robust and accurate in low density flows, for strong shocks, high Mach number flows and complex viscous flows on different types of meshes.

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