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Highlights

- Solving the Helmholtz equation with the minimum numerical dispersion.
- Solving the anisotropic acoustic equation without s-wave artifacts.
- Keeping the same sparse property as low order method.

A highly accurate finite-difference method with minimum dispersion error for solving the Helmholtz equation

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Abstract

Numerical simulation of the acoustic wave equation in either isotropic or anisotropic media is crucial to seismic modeling, imaging and inversion. Actually, it represents the core computation cost of these highly advanced seismic processing methods. However, the conventional finite-difference method suffers from severe numerical dispersion errors and S-wave artifacts when solving the acoustic wave equation for anisotropic media. We propose a method to obtain the finite-difference coefficients by comparing its numerical dispersion with the exact form. We find the optimal finite difference coefficients that share the dispersion characteristics of the exact equation with minimal dispersion error. The method is extended to solve the acoustic wave equation in transversely isotropic (TI) media without S-wave artifacts. Numerical examples show that the method is highly accurate and efficient.

Keywords: Acoustic wave equation, Anisotropic, Finite difference, Dispersion error.

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1. Introduction

Wave extrapolation is crucial to seismic modeling, imaging and inversion. The finite difference approximation of the wave equation in time admits an extrapolation approach that is favored widely because of its relative efficiency and simplicity [1, 2]. However, the standard finite-difference method could suffer from numerical dispersion errors. In this case, the numerical propagation velocity is a function of frequency even when we assume non-dispersive material. There are two main strategies to counter this problem. One strategy involves reducing the grid interval in space and time. The other strategy involves using high-order finite difference approximations. However, both strategies result in considerable additional cost to modeling.

In order to reduce the numerical dispersion error in time domain modeling, many algorithms has been proposed. [1] proposed an efficient high-order finite difference method. [3] and [4] proposed to reformulate the original equation to a system of equations and then use the compact finite difference scheme for the system to obtain highly accurate seismic modeling. [5] proposed the low-rank method for reducing the dispersion error caused by time sampling. [6] proposed to obtain the low-rank coefficients through solving an optimization problem. Thus, they obtained an unconditionally stable algorithm.

Reducing the numerical dispersion error in seismic modeling in frequency domain requires less effort. For example, [7] combine two discretization of the Laplacian operators on the classical Cartesian coordinate system and the 45° rotated system to mitigate the anisotropy of the stencil. In addition, weighted averaging of the mass acceleration term over the grid-points of the FD stencil [8] helps obtain a compact stencil to reduce the numerical dispersion error and obtain the so called Mixed-grid finite-difference methods. The extension of the mixed-grid method [7] to higher accuracy was developed by [9]. They applied a 25-point stencil to the acoustic wave equation. Four directions (0° , 26.6° , 45° and 63.4°) were considered to discretize the Laplacian operator. Another extension of the mixed-grid method was proposed by [10]. The mixed-grid

method was also extended to solve the three dimensional isotropic [11] and anisotropic acoustic wave equations [12, 13, 14]. [15] developed an average-derivative optimal scheme to deal with varying directional sampling intervals to enable the usage of rectangular grid cells with different grid spacing in the x - and z -directions. However, most of these methods are based on utilizing the conventional discrete method in many directions and utilizing weight factors. As such, these methods inherently rely on the discretization approximation and they are difficult to extend to solve the acoustic wave equation in anisotropic media.

For seismic modeling in the frequency domain, especially for a smooth right hand side, the so called compact high-order method [16, 17, 18, 19, 20, 21] provides an efficient and accurate solution. These methods maintain the same sparsity of the operator of the second-order finite-difference approximation and admit the accuracy of the sixth-order approximation [21]. They can deal with not only the homogeneous medium case, but also the inhomogeneous medium case [21], which is needed for most seismic imaging and inversion applications. The compact high-order method can even be extended to solve the elliptic-anisotropic equation in the frequency domain[19]. The absorbing boundary condition [22] can also be incorporated efficiently in such methods.

Alternatively, a linear system is formulated from the discrete approximation of Helmholtz equation in the frequency domain. Solving such a system numerically falls mainly under two categories of methods. The more popular direct methods, which seek to find an efficient way to represent the inverse of the linear system. As soon as such an inverse, though costly, is established, it can be used to solve efficiently for as many source functions as needed. However, direct methods suffer from the large cost, especially in 3D media, of finding the inverse. However, recent advances in direct solvers using block low-rank simplifications [23, 24, 25, 26] reduce the cost of Lower-Upper decomposition. Another category for solving the discrete approximation of Helmholtz equation are the so-called iterative methods. In such methods, the key is to utilize an effective preconditioner. [27] proposed a multi-grid method with a heavily damped wave

equation as a preconditioner. After that, [28] applied it to full-waveform inversion. [29] proposed a sweeping preconditioner by an approximate Lower-Upper decomposition for a smaller layered region, which eliminates the unknowns layer
65 by layer. [30] extended the idea to unstructured grid using finite element methods. In our proposal in handling more complex physics, our approach can be incorporated in direct or iterative solvers. Nevertheless, for simplicity, we will use direct solvers in our numerical implementation.

To keep frequency domain solutions practical, we propose, in this paper, to
70 incorporate the anisotropic assumption of the medium in the Helmholtz solver at an isotropic cost. [31] and [32] introduced elegant formulas describing P -wave propagation in anisotropic acoustic media. However, its direct implementation in the frequency domain usually suffers from the S -wave artifacts [12, 13, 14]. These S -wave artifacts are relatively weak when the source is located in isotropic
75 media [32]. However, in considering scattering like those used in Reflected waveform inversion [33, 34, 35, 36], the source can be in anisotropic media. In these cases, we do need a discrete method for anisotropic wave equation free of S -wave artifacts.

This paper is organized as follows. We first derive the new highly accurate
80 finite-difference method with minimum dispersion error. Then the method is extended to extrapolate the wavefield free of S -wave artifacts in transversely isotropic (TI) media. After that, several numerical examples are shown to demonstrate the accuracy of the proposed method and compare it with existing methods. At last, we apply the proposed method to reverse time migration
85 (RTM) of the benchmark BP model to show its power.

2. Optimizing the Helmholtz coefficients in isotropic media

Let us consider the general form of the discretized Helmholtz wave equation in the frequency domain:

$$L(\mathbf{p}, \omega)u = f, \quad (1)$$

where f is a given source function. The Helmholtz operator $L(\mathbf{p}, \omega)$ in its discrete form is given by a matrix, referred to as the stiffness matrix, and depends on the medium parameters, $\mathbf{p}$, and angular frequency ω . Our objective is to solve for the complex wavefield, u , using this linear system. In the case of a three-dimensional isotropic medium, $L(\mathbf{p}, \omega) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{\omega^2}{v^2}$. Applying Fourier transform of the operator L yields the corresponding dispersion relation

$$-(k_x^2 + k_y^2 + k_z^2) + \frac{\omega^2}{v^2} = 0, \quad (2)$$

where $\vec{k} = (k_x, k_y, k_z)$ is the wavenumber vector. The general finite-difference approximation at grid point (i, j, k) can be written as:

$$\begin{aligned} (L_h u)_{ijk} \approx & \sum_{i_1=-m_1}^{i_1=m_1} \sum_{j_1=-m_2}^{j_1=m_2} \sum_{k_1=-m_3}^{k_1=m_3} a_{i_1 j_1 k_1} u_{i+i_1, j+j_1, k+k_1} \\ & + \frac{\omega^2}{v^2} \sum_{i_1=-m_1}^{i_1=m_1} \sum_{j_1=-m_2}^{j_1=m_2} \sum_{k_1=-m_3}^{k_1=m_3} b_{i_1 j_1 k_1} u_{i+i_1, j+j_1, k+k_1}. \end{aligned} \quad (3)$$

In the above formulation, the parameter $\{m_i\}_{i=1}^3$ will decide the accuracy of the approximation and $a_{i_1 j_1 k_1}$ and $b_{i_1 j_1 k_1}$ are the finite-difference coefficients obtained using a Taylor series expansion of the standard finite-difference approximation. As shown in Figure 1, the higher $\{m_i\}_{i=1}^3$ results in smaller dispersion errors. However, high-order finite difference approximations usually result in a discrete matrix with larger bandwidth and less sparsity. This will result in a more expensive Lower-Upper decomposition as shown in Table 1. This is the reason why most frequency domain implementations are based on low-order finite-difference approximations. Here, we want to maintain the sparsity and bandwidth of the lower order discretization and reduce the dispersion error at the same time. The dispersion relation of the discrete approximation is :

$$\begin{aligned} 0 = & \sum_{i_1=-m_1}^{i_1=m_1} \sum_{j_1=-m_2}^{j_1=m_2} \sum_{k_1=-m_3}^{k_1=m_3} a_{i_1 j_1 k_1} e^{I i_1 \Delta x k_x} e^{I j_1 \Delta y k_y} e^{I k_1 \Delta z k_z} \\ & + \frac{\omega^2}{v^2} \sum_{i_1=-m_1}^{i_1=m_1} \sum_{j_1=-m_2}^{j_1=m_2} \sum_{k_1=-m_3}^{k_1=m_3} b_{i_1 j_1 k_1} e^{I i_1 \Delta x k_x} e^{I j_1 \Delta y k_y} e^{I k_1 \Delta z k_z}, \end{aligned} \quad (4)$$

where I is the imaginary unit of complex number. Comparing the exact dispersion relation 2 with the approximate dispersion relation 4, we obtain

$$R(k_x, k_y, k_z) = \sum_{i_1=-m_1}^{i_1=m_1} \sum_{j_1=-m_2}^{j_1=m_2} \sum_{k_1=-m_3}^{k_1=m_3} a_{i_1 j_1 k_1} e^{I i_1 \Delta x k_x} e^{I j_1 \Delta y k_y} e^{I k_1 \Delta z k_z} + |\vec{k}|^2 \sum_{i_1=-m_1}^{i_1=m_1} \sum_{j_1=-m_2}^{j_1=m_2} \sum_{k_1=-m_3}^{k_1=m_3} b_{i_1 j_1 k_1} e^{I i_1 \Delta x k_x} e^{I j_1 \Delta y k_y} e^{I k_1 \Delta z k_z} = 0, \quad (5)$$

110 where $|\vec{k}| = \sqrt{k_x^2 + k_y^2 + k_z^2}$. Since we have only limited coefficients $a_{i_1 j_1 k_1}$ and $b_{i_1 j_1 k_1}$, equation 5 can not be satisfied for any (k_x, k_y, k_z) . To obtain the finite-difference coefficients with the minimal dispersion error, we propose to obtain the finite difference coefficients by solving the following optimization problem:

$$\min_{a_{i_1 j_1 k_1}, b_{i_1 j_1 k_1}} \max_{k \in \Omega} \sqrt{|R_1(k_x, k_y, k_z)|}, \quad (6)$$

where the dispersion error $R_1(k_x, k_y, k_z)$ is defined as:

$$\sqrt{\frac{-\sum_{i_1=-m_1}^{i_1=m_1} \sum_{j_1=-m_2}^{j_1=m_2} \sum_{k_1=-m_3}^{k_1=m_3} a_{i_1 j_1 k_1} e^{I i_1 \Delta x k_x} e^{I j_1 \Delta y k_y} e^{I k_1 \Delta z k_z}}{\sum_{i_1=-m_1}^{i_1=m_1} \sum_{j_1=-m_2}^{j_1=m_2} \sum_{k_1=-m_3}^{k_1=m_3} b_{i_1 j_1 k_1} e^{I i_1 \Delta x k_x} e^{I j_1 \Delta y k_y} e^{I k_1 \Delta z k_z}}} - |\vec{k}|^2}. \quad (7)$$

115 Before defining the approximate domain Ω , we need to impose some constraints on the coefficients $a_{i_1 j_1 k_1}$ and $b_{i_1 j_1 k_1}$ so we can obtain a unique solution. Actually, if we multiply any number to both $a_{i_1 j_1 k_1}$ and $b_{i_1 j_1 k_1}$, the objective function will admit the same value. So we add the following constraints to the above optimization problem:

$$\sum_{i_1=-m_1}^{i_1=m_1} \sum_{j_1=-m_2}^{j_1=m_2} \sum_{k_1=-m_3}^{k_1=m_3} a_{i_1 j_1 k_1} = 0, \quad \sum_{i_1=-m_1}^{i_1=m_1} \sum_{j_1=-m_2}^{j_1=m_2} \sum_{k_1=-m_3}^{k_1=m_3} b_{i_1 j_1 k_1} = 1. \quad (8)$$

120 If we discretize n_p points per wavelength, we end up with the approximation domain of the wavenumber as:

$$\Omega_{n_p} = \{(k_x, k_y, k_z) | |k_x \Delta x| < \frac{2\pi}{n_p}, |k_y \Delta y| < \frac{2\pi}{n_p}, |k_z \Delta z| < \frac{2\pi}{n_p}\}. \quad (9)$$

Table 1: CPU time comparison. The first (second, third) row is the relative CPU time by 2nd(4th, 8th) order finite difference method. The last row is the CPU time by optimized compact finite difference.

Method	Decomposition(s)	Solve (s)	Total (s)
2nd order	59	2	61
4th order	340	5	345
8th order	5719	13	5732
Proposed	59	3	62

The optimization problem 6 is a nonlinear optimization. However, we can obtain the initial guess from the following optimization problem:

$$\min_{a_{i_1 j_1 k_1}, b_{i_1 j_1 k_1}} \sum_{k \in \Omega} |R(k_x, k_y, k_z)|^2. \quad (10)$$

The solution of 10 can be easily obtained by solving a least squares problem.

125 The resulting optimal coefficients and the dispersion error for different n_p is shown in Table 2 for a space sampling $dy = dz$ in the two dimensional case. Figure 2(a) (2(b) and 2(c)) shows the dispersion error for different methods at $k_y = k_z = 0$ ($k_x = k_y, k_z = 0, k_x = k_y = k_z$). As we can see from these figures, the optimal finite-difference method obtains dispersion errors on an order similar
130 to the 8th-order finite-difference approximation, even though the cost will be the same as the 2nd order finite-difference method. For comparison, we also show the dispersion error of compact high-order finite-difference method[21] in Figure 2. The compact high-order finite difference method[21] has been mathematically proven to be at least 6th order.

135 3. Extension to the TI case

Compared to the isotropic case, solving the acoustic wave equation in anisotropic media is challenging. The acoustic wave equation in anisotropic media was introduced by [37] for the purpose of resolving the quasi-P wave propagation in transversely isotropic (TI) media. As a simple example of extending the proposed method to solve the acoustic wave equation for general anisotropic media,
140

Table 2: The resulting finite difference coefficients and the dispersion error for different number of points n_p in one wavelength.

n_p	$a_{-1,-1}$	$a_{-1,0}$	$b_{-1,-1}$	$b_{-1,0}$	Error
5	0.171609	0.653336	0.017071	0.057915	0.000473
4	0.186135	0.619432	0.013222	0.070794	0.001638
3.3	0.162905	0.652484	0.025109	0.055464	0.004324
3	0.168296	0.630439	0.025731	0.059813	0.007570

(a)

Figure 1: The dispersion error for the finite difference method of different order.

Figure 2: The dispersion error comparison the finite difference method of different order and the optimal finite difference. (a) $k_y = k_z = 0$. (b) $k_x = k_y, k_z = 0$. (c) $k_x = k_y = k_z$.

we now consider the wave extrapolation in transversely isotropic (TI) medium. The TI assumption, as one of the most popular anisotropic assumptions, provides a good approximation to the wave propagation in the Earth when gravity is the dominant factor in the sedimentation. It assumes that the properties of the medium in a given direction depend only on the angle between that direction and the symmetry axis [38]. We use the pure P-wave dispersion relation proposed by [31, 32, 39]:

$$\omega = \sqrt{\frac{v_x^2 \hat{k}_x^2 + v_z^2 \hat{k}_z^2}{2} + \frac{1}{2} \sqrt{(v_x^2 \hat{k}_x^2 + v_z^2 \hat{k}_z^2)^2 - \frac{8\eta}{1+2\eta} v_x^2 v_z^2 \hat{k}_x^2 \hat{k}_z^2}}, \quad (11)$$

where v_x is the P-wave phase velocity in the symmetry plane, v_z is the P-wave phase velocity in the direction normal to the symmetry plane, η is the anellipticity parameter [40] related to Thomsen's anisotropic parameters as $\frac{1+2\delta}{1+2\epsilon} = \frac{1}{1+2\eta}$. For tilted TI, $\hat{k}_x$ and $\hat{k}_z$ are the wavenumbers corresponding to the axis of symmetry and related to the wavenumbers corresponding to our grid as follows:

$$\hat{k}_x = k_x \cos \theta + k_z \sin \theta, \quad \hat{k}_z = k_z \cos \theta - k_x \sin \theta. \quad (12)$$

Similar to the case of isotropic media, we can rewrite the dispersion relation as:

$$\frac{\omega^2}{v_z^2} = K_a^2 = \frac{B^2 \hat{k}_x^2 + \hat{k}_z^2}{2} + \frac{1}{2} \sqrt{(B^2 \hat{k}_x^2 + \hat{k}_z^2)^2 - \frac{8\eta}{1+2\eta} B^2 \hat{k}_x^2 \hat{k}_z^2}, \quad (13)$$

where B is defined as the ratio between v_x and v_z , $B = \frac{v_x}{v_z}$. The general discretization scheme can be formulated as:

$$\begin{aligned} (L_h^a u)_{ijk} &\approx \sum_{i_1=-m_1}^{i_1=m_1} \sum_{j_1=-m_2}^{j_1=m_2} \sum_{k_1=-m_3}^{k_1=m_3} a_{i_1 j_1 k_1} u_{i+i_1, j+j_1, k+k_1} \\ &+ \frac{\omega^2}{v_z^2} \sum_{i_1=-m_1}^{i_1=m_1} \sum_{j_1=-m_2}^{j_1=m_2} \sum_{k_1=-m_3}^{k_1=m_3} b_{i_1 j_1 k_1} u_{i+i_1, j+j_1, k+k_1}. \end{aligned} \quad (14)$$

Similarly, we can define the dispersion error $R_1(k_x, k_y, k_z)$ as:

$$\sqrt{\frac{-\sum_{i_1=-m_1}^{i_1=m_1} \sum_{j_1=-m_2}^{j_1=m_2} \sum_{k_1=-m_3}^{k_1=m_3} a_{i_1 j_1 k_1} e^{I i_1 \Delta x k_x} e^{I j_1 \Delta y k_y} e^{I k_1 \Delta z k_z}}{\sum_{i_1=-m_1}^{i_1=m_1} \sum_{j_1=-m_2}^{j_1=m_2} \sum_{k_1=-m_3}^{k_1=m_3} b_{i_1 j_1 k_1} e^{I i_1 \Delta x k_x} e^{I j_1 \Delta y k_y} e^{I k_1 \Delta z k_z}}} - K_a. \quad (15)$$

However, the anisotropic wavenumber K_a is not space independent. It depends on the anisotropic parameters B , η , θ . In this case, we need to solve the

optimization problem at each point. Actually, we do not need to solve the optimization problem for each B , η , θ . In stead, we can divide the distribution of B , η , θ into several small intervals and choose one representation for each small interval. The key difference between the TI case and isotropic case is that the coefficients $a_{i_1 j_1 k_1}$ and $b_{i_1 j_1 k_1}$ are no longer constant over space. The resulting coefficients are space dependent.

4. Numerical examples

We first compare the proposed approach with standard finite-difference methods. We consider a region of 1 km *1 km. The space sampling rate is 0.01 km. The velocity used in this experiment is 1.5 km/s. We solve for a frequency of 45 Hz, which results in a wavefield representation of about 3.3 points per wavelength. The real and imaginary parts of the modeled wavefield with the new method are shown in Figures 3(a) and 3(b). As we can see, the resulting wavefield shown in Figures 3(a) and 3(b) is clean and a good approximation of a circle shape. For comparison, the real and imaginary parts of the modeled wavefield with the second-order finite difference method are shown in Figures 3(c) and 3(d), respectively. Due to the severe numerical dispersion error caused by low-order finite difference method, the resulting wavefields in Figures 3(c) and 3(d) no longer have a circular shape. Actually, the wavefield in Figures 3(c) and 3(d), which is obtained using the second-order finite-difference method are mainly artifacts due to the large dispersion errors. Figures 3(e) and 3(f) show, respectively, the real and imaginary parts of the modeled wavefield using the fourth-order finite difference approximation. As we increase the order of the finite difference method, artifacts are reduced. However, the resulting wavefield is still nor circular in shape, because of the numerical anisotropic phenomenon. Figures 3(g) and 3(h) show, respectively, the real and imaginary parts of the modeled wavefield using the eighth-order finite difference method. As a result, the numerical anisotropic phenomenon is reduced. However, it is still not as good as our newly proposed method. To compare the different methods more

generally, we calculate the wavefield for all frequencies and obtain the relative time domain wavefield. The source wavelet we are using is shown in Figure 4(a). The relative frequency distribution of the source wavelet is shown in Figure 4(b). The snapshots at time $t = 0.5$ s obtained using the new method is shown in Figure 5(a). For comparison, the snapshots at time $t = 0.5$ s obtained using the second-order finite difference method is shown in Figure 5(b). The accuracy of the new method is much higher than the second-order finite difference method, even though they cost almost the same.

Next, we will compare the accuracy of our proposed method with the low-rank method for a model with an interface. The investigation area is 6 km by 6 km. The velocity in the top layer is 1.5 km/s and the velocity in the bottom layer is 3 km/s. The source is located at a depth of 1 km and a distance of 3 km. The snapshot at 1.875 s of the proposed method is shown in Figure 6(a). For comparison, the snapshot of low-rank method is shown in Figure 6(b). It indicates that the proposed method is reasonably accurate for a discontinuous medium. To compare more clearly, the slice at distance 3 km is shown in Figure 7.

Then, we compare the cost of the method with finite-difference methods for a three dimensional medium. The investigation area is 1 km^3 with a space sampling of 0.01 km in all three directions. The number of grid points in each direction is 100, which includes 15 PML points for each side. Assuming four points per wavelength, the maximum wavenumber we can approximate is given by $k_{max} = \frac{\pi}{2\Delta x}$. So we use a frequency of $\frac{k_{max}v}{2\pi} = 25$ Hz. Figures 8(a) and 8(b) show the real and imaginary components of the modeled wavefield using the optimal finite-difference method. To demonstrate the accuracy of the obtained wavefield, we model the wavefield with a 12th order finite-difference method and consider it as an exact solution. We compare the phase errors $\sin(\arg(u))$ of different methods. The phase errors for the 2nd (4th and 8th) order finite difference approximation is shown in Figure 8(c) (Figure 8(d) and Figure 8(e)). As shown in these Figures, the higher order of the finite difference method will result in less dispersion error. However, due to having only 4 points per

Figure 3: (a) The real and (b) imaginary parts of the wavefield modeled by the new method. (c) The real and (d) imaginary parts of the wavefield modeled by second-order finite difference method. (e) The real and (f) imaginary parts of the wavefield modeled by fourth-order finite difference method. (g) The real and (h) imaginary parts of the wavefield modeled by eighth-order finite difference method.

Figure 4: (a) The source wavelet and (b) its frequency distribution.

Figure 5: The snapshot at $t = 0.5$ s. (a) The proposed method. (b) second-order finite difference method.

Figure 6: The snapshot at 1.875 s. (a) The proposed method. (b) Low-rank method.

Figure 7: The slice at distance of 3 km of the wavefield. red: the low-rank method; blue: the proposed method.

wavelength, the dispersion errors are still visible. For comparison, the dispersion error of the proposed optimal finite-difference method is shown in Figure 8(f). As we can see, the phase error of the optimal finite difference method (Figure 8(f)) is even smaller than the 8th order finite difference approximation (Figure 8(e)). This result complies with the conclusion made in Figures 2(a), 2(b) and 2(c).

After that, we show the ability of the method in modeling an anisotropic acoustic wave. Due to the more complicated physics, we need 5 points per wavelength and more neighborhood points as shown in Figure 9(a) for anisotropic wavefield modeling compared to Figure 9(b) for isotropic media. Figures 10(a) and 10(b) show the real and imaginary parts of the wavefield modeled with elliptic anisotropic media of $v_x = 1.8 \text{ km/s}$ and $v_z = 1.5 \text{ km/s}$. Figures 10(c) and 10(d) show the real and imaginary parts of the wavefield modeled in a VTI medium of $v_x = 1.8 \text{ km/s}$, $v_z = 1.5 \text{ km/s}$ and $\eta = 0.2$. Figure 10(e) and 10(f) show the real and imaginary parts of the wavefield modeled with TTI media of $v_x = 1.8 \text{ km/s}$, $v_z = 1.5 \text{ km/s}$, $\eta = 0.2$ and $\theta = 45^\circ$. As we can see from these figures, there is no S-wave artifacts which often appear in the conventional finite difference modeling of the anisotropic acoustic wave equation. Also, the modeled wavefield has reasonable accuracy.

We, next, investigate the solution in anisotropic layered medium. The in-

Figure 8: The real (a) and imaginary (b) parts of the wavefield modeled by the proposed optimal finite difference method. (c) The phase error of 2nd order finite difference method. (d) The phase error of 4th order finite difference method. (e) The phase error of 8th order finite difference method. (f) The phase error of the optimal finite difference method with $m = 1$.

Figure 9: The neighborhood used to approximate the equation for (a) VTI and TTI wave equation and (b) isotropic wave equation.

Figure 10: The real (a) and imaginary (b) parts of the modeled wavefield by elliptic anisotropic media. The real (c) and imaginary (d) parts of the modeled wavefield by VTI anisotropic media. The real (e) and imaginary (f) parts of the modeled wavefield by TTI anisotropic media.

Figure 11: The snapshot at 1.6 s for TTI interface problem.

vestigation area is 6 km by 6 km. The vertical velocity v , horizontal velocity v_x , η and θ in the top layer are 1.5 km/s, 1.8 km/s, 0.05 and 45° , respectively. The vertical velocity v , horizontal velocity v_x , η and θ in the bottom layer are 3 km/s, 3.6 km/s, 0.1 and 45° , respectively. The source is located at a depth of 1 km and a distance of 3 km. A snapshot at 1.6 s of the wavefield for the proposed method is shown in Figure 11. It is free of S -wave artifacts.

Finally, we apply this method as a wave propagator for reverse time migration of a benchmark dataset. This dataset was released by BP in 2007 to serve as an Anisotropic Velocity Analysis Benchmark dataset. The vertical, horizontal velocity, η and θ fields are shown in Figures 12(a), 12(b), 12(c) and 12(d), respectively. The space sampling we use for this example is 6.25 m by 6.25 m. The relative reverse time migration result is shown in Figure 13. The resulting reverse time migration image is well focused.

5. Conclusions

We developed a highly accurate approach to solve the Helmholtz wave equation that is of optimal dispersion error at a reduced cost. The approach is based on finding an optimal reduced set of coefficients to represent the dispersion relation. The resulting algorithm requires 3.3 points per wavelength for solving the isotropic wave equation. Even though the cost is equivalent to the cost of the second order finite difference approximation, it is of similar accuracy to the

Figure 12: The material parameters we are using for reverse time migration of BPTTI model.
 (a) v_z . (b) v_x . (c) η . (d) θ .

Figure 13: The reverse time migration imaging of BPTTI model.

8th order finite difference approximation. The method can be easily extended to solve the anisotropic acoustic wave equation. The resulting wavefield has no *S*-wave artifacts and very low dispersion error. Numerical examples show that the proposed method is highly accurate.

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