Linear Analysis of Converging Richtmyer-Meshkov Instability in the Presence of an Azimuthal Magnetic Field

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ABSTRACT

We investigate the linear stability of both positive and negative Atwood ratio interfaces accelerated either by a fast magnetosonic or hydrodynamic shock in cylindrical geometry. For the magnetohydrodynamic (MHD) case, we examine the role of an initial seed azimuthal magnetic field on the growth rate of the perturbation. In the absence of a magnetic field, the Richtmyer-Meshkov growth is followed by an exponentially increasing growth associated with the Rayleigh-Taylor instability. In the MHD case, the growth rate of the instability reduces in proportion to the strength of the applied magnetic field. The suppression mechanism is associated with the interference of two waves running parallel and anti-parallel to the interface that transport vorticity and cause the growth rate to oscillate in time with nearly a zero mean value.

1 Introduction

The Richtmyer-Meshkov instability (RMI) occurs when a shock wave impulsively accelerates a perturbed density interface separating two different fluids [1, 2]. RMI may not be classified as a classical hydrodynamic instability with modal analysis because the perturbations grow algebraically (linearly in time) rather than exponentially. The RMI occurs in a variety of physical applications. It is important in astrophysical phenomena such as supernova explosions, supersonic combustion in hypersonic air-breathing engines, and inertial confinement fusion (ICF) [3, 4, 5]. In direct-drive ICF, small capsules containing a deuterium and tritium fuel are compressed by laser-generated shock waves, where achieving sufficiently high pressures and temperatures inside the compressed ICF targets to ignite the fuel is the primary goal of this process. On the other hand, in indirect drive approach to ICF, the driver energy (from laser beams or ion beams) is first absorbed in a high-Z enclosure (a hohlraum), which surrounds the capsule. The material heated by the driver releases X-rays, which drive the capsule implosion [6]. The mixing between the outer shell and the inner fuel inhibits the fusion reaction due to RMI and RTI (Rayleigh Taylor instability) which limits the energy production [7, 8, 9].

The RMI in the presence of a magnetic field initially oriented parallel to the incident shock direction has been investigated numerically and theoretically. Nonlinear ideal magnetohydrodynamics (MHD) simulations of a shock interaction with an oblique planar contact discontinuity were discussed by Samtaney [10]. It was demonstrated that the RMI is suppressed by the existence of such a magnetic field. The primary suppression mechanism may be attributed to transport of baroclinic vorticity away from the density interface by slow- or intermediate mode shocks reducing the vorticity at the interface. This results were reconfirmed in the context of incompressible MHD linear stability analysis by Wheatley *et al.* [11]. They demonstrated that vorticity is transported away from the interface by Alfvén fronts. More detailed nonlinear MHD simulations extending the parameter ranges of earlier investigations [12] demonstrated the suppression of RMI. The effect of transverse

magnetic field, oriented parallel to the interface, recently has been investigated analytically and numerically [13, 14, 15, 16]. In fact, in a super-conductive plasma, the existence of a transverse magnetic field behaves in a fashion somewhat similar to surface tension on the interface. The transverse magnetic field inhibits the growth of surface perturbations and as a result, stabilizes the instability [13]. Cao *et al.* [14] studied this case in the framework of incompressible media through an analytical development of a magnetized impulsive model. They also found an inhibition of the RMI growth with oscillations of the perturbation amplitude. Wheatley *et al.* [16] investigated nonlinear compressible simulations and incompressible model of MHD-RMI for the case of a transverse magnetic field, i.e. there was no normal component of the magnetic field at the interface and found oscillatory solutions of the growth rates with the RMI still being suppressed.

We now discuss some previous studies in the literature of RMI in converging geometry. One of the early investigations of hydrodynamic RMI in cylindrical geometry was carried out by Zhang & Graham [17], in which the imploding and exploding shocks were examined for positive and negative Atwood ratio interfaces, which is defined as $A = (\rho_2 - \rho_1)/(\rho_2 + \rho_1)$, where ρ_1, ρ_2 are the densities of the fluids under investigation. Lombardini & Pullin [18] investigated the RMI in cylindrical and spherical geometries with azimuthal and axial perturbations and developed a linear incompressible theory for the case of an imploding or exploding shock. In particular, they developed a theory for the asymptotic growth rate of perturbations and further derived a unified expression for the asymptotic impulsive growth rate in planar, cylindrical and spherical cases. In their work, the linear analysis was complemented and verified by compressible nonlinear simulations with a small amplitude perturbation. In converging geometry, the RMI is typically followed by the Rayleigh-Taylor instability (RTI) due to the continued acceleration or deceleration of the interface towards the center of convergence, depending on sign of Atwood number, which was not captured by the model of Lombardini & Pullin. Mikaelian investigated the linear RMI and RTI of incompressible flow for a chosen number of concentric spherical shells affected by a radial implosion or explosion [19]. Moreover, he proposed a theoretical model for a turbulent mix of different fluids in spherical geometry. Likewise, he also investigated the linear stability of RTI and RMI in concentric cylindrical shells via numerical simulations and developed a simple model for the evolution of turbulent mix [20]. The physical configuration that is relevant to ICF is one where the main incident shock is a converging shock. So it is of interest to examine the RM instability in a converging geometry, especially in the presence of a magnetic field and investigate the physical mechanisms of suppression if any.

Pullin *et al.* [21] examined the behavior of a converging cylindrical fast magnetosonic MHD shock onto a time-wise constant line current within ideal MHD using Whitham's theory of geometrical shock dynamics (GSD) [22], where the shock is moving into a region with a spatially varying magnetic field. This shock can weaken in terms of both the pressure ratio and Mach number as it reaches the origin, contrary to the expected behavior under similar conditions of a gas-dynamic shock [23]. Mostert *et al.* [24] investigated the behavior of a cylindrical fast MHD shock that collapses onto an axial line where a time-dependent current generates an azimuthal magnetic field that has a power law time behavior. The time variation of this magnetic field is tuned in a manner such that it becomes zero at the instant where the shock reaches the axis. It was recently made evident by the work of Mostert *et al.* [25] that the azimuthal, and spherical symmetry may be violated in cylindrical and spherical geometry in the presence of a magnetic field, respectively. In these aforementioned studies, the emphasis was on the shock behavior as it collapses on to the central axis. Mostert *et al.* [26] have carried out an extensive investigation of nonlinear RMI in cylindrical and spherical geometry for a various initial seed magnetic field configurations. All cases showed suppression of the RM and RT instabilities. A linear stability investigation of MHD RMI in cylindrical geometry with a magnetic field normal to the interface showed a significant suppression of the instability for $\beta \leq 4$, related to the transport of vorticity away from the interface by Alfvén fronts [27].

Presently, we consider the case of a cylindrically collapsing fast MHD shock interacting with an azimuthally perturbed interface. Clearly, there are a wealth of configurations and parameters for which the cylindrically converging MHD RMI could be investigated. This investigation is restricted to fall within the framework of linear stability where the perturbations are solved with a linearized set of ideal MHD equations whereas the background, purely radial, interaction between the converging MHD shock and unperturbed interface follows the usual nonlinear ideal MHD equations albeit in one dimension. Our linear simulations are carried out for cases where the incident shock propagates from a light gas to a heavy one and vice versa, corresponding to a positive/negative Atwood numbers ($A > 0/A < 0$).

The paper is organized as follows: in Section 2, the physical setup and the ideal MHD equations are introduced. Here we extend the earlier numerical linear stability analysis by Samtaney [28] to MHD in cylindrical geometry. Linearizing the ideal MHD equations about a time-dependent base state we derive a set of hyperbolic equations with source terms for the perturbed quantities. These equations are solved using an upwind technique. The details of the numerical method are previously given in References [27, 21] and hence not repeated here. In Section 3, we present numerical results for both hydrodynamics and MHD for positive/negative Atwood ratio interface, with azimuthal perturbations, accelerated by an MHD converging shock. Finally, some conclusions and potential future work are discussed in Section 4.

2 Physical Setup and Governing Equations

2.1 Physical Setup

The schematic of the physical setup is shown in Fig. 1 for (r, θ) -section with fixed axial coordinate z . A cylindrical annulus domain is confined by left and right boundaries at $r = R_l$ and $r = R_r$, respectively. A density interface of azimuthal perturbation wavenumber m is located at $r = R_0$ that separates two conducting fluids of different densities ρ_1 and ρ_2 . A converging cylindrical fast magnetosonic MHD shock with Mach number M (based on the fast magnetosonic speed) is located at $r = R_s$, where the setup of the initial condition is given in Section 2.2. The spatially varying azimuthal magnetic field is characterized by its non-dimensional strength using the parameter beta $\beta = 2p/(B_{\theta,0})^2$, where p is the initial pressure ahead of the shock and $B_{\theta,0}$ is the azimuthal component of the magnetic field at $r = R_0$. Five parameters characterize the problem: the Mach number M of MHD shock, the Atwood number $A = (\rho_2 - \rho_1)/(\rho_2 + \rho_1)$, the azimuthal perturbation wavenumber m , the plasma β and the distance ratio R_s/R_0 . These five parameters $(M, A, m, \beta, R_s/R_0)$ essentially span the space of numerical results. To limit the scope of our investigation, our simulation domain is chosen to be $R_l = 0.1, R_r = 2.6$, and we fix $M = 2.0$, $A = \pm 0.667$, the positive Atwood ratio corresponding to air-SF₆ interface (vice verse for negative Atwood ratio) and $R_s/R_0 \simeq 1.2$. Mainly, we study the cases of different perturbation wavenumber m and plasma β .

2.2 Initial Conditions

The details of the setup of a converging cylindrical ideal fast magnetosonic shock may be found in Reference [21]. We setup a planar MHD shock at a very large radius and let this shock collapse onto the axis of convergence. When the shock reaches its desired location at $r = R_s$, we extract the profiles and use that as the initial shock profile. The incident converging shock may not be simply setup as a planar shock because it is necessary to have correct radial variation of the flow variables behind the shock. Figure 2 shows the initial profile of density, magnetic field, pressure, and radial component of velocity of the base flow incident shock as a function of radius. The perturbed interface in our model is affected by an impulse due to the shock interaction together with the acceleration effects caused by the slow pressure increase with radius behind the shock.

2.3 Governing Equations

It is expected that the diffusion time-scale would be much larger than the convective time-scale in this converging RMI scenario. Hence we neglect viscous, thermal and plasma resistivity effects. The flow is considered to be cylindrically symmetric with non-zero velocity in the radial direction. The governing equations of ideal MHD are written in conservation form in cylindrical coordinates as follows:

$$\frac{\partial U}{\partial t} + \frac{1}{r} \frac{\partial(rF(U))}{\partial r} + \frac{1}{r} \frac{\partial G(U)}{\partial \theta} = S, \quad (1)$$

where $U \equiv U(r, \theta, t) = \{\rho, \rho u_r, \rho u_\theta, B_r, B_\theta, e, \rho\phi\}^T$ is the solution vector, ρ is the density, $(\rho u_r, \rho u_\theta)$ is the momentum, (B_r, B_θ) is the magnetic field, e is the total energy per unit volume and ϕ is a passive scalar used to track the density interface. These equations are already expressed in non-dimensional form by choosing a reference density, pressure, and length scale (for example, see [16] for the non-dimensionalization). $F(U)$ and $G(U)$ are the fluxes of mass, momentum, magnetic flux and energy in the r and θ directions given as

$$F = \begin{bmatrix} \rho u_r \\ \rho u_r^2 + \tilde{p} - B_r^2 \\ \rho u_r u_\theta - B_r B_\theta \\ 0 \\ u_r B_\theta - u_\theta B_r \\ (e + \tilde{p})u_r - (\mathbf{B} \cdot \mathbf{u})B_r \\ \rho u_r \phi \end{bmatrix}, G = \begin{bmatrix} \rho u_\theta \\ \rho u_r u_\theta - B_r B_\theta \\ \rho u_\theta^2 + \tilde{p} - B_\theta^2 \\ u_\theta B_r - u_r B_\theta \\ 0 \\ (e + \tilde{p})u_\theta - (\mathbf{B} \cdot \mathbf{u})B_\theta \\ \rho u_\theta \phi \end{bmatrix}, \quad (2)$$

where $\tilde{p} = p + \frac{1}{2}(\mathbf{B} \cdot \mathbf{B})$, is the sum of gas and the magnetic pressures. S is a source term arising in cylindrical coordinates, given by

$$S = -\frac{1}{r} \begin{bmatrix} 0 \\ B_\theta^2 - \rho u_\theta^2 - \tilde{p} \\ \rho u_r u_\theta - B_r B_\theta \\ 0 \\ u_\theta B_r - u_r B_\theta \\ 0 \\ 0 \end{bmatrix}. \quad (3)$$

To close the system we use the perfect gas equation of state so that $e = \frac{p}{\gamma-1} + \frac{1}{2}(\mathbf{B} \cdot \mathbf{B}) + \frac{1}{2}\rho(\mathbf{u} \cdot \mathbf{u})$, where γ is the ratio of specific heats fixed at 5/3. The initial magnetic field has an inverse radial dependence. Close to the origin, large numerical errors are encountered owing to a lack of analytical cancellation of a term proportional to r^{-3} in the radial momentum equation. These errors are eliminated by rewriting the azimuthal component of the magnetic field as $B_\theta = B_\theta^* + B_\theta^1$ where $B_\theta^* = \frac{\beta}{r}$ is a curl-free portion of the magnetic field. The ideal MHD equations are rewritten as

$$\frac{\partial \tilde{U}}{\partial t} + \frac{1}{r} \frac{\partial(r\tilde{F}(\tilde{U}))}{\partial r} + \frac{1}{r} \frac{\partial \tilde{G}(\tilde{U})}{\partial \theta} = \tilde{S}, \quad (4)$$

where $\tilde{U} = \{\rho, \rho u_r, \rho u_\theta, B_r, B_\theta^1, \tilde{e}, \rho\phi\}^T$. Also, the modified fluxes and source terms are

$$\tilde{F} = \{\rho u_r, \rho u_r^2 + \tilde{p}_t - B_r^2 + B_\theta^{*1} B_\theta^1, \rho u_r u_\theta - B_r B_\theta, 0, u_r B_\theta - u_\theta B_r, (\tilde{e} + \tilde{p}_t + B_\theta^* B_\theta^1) u_r - (\mathbf{B} \cdot \mathbf{u}) B_r, \rho u_r \phi\}^T,$$

$$\tilde{G} = \{\rho u_\theta, \rho u_\theta u_\theta - B_r B_\theta, \rho u_\theta^2 + \tilde{p}_t - (B_\theta^1)^2 - B_\theta^{*1} B_\theta^1, u_\theta B_r - u_r B_\theta, 0, (\tilde{e} + \tilde{p}_t + B_\theta^* B_\theta^1) u_\theta - (\mathbf{B} \cdot \mathbf{u}) B_\theta, \rho u_\theta \phi\}^T,$$

$$\tilde{S} = -\frac{1}{r} \{0, (B_\theta^1)^2 - \rho u_\theta^2 - \tilde{p}_t + B_\theta^{*1} B_\theta^1, \rho u_r u_\theta - B_r B_\theta, 0, u_\theta B_r - u_r B_\theta, 0, 0\}^T,$$

where $\tilde{e} = \frac{p}{\gamma-1} + \frac{1}{2}(B_r^2 + (B_\theta^1)^2) + \frac{1}{2}\rho(u_r^2 + u_\theta^2)$ and $\tilde{p}_t = p + \frac{1}{2}(B_r^2 + (B_\theta^1)^2)$.

2.4 Linear Stability Analysis

If a small parameter is present, then the governing equations can be linearized using an expansion in terms of the small parameter, retaining terms up to first order and neglecting higher order terms. For the physical situation of a shock interacting with a perturbed density interface, the ratio of the perturbation amplitude to wavelength may be considered a small parameter. In general, linear stability can be broadly classified into normal mode analysis or an initial value problem (IVP) approach. The normal mode approach is suitable when the instability grows exponentially in time (as in the Rayleigh-Taylor instability case). The IVP approach is the chosen method when we have algebraic growth (as in the RMI case) or when transient growth is investigated. Our approach detailed below is somewhat similar in spirit to the original analysis of Richtmyer [1] and Yang *et al.* [29]. Richtmyer derived a set of linear equations for the case of a shock interacting with a density interface wherein the reflected and transmitted waves after the shock refraction were both shocks. Yang *et al.*, on the other hand derived a set of linear equations, for the case where the reflected wave is a rarefaction. In both analyses, the amplitude of the interface itself is not present; and in both cases the growth rate (or rather the time derivative of the interface amplitude) is computed numerically. In this sense, both these were instances of linear stability analysis as IVPs. In our investigation, an analytical process similar to Richtmyer and Yang *et al.* is not tractable owing to the complexity of the refractions in MHD, and the converging geometry. Instead we follow the approach of Samtaney [28] who proposed a numerical method (also an IVP approach) for solving the linearized equations resulting in stability analysis of shocked interfaces in gas dynamics and magnetohydrodynamics.

To examine the linear stability of a converging shock interacting with a perturbed interface, we linearize the ideal MHD equations by splitting the solution into a background one dimensional (radial) solution, with $\{\}$ notation, and perturbed

solutions, with $\{\}$ notation, as $\tilde{U}(r, \theta, t) = \hat{U}(r, t) + \bar{\epsilon} \hat{U}(r, t) e^{im\theta}$, where $\bar{\epsilon}$ is the small perturbation (small amplitude to wavelength ratio), m is the azimuthal perturbation wavenumber, and $i \equiv \sqrt{-1}$. The time-dependent background state is a function of the radial coordinate alone and comprises of MHD shocks, and a contact discontinuity. We term this background state as the "base" state (even though, because it is time dependent, it is strictly speaking not a base state of standard hydrodynamic stability analysis). Details of the base state are presented later. Substituting the base and perturbed solutions into the governing equations, we get the partial differential equations governing the base and perturbed states as follows:

$$\frac{\partial \hat{U}}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (r F(\hat{U})) = \hat{S}, \quad (5a)$$

$$\frac{\partial \hat{U}}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (r M(\hat{U}) \hat{U}) + \frac{im}{r} N(\hat{U}) \hat{U} = \hat{S}, \quad (5b)$$

where

$$\hat{S} = -\frac{1}{r} \begin{bmatrix} 0 \\ (\hat{B}_\theta^1)^2 - \hat{\rho} \hat{u}_\theta^2 - \hat{p}_t + B_\theta^* \hat{B}_\theta^1 \\ \hat{\rho} \hat{u}_r \hat{u}_\theta - \hat{B}_r \hat{B}_\theta \\ 0 \\ \hat{u}_\theta \hat{B}_r - \hat{u}_r \hat{B}_\theta \\ 0 \\ 0 \end{bmatrix}, \quad \hat{S} = -\frac{1}{r} \begin{bmatrix} 0 \\ 2\hat{B}_\theta^1 \hat{B}_\theta - 2\hat{\rho} \hat{u}_\theta \hat{u}_\theta - \hat{\rho} \hat{u}_\theta^2 - \hat{p}_t + B_\theta^* \hat{B}_\theta^1 \\ \hat{\rho} \hat{u}_r \hat{u}_\theta + \hat{\rho} \hat{u}_r \hat{u}_\theta + \hat{\rho} \hat{u}_r \hat{u}_\theta - \hat{B}_r \hat{B}_\theta - \hat{B}_r \hat{B}_\theta \\ 0 \\ \hat{u}_\theta \hat{B}_r + \hat{u}_\theta \hat{B}_r - \hat{u}_r \hat{B}_\theta - \hat{u}_r \hat{B}_\theta \\ 0 \\ 0 \end{bmatrix}, \quad (6)$$

where $M(\hat{U})$ and $N(\hat{U})$ are the Jacobian matrices of the fluxes $\tilde{F}$ and $\tilde{G}$ with respect to the base state. The total pressures $\hat{p}_t, \hat{p}_t$ are specified in terms of other variables in Eqns (7) and (8).

$$\hat{p}_t = (\gamma - 1) \hat{e} - \frac{\gamma - 1}{2} \hat{\rho} (\hat{u}_r^2 + \hat{u}_\theta^2) - \frac{\gamma - 2}{2} (\hat{B}_r^2 + (\hat{B}_\theta^1)^2), \quad (7)$$

$$\hat{p}_t = (\gamma - 1) \hat{e} - \frac{\gamma - 1}{2} \hat{\rho} (\hat{u}_r^2 + \hat{u}_\theta^2) - (\gamma - 2) (\hat{B}_r \hat{B}_r + \hat{B}_\theta^1 \hat{B}_\theta^1). \quad (8)$$

We note here that the above equations, derived from ideal MHD equations, are scale-free, and the choice of the radial domain can be arbitrary. Equations (5a) and (5b), governing the base and perturbed state, respectively, are both hyperbolic partial differential equations which are solved numerically with an explicit time marching upwind procedure the details of which may be found in References [21, 27].

3 Results and Discussions

In this section, we present results from linear stability simulations. First, a convergence test is performed to quantify the order of accuracy of the numerical method. Then, we examine the time-dependent radially varying background state solution, followed by the investigation of azimuthal perturbation quantities for hydrodynamics and MHD for both $A > 0$ and $A < 0$. Adopting the viewpoint that the main driving mechanism of the RMI to be the vorticity generated at the interface due to baroclinic source term induced by the incident shock, we focus on the vorticity in the domain and examine the suppression of RMI in terms of vorticity.

3.1 Convergence Test

To examine the convergence of our numerical results, we chose a test case with plasma beta $\beta = 16$, and perturbation wavenumber $m = 128$ and $A = 0.667$. The solution is computed with different mesh sizes: 800, 1600, 3200, 6400 and 12800 zones in the radial domain. The unscaled growth rate of the perturbed interface, (essentially the time derivative of the perturbation amplitude) $\dot{h}$, computed as the real value $Re(\hat{u}_r)$ at the interface position, is plotted in Fig. 3. The finest mesh

resolution, 12800 cells, is designated as the “exact” numerical solution for our simulations. We fix the time $t = 0.3$ to obtain the relative error of the growth rates which is plotted in Fig. 4. It shows that as the mesh resolution increased the error is decreased which implies that the growth rates are converging. The order of accuracy is 1.4914, which is between the first and second order: a value between one and two is generally considered acceptable for flows dominated by shocks and contact discontinuities. In ideal MHD, with the lack of a physical dissipation, one cannot expect a point-wise convergence of the solution with grid refinement. Numerical diffusion decreases as the grid is refined, which causes all discontinuities to be more sharply resolved: this is also observed in nonlinear ideal MHD simulation of planar shock interactions with a perturbed interface with an initially transverse magnetic field [16]. To more clearly illustrate the wave structures in the vicinity of the interface, results in our simulations are obtained on a mesh with 6400 grid cells.

3.2 Base State Solution

Here we present the solution of the base state equations for MHD RMI. The time-dependent base state equations (5a) are independent of the azimuthal wavenumber m . It turns out that at $t = 0.0$ the density profile is a mildly increasing with radius just behind the shock (Fig. 5). A similar behavior is seen for the radial velocity, $\hat{u}_r$, and the pressure, $\hat{p}$, which are increasing in magnitude behind the shock front. Next, the density profiles for positive, negative Atwood ratios ($A > 0, A < 0$) are examined, respectively.

3.2.1 Positive Atwood Number

The base state is examined using the density profiles at time $t = 0.0$ (initial condition) and $t = 0.5$ as shown in Fig. 5 for $\beta = 16$. The density spacetime diagram illustrates the wave diagram of the base state more clearly. The contact discontinuity (CD) can be seen as vertical line at initial position R_0 in Fig. 6 indicating that it is nearly stationary until the incident MHD shock (IS) overruns it. The IS bifurcates into transmitted fast (TF), and a reflected fast (RF) MHD shocks. In one-dimensional flow and the presence of azimuthal magnetic field with $u_\theta = B_r = 0$, the slow and intermediate speeds collapse to zero [21], and hence the slow-mode shocks are not seen. The IS provides an impulse to the CD, thereby accelerating it. The CD is located between the two shocks RF and TF. We note that at $t = 0.5$, the transmitted fast shock is about to leave the domain so, all simulations are stopped at that time. There is also an acceleration effect from the converging geometry, and due to the pressure magnitude changing behind the shock.

3.2.2 Negative Atwood Number

The density profiles at time $t = 0.0$ (initial condition) and $t = 0.2$ are plotted in Fig. 7 as solutions for the base state equations for $\beta = 4$. At $t = 0.0$ the CD is seen as vertical line at initial position R_0 indicating that it is nearly stationary until it is overrun by the IS. Then, the IS bifurcates into TF, and fast mode rarefaction wave (RR) moving opposite direction of the shock. The IS provides an impulse to the CD which is located between the TF shock and RR waves. The density spacetime diagram shows these waves clearly in Fig 8. The small blip in the profile discerned near $r = 0.9$ in density profile is attributed to a startup error due to a sharp shock initialization. We note that at $t = 0.2$, the transmitted fast shock is about to leave the domain and therefore, all simulations are stopped at that time.

3.3 Evolution of the Perturbations

In this section, we present the results for both hydrodynamics (HD) (i.e., $\beta = \infty$), and MHD (i.e., finite β) cases with azimuthal perturbations. In principle, both azimuthal and axial perturbation modes should also be examined, either separately and in combination, for the sake of completeness. However, in previous investigations by Bakhsh *et al.* [27] inclusion of both axial and azimuthal perturbations showed somewhat similar results. Hence, we limit our investigation to azimuthal perturbations at this time. Following Lombardini & Pullin [18] we non-dimensionalize the time scale using $1/(a_0(m/R_0))$ as our reference time scale, where a_0 is the sound speed in the unshocked medium ahead of the incident shock. This time scale corresponds to the time taken by an acoustic wave to traverse a wavelength of the unshocked interface. The perturbation growth rate $\hat{h}$ is normalized by the theoretical growth rate $\hat{h}_\infty$ expressed as

$$\hat{h}_\infty = \frac{F_M h_0^+ \Delta W (1 + mA^+)}{R_0}, \quad (9)$$

where ΔW is the impulsive change in the interface velocity, $A^+ = \frac{\rho_2^+ - \rho_1^+}{\rho_2^+ + \rho_1^+}$ is the post-shock Atwood number, where ρ_2^+ is the post shock density of the fluid the shock is incident on, ρ_1^+ is the post shock density of the fluid the shock is incident from, F_M is a parameter depending on Mach number of the incident shock ($F_M = 0.85$ for $A > 0$ case; [18]). h_0^+ is the post-shock initial perturbation amplitude, which in linear stability cancels out, it is chosen as unity for convenience. In the case of $A < 0$,

a modified $\hat{h}_{\infty*}$ is used to scale $\hat{h}$, where the post-shock initial amplitude is replaced by the average $\frac{h_0^+ + h_0^-}{2}$, where h_0^- is the pre-shock initial amplitude [30], and $F_M = 1$ for this case. We note that there is indeed a small difference in ΔW between the MHD and HD base states. However, in order to compare results between HD and MHD cases with different magnetic field strengths, we use the same theoretical growth rate in equation 9 to normalize the growth rate in both HD and MHD cases.

3.3.1 Hydrodynamic Case

We begin our discussion with the HD case for both positive and negative Atwood numbers followed by MHD results for the perturbation wavenumber $m = 256$. The growth rate in HD is similar to that obtained in the Cartesian geometry case until about scaled time $a_0 t m / R_0 \simeq 30$ as shown in Fig. 9. The growth rate increases from zero and after a small time lag of the order of a few acoustic time periods (the acoustic time period is the time taken by a sound wave to travel the distance equal to the wavelength in the azimuthal direction) reaches a value of unity and then oscillates around unity; this would correspond to the asymptotic growth rate obtained in Cartesian planar geometry and this is the growth rate associated with the RMI. The shock provides an impulsive acceleration to the interface. But in the cylindrical case, after the shock passage there is another acceleration due to the pressure rise behind the shock (see Fig. 2(c)). Moreover, the interface continues to accelerate (resp. decelerate) in the negative (resp. positive) Atwood ratio case for a certain time duration which the acceleration vector points in the direction from the heavy to the light fluid (i.e. RT unstable stratification). Therefore, in the cylindrical case, as the interface approaches the origin, the RTI manifests itself, and we observe an exponential increase in the magnitude of the growth rate. The RTI is associated with a phase change for the positive Atwood interface. For the $A > 0$ interface, this growth rate is opposite in sign to the model growth rate $\hat{h}_{\infty*}$. For the $A < 0$ interface, the RMI itself undergoes a phase change and the RTI reinforces this with the corresponding exponential increase in growth rate as seen in Fig. 9. In reality, the instability of the interface is a combination of both RM and RT instabilities. However, the RMI phase dominates the early part of the growth rate curve, and the RTI dominates the later portion of the growth rate curve. Hence the demarcation between RMI and RTI phases in Fig. 9 is somewhat artificial; nonetheless after the scaled time ≈ 30 the exponential growth of the RTI is evident. Another subtle feature is that the frequency of oscillation of the growth rate increases with time: this is because, although the wavenumber m is fixed, the wavelength $2\pi R/m$ decreases as interface moves radially inwards.

3.3.2 Effect of Azimuthal Seed Magnetic Field

In this section, we investigate the effect of varying β parameter which characterizes the strength of the seed magnetic field in the azimuthal direction, as well as effect of varying the azimuthal perturbation wavenumber m on the time history of scaled growth rates in MHD. Note that β is a non-dimensional parameter that is proportional to square of ratio of the acoustic speed in the unshocked incident gas and the Alfvén speed. In our investigation, β is larger than unity so that the Alfvén speed is always lower than the acoustic sound speed. First, the effect of the varying the magnetic field strength is presented in the left and right panel in Fig. 10(a), 10(b) for $A > 0$ and $A < 0$ interfaces, respectively, for $\beta = \infty, 256, 128, 16, 4$ for fixed wavenumber $m = 256$. For finite β we note two frequencies: the higher frequency oscillations are attributed to the acoustic waves traveling in the azimuthal direction. These are clearly discerned in the $\beta = \infty$ or hydrodynamic cases discussed earlier and also present for finite β cases (e.g. see the $\beta = 128, 256$ cases where these are more clear especially at late times). For finite magnetic field strength or finite β the oscillations corresponding to the propagation of Alfvén waves in the azimuthal direction occur at a lower frequency than those associated with acoustic modes. The frequency of the Alfvén mode oscillations may be predicted approximately by the model proposed by Wheatley *et al.* [16] for the planar case. An obvious effect of the azimuthal seed field is that it suppresses the instability or growth of the interface. The larger the magnetic field (smaller β) the larger the suppression: the amplitude of the growth rate decreases with decreasing β and the oscillation frequency of the Alfvén mode increases. Here, by instability we mean a combination of RMI and RTI and the magnetic field affects both. Later in this section we will estimate the minimum field strength required to suppress the RTI. For any given β , as time progresses, we see a decrease in the amplitude of the growth rate. We note that the reduction of growth rate amplitude with time is attributed to two reasons: (a) the dominant reason is that as the density interface moves radially inwards the magnetic field strength increases in proportion to $1/r$ and this reduces the growth amplitude; and (b) the second reason is that our numerical method based on upwind schemes has implicit numerical diffusion which results in a diffusion of the baroclinic vorticity away from the interface causing a decrease in the growth rate of the interface perturbations. As discussed for the hydrodynamic cases, there is a short time lag before the RMI fully develops (or scaled growth reaches unity). For finite but relatively high β cases ($\beta \geq 128$) we note that the growth rate reaches unity before reducing. The lower β cases see a suppression of the growth rate very early so that the scaled growth rate never even approaches unity. We next elucidate the physical mechanism of the instability suppression.

At the CD, for finite β , the vorticity distribution breaks up into two vorticity carrying waves that propagate parallel and anti-parallel to the magnetic field, causing the amplitude of interface to oscillate in time. This effect is similar to that observed by Wheatley *et al.* [16] in planar nonlinear ideal MHD simulations. Moreover, simulations presented here in cylindrical geometry show that as the magnetic field strength increases the interface will oscillate with higher frequencies

and smaller amplitudes. This can be due to the fact that vorticity carrying waves propagate faster at the interface causing more rapid or higher frequency oscillations. Furthermore, as in the classic treatment of the RTI with a magnetic field parallel to the interface where Chandrasekhar [31] suggests that the magnetic field acts as a surface tension term, a higher field strength (lower β) is correlated with the lower amplitude of the growth rate. These results also agree with Levy *et al.* [15] who examined the cases of $A < 0$, $A > 0$ interfaces in the presence of parallel and transverse magnetic field. They found that a higher magnetic field corresponds to a longer oscillation period, and a smaller perturbation amplitude.

To further illustrate the vorticity dynamics in the vicinity of the interface, we compute z-component of vorticity $\hat{\omega}_z = \text{Im}(\frac{1}{r} \frac{\partial(r\hat{u}_\theta)}{\partial r} - \frac{1}{r} \text{im}\hat{u}_r)$ (where $\text{Im}(x)$ indicates the imaginary value of x) evaluated at the interface for the case of $A > 0$, $\beta = 16$ and $m = 256$. Using this, we recreate a two dimensional vorticity field in the vicinity of the interface to examine the vorticity as $\omega_z(r, \theta, t) = \hat{\omega}_z(r, t) \sin(m\theta)$ in the range of $\theta \in [-2\pi/m, 2\pi/m]$. Figure 11(b-f) shows this two dimensional vorticity field (two wavelengths shown) at scaled times $t = 18, 23, 27, 33$ and 40 which correspond to a full oscillation cycle of the growth rate (shown in Figure 11(a)). The dashed line in the 2D vorticity images corresponds to the mean location of the interface. There is a clear correspondence, somewhat obvious from a kinematics viewpoint, between the maximum positive growth rate and positive vorticity on the interface (for half-wavelength shown in $\theta \in [0, \pi/m]$), zero growth when interface vorticity is zero on the interface, and negative growth with negative vorticity on the interface ($\theta \in [0, \pi/m]$). We note that the dominant effect is that the vorticity on either side of the interface is transported by waves propagating at the local Alfvén speed, parallel and anti-parallel to the interface. The interference of these waves, with alternating phase causes constructive and destructive interference and vorticity patterns alternate with changing signs. The net result is that the growth rate of the interface perturbation oscillates from positive to negative. In addition, the periodic interference of these waves continues with time and causes the interface growth rate to oscillate. As mentioned earlier, the process is very similar to that seen in planar nonlinear simulations of Wheatley *et al.* (See Figure 6 in Reference [16]).

Next, for a constant magnetic field strength $\beta = 4$ we plot the time history of scaled growth rates by varying the azimuthal perturbation wavenumber $m = 16, 64, 128, 256$ in Figures 12(a), 12(b) for $A > 0$, $A < 0$ interfaces, respectively. The growth rates oscillate in time with a frequency that is proportional to m so that on the scaled time plot the frequency is nearly the same for all wavenumbers. However, the amplitude of the growth rate decreased with increasing wavenumber, the highest wavenumber case ($m = 256$) with the smallest amplitudes compared with other cases. This indicates the level of suppression of the instability increases with wavenumber. Besides, for the MHD cases, the onset of RTI is not as well demarcated from the RMI phase as it was for the hydrodynamic cases.

From the discussion earlier, it is evident that a magnetic field strength corresponding to $\beta = 4$ effectively suppresses the instability for both positive and negative Atwood ratio interfaces, for all wavenumbers $m \geq 64$ within several Alfvén wave transit times. We now examine the behavior of the instability for magnetic fields which are considerably weaker (large β cases) than the one examined above. The objective is to determine, albeit numerically, a critical value of the field as β_{crit} that would effectively have a suppressing effect on the instability. We plot the growth rate in Fig. 13(a), 13(b) for both Atwood ratio interface for large values of β for $m = 256$. It is clear that $\beta = 20,000$ has virtually no effect (apart from slightly delaying the onset of RTI for $A > 0$). However, as the field strength increases, we observe a strong effect at $\beta = 1000$.

The classic dispersion relation based on normal mode analysis (the solution is proportional to $\exp(nt)$) for magnetic Rayleigh-Taylor from Chandrasekhar [31] for the case of a magnetic field that is parallel to the interface is expressed below for a 2D planar case:

$$n^2 = gk \left(\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} - \frac{2B^2}{(\rho_1 + \rho_2)gk} \right), \quad (10)$$

where k is the perturbation wavenumber, and g is the acceleration in the direction of the heavy-to-light fluid. This result strictly is not valid for our instability study, because of the difference in geometry (planar vs. cylindrical), and because ours is a combination RM/RT instability with no constant acceleration. Nonetheless, the above result can be used for the $A < 0$ interface to estimate the strength of the critical magnetic field B_{crit} , which can suppress the instability at least during the RT phase. For zero growth, we estimate this as:

$$B_{crit}^2 = \frac{(\rho_1^+ - \rho_2^+)g(\tau)R(\tau)}{2m}, \quad \beta_{crit} = \frac{2p_0}{B_{crit}^2}, \quad (11)$$

where the superscript $+$ indicates post-shock density values, and τ is chosen as the time when the RTI begins to manifest itself, $g(\tau)$ is the acceleration experienced by the interface located at radius $R(\tau)$ at that time. Choosing scaled time $\tau = 20$, at which $g(\tau) = 0.4$, and $R(\tau) = 0.8$, and using the post-shock values as $\rho_1^+ = 1.9$, and $\rho_2^+ = 0.72$, we obtain $\beta_{crit} = 2712$. An examination of Fig. 13(b) indicates that this is a reasonable estimate. This estimation of the critical value of the magnetic

field relies on the knowledge of the acceleration and the post-shock density values: these can be obtained from a simple one-dimensional simulation. Alternatively, if we simply use pre-shock conditions, the only unknown is $g(\tau)$: if this is estimated as the maximum acceleration after the initial impulsive acceleration, we obtain $\beta_{crit} = 3200$.

4 Conclusions

In this paper, linearized ideal MHD equations with a time-dependent base state with discontinuities have been solved numerically. The numerical linear stability analysis is performed in cylindrical geometry, with azimuthal perturbations at a positive or negative Atwood number interface that interacts with a fast magnetosonic converging MHD shock. In hydrodynamics, the baroclinic vorticity deposited by the incident shock on the interface is the driving mechanism of the Richtmyer-Meshkov instability. The RMI phase is somewhat short-lived and followed by Rayleigh-Taylor instability that dominates the growth rate. For the MHD cases, the suppression of the instability at the interface is most evident for large wavenumbers and relatively strong magnetic fields strengths. After the shock interacts with the interface, the emerging vorticity breaks up into waves traveling parallel and anti-parallel to the magnetic field. The interference as these waves propagate with alternating phase causing the perturbation growth rate of the interface to oscillate in time. The present research can be further extended in many aspects: linear stability in spherical geometry, simulating flows with multiple interfaces, applying other magnetic field configurations and investigating the nonlinear development of these flows.

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Fig. 1. Schematic diagram of the physical setup: Two fluids of densities ρ_1 and ρ_2 are separated by a perturbed interface located at $r = R_0$. A fast magnetosonic MHD shock is initially located at $r = R_s$, with Mach number M

Fig. 2. Initial conditions of the base flow variables (a) Incident shock position (b) Modified magnetic field (c) Pressure and (d) Radial velocity

Fig. 3. Time history of unscaled growth rate of the shocked interface for $\beta = 16$ wavenumber $m = 128$ using different mesh sizes: 800, 1600, 3200, 6400, 12800 zones in the radial domain

Fig. 4. Error of unscaled growth rate of the shocked $m = 128$ wavenumber interface for $\beta = 16$ at time $t = 0.3$

Fig. 5. Density profiles of base state at $t = 0.0$ (initial conditions) and $t = 0.5$ for $\beta = 16$. IS denotes the incident shock, TF denotes the transmitted fast MHD shock, RF denotes the reflected fast MHD shocks, and CD is the contact discontinuity

Fig. 6. Spacetime diagram of density field of the base state. IS denotes the incident shock, TF denotes the transmitted fast MHD shock, RF denotes the reflected fast MHD shock, and CD is the contact discontinuity

Fig. 7. Density profiles of base state at $t = 0.0$ (initial conditions) and $t = 0.2$ for $\beta = 4$. IS denotes the incident shock, TF denotes the transmitted fast MHD shock, RR denotes the reflected rarefaction fan, and CD is the contact discontinuity

Fig. 8. Spacetime diagram of density field of the base state. IS denotes the incident shock, TF denotes the transmitted fast MHD shock, RR denotes the rarefaction fan, and CD is the contact discontinuity

Fig. 9. Scaled $\dot{h}$ for HD and $m = 256$ for $A < 0$ and $A > 0$

(a) Effect of different β and $m = 256$ for $A > 0$ (b) Effect of different β and $m = 256$ for $A < 0$

Fig. 10. Time history of growth rates for different β and fixed azimuthal wavenumber $m = 256$ (a) $A > 0$ (b) $A < 0$

Fig. 11. 2D Vorticity ω_z in the interface for $\beta = 16, m = 256$ at different scaled times chosen to correspond to a full oscillation period of the growth rate. The arrows in the growth rate plot (a) indicate the times chosen

Fig. 12. Time history of the growth rates for different m and $\beta = 4$ (a) $A > 0$ (b) $A < 0$

(a) Effect of different β on RTI for $A > 0, m = 256$

(b) Effect of different β on RTI for $A < 0, m = 256$

Fig. 13. Time history of growth rates for high β values, and fixed azimuthal wavenumber $m = 256$ (a) $A > 0$ (b) $A < 0$