Accepted Manuscript

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PII: S0167-2789(17)30228-2
DOI: <https://doi.org/10.1016/j.physd.2017.10.015>
Reference: PHYSD 31967

To appear in: *Physica D*

Received date: 27 April 2017
Revised date: 25 October 2017
Accepted date: 29 October 2017

Please cite this article as: K. Bod'ová, J. Haskovec, P. Markowich, Well posedness and Maximum Entropy approximation for the dynamics of quantitative traits, *Physica D* (2017), <https://doi.org/10.1016/j.physd.2017.10.015>

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WELL POSEDNESS AND MAXIMUM ENTROPY APPROXIMATION FOR THE DYNAMICS OF QUANTITATIVE TRAITS

KATARÍNA BOŽDOVÁ, JAN HASKOVEC, AND PETER MARKOWICH

ABSTRACT. We study the Fokker-Planck equation derived in the large system limit of the Markovian process describing the dynamics of quantitative traits. The Fokker-Planck equation is posed on a bounded domain and its transport and diffusion coefficients vanish on the domain's boundary. We first argue that, despite this degeneracy, the standard no-flux boundary condition is valid. We derive the weak formulation of the problem and prove the existence and uniqueness of its solutions by constructing the corresponding contraction semigroup on a suitable function space. Then, we prove that for the parameter regime with high enough mutation rate the problem exhibits a positive spectral gap, which implies exponential convergence to equilibrium.

Next, we provide a simple derivation of the so-called Dynamic Maximum Entropy (DynMaxEnt) method for approximation of observables (moments) of the Fokker-Planck solution, which can be interpreted as a nonlinear Galerkin approximation. The limited applicability of the DynMaxEnt method inspires us to introduce its modified version that is valid for the whole range of admissible parameters. Finally, we present several numerical experiments to demonstrate the performance of both the original and modified DynMaxEnt methods. We observe that in the parameter regimes where both methods are valid, the modified one exhibits slightly better approximation properties compared to the original one.

Dedicated to E. Titi's 60. Birthday

And if you listen very hard
The tune will come to you at last
When all are one and one is all
To be a rock and not to roll

Excerpt from: *Stairway to Heaven*
written by Jimmy Page and Robert Plant

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1. INTRODUCTION

Quantitative genetics studies evolution of biologically important traits which are continuous (e.g., size, weight, lifespan, color) and depend on many loci; some of the loci contribute to the trait with a large effect but majority contribute with very small effect. Selection, mutation and stochasticity affect the evolution of quantitative traits and lead to complex dynamics that is often studied using mathematical models. However, most of these models are discrete and stochastic and their complexity obscures rigorous analysis of the effects of various parameters, that are key for biological interpretation. Therefore, the discrete processes are often replaced by continuous approximation, leading to the so-called *diffusion theory*, which accurately approximates multiple classes of population genetics models [17, 18, 12, 10].

The diffusion theory describes the dynamics of allele frequencies $\mathbf{x} = (x_1, \dots, x_L)$, where L is the number of loci that contribute to the trait. Each locus allows two possible alleles (type 0 and 1) and x_i is the proportion of alleles 1 at the i -th locus in a population of N diploid individuals. The joint probability density of allele frequencies $u = u(t, \mathbf{x})$ then satisfies the linear Fokker-Planck equation (i.e., the deterministic forward Kolmogorov equation) of a form

$$\frac{\partial u}{\partial t} = -\frac{1}{2} \sum_{i=1}^L \frac{\partial}{\partial x_i} \left(\xi_i \frac{\partial(\boldsymbol{\alpha} \cdot \mathbf{A})}{\partial x_i} u \right) + \frac{1}{4N} \sum_{i=1}^L \frac{\partial^2}{\partial x_i^2} (\xi_i u), \quad (1.1)$$

on $\Omega_{\mathbf{x}} := (0, 1)^L$, where we denoted $\xi_i := \xi(x_i) = x_i(1 - x_i)$ for $i = 1, \dots, L$. The diffusion term captures the stochasticity of the allele frequencies arising

from random sampling. Here we assume that linkage disequilibria are negligible, otherwise this term would be of cross-diffusion type, reflecting correlations between loci [6]. This biologically corresponds to a sufficiently fast recombination such that the population can be represented by allele frequencies. The equation (1.1) describes a broad range of evolutionary scenarios. These depend on the form of the drift term, which captures deterministic effects on allele frequencies that are described by a vector of coefficients α and a vector of complementary quantities $\mathbf{A}$. We assume the mathematically simplest form of directional selection, extended to dominance, which adds selection on heterozygous individuals. Moreover, we assume symmetrical mutation (easily generalizable to asymmetrical mutation), which together, using the notation of [6], corresponds to the choice

$$\mathbf{A} = (\xi'_1, \dots, \xi'_L, \xi_1, \dots, \xi_L, \ln \xi_1, \dots, \ln \xi_L)$$

and

$$\alpha \cdot \mathbf{A} = -\beta \sum_{i=1}^L \gamma_i \xi'_i + 2h \sum_{i=1}^L \eta_i \xi_i + 2\mu \sum_{i=1}^L \ln \xi_i, \quad (1.2)$$

where the nondimensional parameters $\beta, h, \gamma_i, \eta_i \in \mathbb{R}$ represent the effects of loci on the traits, $\mu > 0$ is the mutation rate, and $\xi'_i := \xi'(x_i) = 1 - 2x_i$. For notational simplicity and without loss of generality, we set $\beta = h = 1$ in the sequel, so that

$$\alpha = (-\gamma_1, \dots, -\gamma_L, 2\eta_1, \dots, 2\eta_L, 2\mu, \dots, 2\mu) \in \mathbb{R}^{3L}.$$

The main difficulty for analysis of the Fokker-Planck equation (1.1) is the degeneracy of the diffusion coefficients $\xi_i = x_i(1 - x_i)$ at the boundary of $\Omega_{\mathbf{x}}$, which arises due to finite population size. Consequently, the task of prescribing boundary conditions that lead to a well-posed problem is far from obvious; see also [7, 8] for related issues in population genetics problems. As noted above, we aim at interpreting the solution u as a time-dependent probability density, which calls for a no-flux boundary condition. In Section 2 we argue that the standard no-flux boundary condition is indeed appropriate for (1.1). In Section 3 we derive the weak formulation of (1.1) subject to the no-flux boundary condition and prove the existence and uniqueness of its solutions by constructing the corresponding contraction semigroup. Then, in Section 4 we prove that for the parameter regime with high enough mutation rate the problem exhibits a positive spectral gap, which implies exponential convergence to equilibrium.

The main challenge of quantitative genetics is to predict the dynamics of the complex traits, particularly when the genetic forces are changing in time. If all evolutionary forces α (with a possible time-dependence) are known one may simply solve the Fokker-Planck equation (1.1) and use the allele frequency distribution $u(t, \mathbf{x})$ to obtain the trait mean and other observables

of interest by computing expectations of functions of allele frequencies subject to $u(t, \mathbf{x})$. However, equation (1.1) is too complex both for analytical solutions and for effective numerical methods in the multi-locus case. Therefore, we present a method that approximates the macroscopic observables of the Fokker-Planck solution without needing to solve the PDE (for more biological details see [6]). Section 5 is devoted to the study of this approximation, the so-called Dynamic Maximum Entropy (DynMaxEnt). We first show in Section 5.1 that a related constrained entropy maximization is equivalent to a moment-matching problem, which we solve in a simple case. This provides an interpretation of the observables and evolutionary forces as the variational constraints and the corresponding Lagrange multipliers. Then, in Section 5.2 we provide a simple and straightforward derivation of the DynMaxEnt method by adopting a quasi-stationary approximation, by assuming that at every time the allele frequency distribution can be approximated by an equilibrium distribution with some evolutionary parameters $\alpha(t)$, that change in time to match the observables. This results in a nonlinear low-dimensional system of ordinary differential equations. It should be noted that the approximation is exact if the system is in equilibrium. It can be interpreted as a nonlinear Galerkin approximation of the Fokker-Planck equation (1.1).

However, this "original" DynMaxEnt method cannot be applied in the biologically realistic regime of small mutations, i.e., when $4N\mu \leq 1$. This inspires us to introduce a modified version, which is valid for the whole range of admissible parameters, Section 5.3. Finally, in Section 6 we present several numerical experiments to demonstrate the performance of both the original and modified DynMaxEnt methods. We observe that in the parameter regimes where both methods are valid, the modified one exhibits slightly better approximation properties compared to the original one.

The surprisingly good approximation properties of the DynMaxEnt method, as documented by the numerical results in [6] and Section 6 of this paper, suggest that the infinitely-dimensional dynamics of the Fokker-Planck equation (1.1) can be well approximated by suitable finitely-dimensional dynamical systems. This is reminiscent of the recent series of works of E. Titi and collaborators [21, 14, 2, 20, 1] where a data assimilation (downscaling) approach to fluid flow problems is developed, inspired by ideas applied for designing finite-parameters feedback control for dissipative systems. The goal of a data assimilation algorithm is to obtain (numerical) approximation of a solution of an infinitely-dimensional dynamical system corresponding to given measurements of a finite number of observables. In particular, in [21], it has been shown that solutions of the two-dimensional Navier-Stokes equations can be well reconstructed from a relatively low number of low Fourier modes or local averages over finite volume elements. In [14], continuous data assimilation (CPA) algorithm was proposed and analyzed for a two-dimensional Bénard convection problem, where the observables were incorporated as a feedback (nudging) term in the evolution equation of the

horizontal velocity. In [2] CPA was applied for downscaling a coarse resolution configuration of the 2D Bénard convection equations into a finer grid, while in [20] the CPA method is studied for a three-dimensional Brinkman-Forchheimer-extended Darcy model of porous media, and in [1] for the three-dimensional Navier-Stokes- α model. Finally, in [15] numerical performance of the CPA algorithm in the context of the two-dimensional incompressible Navier-Stokes equations was studied. It was shown that the numerical method is computationally efficient and performs far better than the analytical estimates suggest. This is similar to our numerical observations showing very good approximation properties of the DynMaxEnt method applied to the Fokker-Planck equation (1.1).

2. BOUNDARY CONDITIONS FOR THE STATIONARY PROBLEM

As pointed out in Section 1, the degeneracy of the diffusion coefficients $\xi_i = x_i(1 - x_i)$ at the boundary $\partial\Omega_{\mathbf{x}}$ renders the question which type of boundary conditions shall be prescribed (if any at all) to produce a well-posed problem. However, the essential property that we want to impose on the solution is total mass conservation (together with nonnegativity preservation), so that we can interpret it as a probability density. Therefore, the precise question that we ask is: Does (1.1) admit unique nonnegative integrable solutions that conserve total mass?

The question is answered by the Feller classification of one-dimensional diffusions [13, 24], where the prototypical parabolic equation

$$\frac{\partial u}{\partial t} = \partial_{xx}(axu) - \partial_x((bx + c)u) \quad (2.1)$$

is studied on the interval $(0, \infty)$ with constant coefficients $a > 0, b, c \in \mathbb{R}$. Clearly, the behavior of the solutions of (1.1) at the boundary is modeled by (2.1) at $x = 0$. In particular, the case $4N\mu \geq 1$ corresponds to the "entrance case" of [13] ($c \geq a$ in (2.1)), where a unique solution of the initial value problem exists that is positivity and mass preserving, and such that its flux vanishes at the boundary. On the other hand, for $4N\mu < 1$ the "regular case" takes place ($0 < c < a$ in (2.1)), where the situation is almost the same as with nondegenerate equations: boundary and initial conditions have to be prescribed in order to select a unique solution. In particular, there exists a unique nonnegativity and mass preserving solution of the initial value problem defined by the condition that the flux at the boundary vanishes (homogeneous Neumann or "reflecting barrier").

To summarize, in both the cases mentioned above (i.e., as long as $\mu > 0$, which we assume), equation (1.1) posed in the spatially one-dimensional setting indeed admits unique nonnegative and mass preserving solutions. This fact can be also illustrated with the following simple calculation: Let us write the stationary 1D-version of (1.1) in the form

$$\partial_x \left(Du_{\alpha} \partial_x \left(\frac{u}{u_{\alpha}} \right) \right) = g \quad (2.2)$$

for $x \in (0, 1)$, where $g \in L^1(0, 1)$ is a prescribed function, $D = \frac{1}{4N}\xi(x) = \frac{1}{4N}x(1-x)$, and u_α is the solution of the homogeneous problem,

$$u_\alpha = \mathbb{Z}_\alpha^{-1} \xi^{-1} \exp(2N\alpha \cdot \mathbf{A}) = \mathbb{Z}_\alpha^{-1} \xi^{4N\mu-1} \exp(2N\gamma\xi' + 4N\eta\xi).$$

Here $\mathbb{Z}_\alpha$ is the normalization constant (partition function)

$$\mathbb{Z}_\alpha := \int_0^1 \xi^{-1} \exp(2N\alpha \cdot \mathbf{A}) dx. \quad (2.3)$$

Observe that the above integral is finite for $\mu > 0$, which we assumed. Obviously, mass conservation is imposed by choosing g such that $\int_0^1 g(s) ds = 0$.

Observe that the product Du_α behaves like $\xi^{4N\mu}$ close to $x = 0$ and $x = 1$, so that it vanishes at the boundary and leads to a degeneracy in the flux term $F(x) := Du_\alpha \partial_x (u/u_\alpha)$ for $x \in \{0, 1\}$. To avoid possible difficulties due to this degeneracy, we integrate (2.2) for a fixed $x \in (0, 1)$ on the interval $(1/2, x)$,

$$F(x) = Du_\alpha \partial_x \left(\frac{u}{u_\alpha} \right) = \int_{1/2}^x g(s) ds + C_1,$$

where C_1 is an integration constant. Clearly, imposing the formal no-flux boundary condition $F(x) = 0$ at, say, $x = 0$ is equivalent to setting C_1 to the particular value

$$C_1 = \int_0^{1/2} g(s) ds.$$

The assumption $\int_0^1 f(s) ds = 0$ then implies that $F(x) = 0$ is verified also at $x = 1$. Thus, we only need to check that the solution u is integrable. Integrating once again yields

$$u = C_2 u_\alpha + C_1 u_\alpha \int_{1/2}^x \frac{ds}{D(s)u_\alpha(s)} + u_\alpha \int_{1/2}^x \frac{G(s) ds}{D(s)u_\alpha(s)}, \quad (2.4)$$

with $G(s) := \int_{1/2}^s g(r) dr$. Observe that

$$\int_{1/2}^x \frac{ds}{D(s)u_\alpha(s)} \approx \xi^{-4N\mu+1} \quad \text{close to } x \in \{0, 1\},$$

so that the second term in (2.4) is bounded on $[0, 1]$ and thus integrable. Due to the boundedness of $G(s)$, the same holds also for the third term in (2.4). Consequently, the solution u constructed in (2.4) is integrable on $(0, 1)$.

Although the above arguments only apply to the spatially one-dimensional setting, it provides a strong heuristic hint that the conclusion also holds in the multidimensional case.

3. EXISTENCE AND UNIQUENESS OF SOLUTIONS

In this section we construct solutions of the Fokker-Planck equation (3.1), which we rewrite in the form

$$\frac{\partial u}{\partial t} = \nabla_{\mathbf{x}} \cdot \left(D u_{\alpha} \nabla_{\mathbf{x}} \left(\frac{u}{u_{\alpha}} \right) \right) \quad (3.1)$$

with the diagonal diffusion matrix $D = D(\mathbf{x})$, $D_{ij} = \frac{1}{4N} \xi_i \delta_{ij}$ for $i, j = 1, \dots, L$, and u_{α} is the stationary solution taking the form

$$u_{\alpha} = \frac{1}{\mathbb{Z}_{\alpha}} \frac{\exp(2N\alpha \cdot \mathbf{A})}{\prod_{i=1}^L \xi_i}. \quad (3.2)$$

Here $\mathbb{Z}_{\alpha}$ is the normalization constant (partition function)

$$\mathbb{Z}_{\alpha} := \int_{\Omega_{\mathbf{x}}} \frac{\exp(2N\alpha \cdot \mathbf{A})}{\prod_{i=1}^L \xi_i} d\mathbf{x}. \quad (3.3)$$

Observe that the above integral is finite for $\mu > 0$, which we assumed. Equation (3.1) is supplemented with the formal no-flux boundary condition (total mass conservation)

$$D u_{\alpha} \nabla_{\mathbf{x}} \left(\frac{u}{u_{\alpha}} \right) \cdot \nu = 0 \quad \text{a.e. on } \partial\Omega_{\mathbf{x}}, \quad (3.4)$$

where $\nu = \nu(x)$ denotes the unit normal vector to the boundary of $\Omega_{\mathbf{x}}$. Moreover, we prescribe the initial condition

$$u(t=0) = u_0 \quad \text{on } \Omega_{\mathbf{x}}, \quad (3.5)$$

such that $\int_{\Omega_{\mathbf{x}}} u_0 d\mathbf{x} = 1$. Our strategy is to apply the standard coordinate transforms to convert the problem to the Hamiltonian form $(-\Delta + V)$ for a suitable potential V and then construct the corresponding semigroup; see, e.g. [26, 3]. In order to obtain some intuition, we first carry out the transform formally.

3.1. Formal calculations. Setting

$$\mathcal{B}(\mathbf{x}) := \sum_{i=1}^L \ln \xi(x_i) - 2N\alpha \cdot \mathbf{A}(\mathbf{x}),$$

(3.1) is written in the form

$$\frac{\partial u}{\partial t} = \nabla_{\mathbf{x}} \cdot (D(\nabla_{\mathbf{x}} u + u \nabla_{\mathbf{x}} \mathcal{B}))$$

with the boundary condition $D(\nabla_{\mathbf{x}} u + u \nabla_{\mathbf{x}} \mathcal{B}) \cdot \nu = 0$. For $i = 1, \dots, N$ we introduce the Lamperti coordinate transform [19]

$$y_i := y(x_i) := 2\sqrt{N} \int_0^{x_i} \frac{ds}{\sqrt{\xi(s)}} = 4\sqrt{N} \arcsin \sqrt{x_i}, \quad (3.6)$$

and denote $\mathbf{y} := (y_1, \dots, y_L)$. Note that $\mathbf{x} \mapsto \mathbf{y}$ maps $\Omega_{\mathbf{x}} = (0, 1)^L$ onto $\Omega_{\mathbf{y}} := (0, Y_N)^L$ with $Y_N := 2\pi\sqrt{N}$. Introducing the new variable

$$\bar{u}(\mathbf{y}) := J(\mathbf{x}(\mathbf{y}))u(\mathbf{x}(\mathbf{y})), \quad J(\mathbf{x}) := \left(2\sqrt{N}\right)^{-L} \prod_{j=1}^L \xi^{1/2}(x_j) \quad (3.7)$$

transforms (3.1) to the form

$$\frac{\partial \bar{u}}{\partial t} = \nabla_{\mathbf{y}} \cdot (\nabla_{\mathbf{y}} \bar{u} + \bar{u} \nabla_{\mathbf{y}} (\bar{\mathcal{B}} - \ln \bar{J})), \quad (3.8)$$

with $\bar{\mathcal{B}}(\mathbf{y}) := \mathcal{B}(\mathbf{x}(\mathbf{y}))$ and $\bar{J}(\mathbf{y}) := J(\mathbf{x}(\mathbf{y}))$. By $\mathbf{x}(\mathbf{y})$ we denote the componentwise inverse transform $x_i = x(y_i)$. The no-flux boundary condition (3.4) transforms as

$$\frac{1}{2\sqrt{N}} \sum_{i=1}^L \bar{J}^{-1} \sqrt{\xi_i} (\partial_{y_i} \bar{u} + \bar{u} \partial_{y_i} (\bar{\mathcal{B}} - \ln \bar{J})) \nu_i = 0 \quad \text{a.e. on } \partial\Omega_{\mathbf{y}},$$

where we use the shorthand notation $\xi_i = \xi(x_i(y_i))$. Note that the product $\bar{J}^{-1} \sqrt{\xi_i}$ is constant in y_i and positive on the set $\{\mathbf{y} \in \partial\Omega_{\mathbf{y}}; y_i \in \{0, Y_N\}, 0 < y_j < Y_N \text{ for } j \neq i\}$. Consequently, the transformed boundary condition is equivalent to the nondegenerate expression

$$(\nabla_{\mathbf{y}} \bar{u} + \bar{u} \nabla_{\mathbf{y}} (\bar{\mathcal{B}} - \ln \bar{J})) \cdot \nu = 0 \quad \text{a.e. on } \partial\Omega_{\mathbf{y}}, \quad (3.9)$$

which can be also written as $\nu \cdot \nabla_{\mathbf{y}} \ln(\bar{u}/\bar{u}_{\alpha}) = 0$ a.e. on $\partial\Omega_{\mathbf{y}}$. The steady state for (3.8)–(3.9) is

$$\bar{u}_{\alpha} := \bar{\mathcal{Z}}_{\alpha}^{-1} \exp(-(\bar{\mathcal{B}} - \ln \bar{J})), \quad \bar{\mathcal{Z}}_{\alpha} := \int_{\Omega_{\mathbf{y}}} \exp(-(\bar{\mathcal{B}} - \ln \bar{J})) \, d\mathbf{y}.$$

Finally, with the standard unitary transform

$$z(\mathbf{y}) := \bar{u}(\mathbf{y}) / \sqrt{\bar{u}_{\alpha}(\mathbf{y})},$$

the Fokker-Planck equation (3.1) transforms to the Hamiltonian form

$$\frac{\partial z}{\partial t} = \Delta_{\mathbf{y}} z - V(\mathbf{y})z, \quad (3.10)$$

with

$$V(\mathbf{y}) = \frac{\Delta_{\mathbf{y}} \sqrt{\bar{u}_{\alpha}}}{\sqrt{\bar{u}_{\alpha}}} = \frac{1}{2} \frac{\Delta_{\mathbf{y}} \bar{u}_{\alpha}}{\bar{u}_{\alpha}} - \frac{1}{4} \frac{|\nabla_{\mathbf{y}} \bar{u}_{\alpha}|^2}{\bar{u}_{\alpha}^2},$$

which can be further expressed as

$$V(\mathbf{y}) = -\frac{1}{2} \Delta_{\mathbf{y}} (\bar{\mathcal{B}} - \ln \bar{J}) + \frac{1}{4} |\nabla_{\mathbf{y}} (\bar{\mathcal{B}} - \ln \bar{J})|^2. \quad (3.11)$$

The boundary condition (3.9) transforms to

$$\sqrt{\bar{u}_{\alpha}} \left(\nabla_{\mathbf{y}} z + \frac{1}{2} z \nabla_{\mathbf{y}} (\bar{\mathcal{B}} - \ln \bar{J}) \right) \cdot \nu = 0 \quad \text{a.e. on } \partial\Omega_{\mathbf{y}}. \quad (3.12)$$

Let us remark that with (1.2), $\bar{u}_\alpha$ behaves like $\left(\prod_{i=1}^L \xi_i\right)^{4N\mu-1/2}$ close to the boundary, so that for $4N\mu - 1/2 > 0$ the boundary condition (3.12) is degenerate.

Inserting the expression (1.2) for $\alpha \cdot \mathbf{A}$ into (3.11) gives the explicit expression for the potential

$$V = \frac{1}{16N} \left(4N\mu - \frac{1}{2}\right) \left(4N\mu - \frac{3}{2}\right) \frac{(\xi'_i)^2}{\xi_i} + (\text{bounded terms}), \quad (3.13)$$

where (bounded terms) are expressions involving

$$\begin{aligned} \frac{\partial \xi_i}{\partial y_i} &= \frac{\sqrt{\xi_i}}{2\sqrt{N}} \xi'_i, & \frac{\partial^2 \xi_i}{\partial y_i^2} &= -\frac{1}{2N} \xi_i + \frac{1}{4N} (\xi'_i)^2, \\ \frac{\partial \xi'_i}{\partial y_i} &= -\frac{1}{\sqrt{N}} \sqrt{\xi_i}, & \frac{\partial^2 \xi'_i}{\partial y_i^2} &= \frac{1}{4N} \xi'_i, \end{aligned}$$

that are uniformly bounded on $\bar{\Omega}_y$. The unbounded term in V is

$$\frac{(\xi'_i)^2}{\xi_i} = \frac{(1 - 2x_i)^2}{x_i(1 - x_i)},$$

so for the potential to be bounded below, we need $4N\mu \geq 3/2$.

3.2. Construction of solutions for the case $4N\mu \geq 1/2$. In this Section we shall construct weak solutions of the Fokker-Planck equation (3.1) with $4N\mu \geq 1/2$, subject to the no-flux boundary condition (3.4) and the initial datum (3.5). However, since the equivalent form (3.10) is more suitable to study the asymptotic behavior of the solution for large times, we shall work with this formulation. Due to the issues caused by the degeneracy of the boundary condition, we shall start from a weak formulation of (3.1) and carry out the coordinate transform as in previous Section in order to arrive at a weak formulation of (3.10).

To obtain a symmetric form, we multiply (3.1) by φ/u_α , with a test function $\varphi \in C^\infty(\bar{\Omega}_x)$, and integrate by parts, taking into account the no-flux boundary condition (3.4). We arrive at

$$\frac{d}{dt} \int_{\Omega_x} \frac{u}{u_\alpha} \frac{\varphi}{u_\alpha} u_\alpha d\mathbf{x} = - \int_{\Omega_x} D \nabla_{\mathbf{x}} \left(\frac{u}{u_\alpha} \right) \cdot \nabla_{\mathbf{x}} \left(\frac{\varphi}{u_\alpha} \right) u_\alpha d\mathbf{x}.$$

Carrying out the coordinate transform $\mathbf{x} \mapsto \mathbf{y}$ (3.6), with the Jacobian J given by (3.7), yields

$$\frac{d}{dt} \int_{\Omega_y} \frac{\bar{u}}{\bar{u}_\alpha} \frac{\bar{\varphi}}{\bar{u}_\alpha} d\bar{u}_\alpha(\mathbf{y}) = - \int_{\Omega_y} \nabla_{\mathbf{y}} \left(\frac{\bar{u}}{\bar{u}_\alpha} \right) \cdot \nabla_{\mathbf{y}} \left(\frac{\bar{\varphi}}{\bar{u}_\alpha} \right) d\bar{u}_\alpha(\mathbf{y}),$$

with $\bar{u}$ given by (3.7), $\bar{\varphi}(\mathbf{y}) := J(\mathbf{x}(\mathbf{y}))\varphi(\mathbf{x}(\mathbf{y}))$ and $d\bar{u}_\alpha(\mathbf{y}) := \bar{u}_\alpha d\mathbf{y}$. Finally, defining $z := \bar{u}/\sqrt{\bar{u}_\alpha}$ and $\psi := \bar{\varphi}/\sqrt{\bar{u}_\alpha}$, we arrive at

$$\frac{d}{dt} \int_{\Omega_y} \frac{z}{\sqrt{\bar{u}_\alpha}} \frac{\psi}{\sqrt{\bar{u}_\alpha}} d\bar{u}_\alpha(\mathbf{y}) = - \int_{\Omega_y} \nabla_{\mathbf{y}} \left(\frac{z}{\sqrt{\bar{u}_\alpha}} \right) \cdot \nabla_{\mathbf{y}} \left(\frac{\psi}{\sqrt{\bar{u}_\alpha}} \right) d\bar{u}_\alpha(\mathbf{y}) \quad (3.14)$$

We thus define the space

$$\mathcal{H}_{\mathbf{y}} := \left\{ z \in L^2(\Omega_{\mathbf{y}}); \int_{\Omega_{\mathbf{y}}} \left| \nabla_{\mathbf{y}} \frac{z}{\sqrt{\bar{u}_{\alpha}}} \right|^2 d\bar{u}_{\alpha}(\mathbf{y}) < +\infty \right\} \quad (3.15)$$

with the scalar product

$$(z, \psi)_{\mathcal{H}_{\mathbf{y}}} := \int_{\Omega_{\mathbf{y}}} \frac{z}{\sqrt{\bar{u}_{\alpha}}} \frac{\psi}{\sqrt{\bar{u}_{\alpha}}} d\bar{u}_{\alpha}(\mathbf{y}) + \int_{\Omega_{\mathbf{y}}} \nabla_{\mathbf{y}} \left(\frac{z}{\sqrt{\bar{u}_{\alpha}}} \right) \cdot \nabla_{\mathbf{y}} \left(\frac{\psi}{\sqrt{\bar{u}_{\alpha}}} \right) d\bar{u}_{\alpha}(\mathbf{y}) \quad (3.16)$$

and the induced norm $\|z\|_{\mathcal{H}_{\mathbf{y}}}^2 := (z, z)_{\mathcal{H}_{\mathbf{y}}}$. Central for our analysis is the following result.

Lemma 1. *Let $4N\mu \geq 1/2$. Then for every $z \in \mathcal{H}_{\mathbf{y}}$ the inequality holds*

$$\int_{\Omega_{\mathbf{y}}} \left| \nabla_{\mathbf{y}} \frac{z}{\sqrt{\bar{u}_{\alpha}}} \right|^2 d\bar{u}_{\alpha}(\mathbf{y}) \geq \int_{\Omega_{\mathbf{y}}} |\nabla z|^2 + Vz^2 d\mathbf{y},$$

with V defined in (3.11).

Proof: We have

$$\int_{\Omega_{\mathbf{y}}} \left| \nabla_{\mathbf{y}} \frac{z}{\sqrt{\bar{u}_{\alpha}}} \right|^2 d\bar{u}_{\alpha}(\mathbf{y}) = \int_{\Omega_{\mathbf{y}}} \left(|\nabla z|^2 + \frac{1}{4} \frac{|\nabla_{\mathbf{y}} \bar{u}_{\alpha}|^2}{\bar{u}_{\alpha}^2} z^2 - \frac{\nabla_{\mathbf{y}} z \cdot \nabla_{\mathbf{y}} \bar{u}_{\alpha}}{\bar{u}_{\alpha}} z \right) d\mathbf{y}.$$

We integrate by parts in the last term of the right-hand side,

$$\begin{aligned} - \int_{\Omega_{\mathbf{y}}} \frac{\nabla_{\mathbf{y}} z \cdot \nabla_{\mathbf{y}} \bar{u}_{\alpha}}{\bar{u}_{\alpha}} z d\mathbf{y} &= -\frac{1}{2} \int_{\Omega_{\mathbf{y}}} \frac{\nabla_{\mathbf{y}} \bar{u}_{\alpha}}{\bar{u}_{\alpha}} \cdot \nabla_{\mathbf{y}} z^2 d\mathbf{y} \\ &= \frac{1}{2} \int_{\Omega_{\mathbf{y}}} z^2 \nabla_{\mathbf{y}} \cdot \left(\frac{\nabla_{\mathbf{y}} \bar{u}_{\alpha}}{\bar{u}_{\alpha}} \right) d\mathbf{y} - \frac{1}{2} \int_{\partial\Omega_{\mathbf{y}}} z^2 \frac{\nabla_{\mathbf{y}} \bar{u}_{\alpha}}{\bar{u}_{\alpha}} \cdot \nu dS_{\mathbf{y}}. \end{aligned}$$

With (1.2) we have

$$\frac{\nabla_{y_i} \bar{u}_{\alpha}}{\bar{u}_{\alpha}} = \nabla_{y_i} (\ln \bar{J} - \bar{B}) = \frac{\sqrt{\xi_i}}{2\sqrt{N}} \left[\left(4N\mu - \frac{1}{2} \right) \frac{\xi'_i}{\xi_i} - 4N\gamma_i + 4N\eta\xi'_i \right].$$

Since ξ_i vanishes for $y_i \in \{0, Y_N\}$ and ξ'_i is bounded on $[0, Y_N]$, we have

$$-\frac{1}{2} \int_{\partial\Omega_{\mathbf{y}}} z^2 \frac{\nabla_{\mathbf{y}} \bar{u}_{\alpha}}{\bar{u}_{\alpha}} \cdot \nu dS_{\mathbf{y}} = -\frac{1}{4\sqrt{N}} \left(4N\mu - \frac{1}{2} \right) \sum_{i=1}^L \int_{\partial\Omega_{\mathbf{y}}} z^2 \frac{\xi'_i}{\sqrt{\xi_i}} \nu_i dS_{\mathbf{y}}.$$

We write the boundary of the hypercube $\Omega_{\mathbf{y}}$ as an union of the pairs of faces,

$$\partial\Omega_{\mathbf{y}} = \bigcup_{i=1}^L F_i, \quad F_i := \{\mathbf{y} \in \partial\Omega_{\mathbf{y}}, y_j \in \{0, Y_N\}\},$$

then we have

$$\sum_{i=1}^L \int_{\partial\Omega_{\mathbf{y}}} z^2 \frac{\xi'_i}{\sqrt{\xi_i}} \nu_i dS_{\mathbf{y}} = \sum_{i=1}^L \int_{F_i} \left[z^2 \frac{\xi'_i}{\sqrt{\xi_i}} \right]_{y_i=0}^{Y_N} dS_{F_i},$$

where dS_{F_i} denotes the $(L-1)$ -dimensional Lebesgue measure on F_i . Since $x'_i(x(Y_N)) = x'_i(1) = -1$ and $x'_i(x(0)) = x'_i(0) = 1$, we have

$$\left[z^2 \frac{\xi'_i}{\sqrt{\xi_i}} \right]_{y_i=0}^{Y_N} \leq 0.$$

Therefore, if $4N\mu - \frac{1}{2} \geq 0$,

$$-\frac{1}{2} \int_{\partial\Omega_y} z^2 \frac{\nabla_y \bar{u}_\alpha}{\bar{u}_\alpha} \cdot \nu dS_y \geq 0.$$

Consequently,

$$\begin{aligned} \int_{\Omega_y} \left| \nabla_y \frac{z}{\sqrt{\bar{u}_\alpha}} \right|^2 d\bar{u}_\alpha(\mathbf{y}) &\geq \int_{\Omega_y} \left(|\nabla z|^2 + \frac{1}{4} \frac{|\nabla_y \bar{u}_\alpha|^2}{\bar{u}_\alpha^2} z^2 + \frac{1}{2} z^2 \nabla_y \cdot \left(\frac{\nabla_y \bar{u}_\alpha}{\bar{u}_\alpha} \right) \right) d\mathbf{y} \\ &= \int_{\Omega_y} |\nabla z|^2 + V z^2 d\mathbf{y}. \end{aligned}$$

Finally, the above formal calculation are made rigorous by replacing $\bar{u}_\alpha$ by $\bar{u}_\alpha^\varepsilon := \bar{u}_\alpha + \varepsilon$ for $\varepsilon > 0$ and subsequently passing to the limit $\varepsilon \rightarrow 0$. ■

Lemma 2. *Let $4N\mu \geq 1/2$. Then the space $\mathcal{H}_y$ defined in (3.15) with the scalar product $(\cdot, \cdot)_{\mathcal{H}_y}$ is a Hilbert space, and is densely embedded into $L^2(\Omega_y)$.*

Proof: The definition (3.16) of the scalar product in $\mathcal{H}_y$ directly implies that $\|\cdot\|_{L^2(\Omega_y)} \leq \|\cdot\|_{\mathcal{H}_y}$. Therefore, if z_k is a Cauchy sequence in $\mathcal{H}_y$, then it is also Cauchy in $L^2(\Omega_y)$ and a limit $z \in L^2(\Omega_y)$ exists. Since the sequence $\|z_k\|_{\mathcal{H}_y}$ is bounded, $z \in \mathcal{H}_y$ and, therefore, the space $\mathcal{H}_y$ is complete. The density of the embedding into $L^2(\Omega_y)$ is due to the fact that the set of smooth functions with compact support is dense in $\mathcal{H}_y$ (note that the possible singularity of the weight u_α at the boundary $\partial\Omega_y$ does not pose any difficulties due to the compactness of the support of the approximating functions). ■

Definition 1. *We call $z \in L^2((0, T); \mathcal{H}_y) \cap C([0, T]; L^2(\Omega_y))$ a weak solution of (3.10) on $[0, T)$ subject to the boundary condition (3.12) if (3.14) holds for every $\psi \in \mathcal{H}_y$ and almost all $t \in (0, T)$, and the initial condition is satisfied by continuity in $C([0, T]; L^2(\Omega_y))$.*

We remark that a formal integration by parts in the right-hand side of (3.14) gives

$$\begin{aligned} & - \int_{\Omega_y} \nabla_y \left(\frac{z}{\sqrt{\bar{u}_\alpha}} \right) \cdot \nabla_y \left(\frac{\psi}{\sqrt{\bar{u}_\alpha}} \right) d\bar{u}_\alpha(\mathbf{y}) \\ &= - \int_{\partial\Omega_y} \left[\sqrt{\bar{u}_\alpha} \left(\nabla_y z + \frac{1}{2} z \nabla_y (\bar{\mathcal{B}} - \ln \bar{\mathcal{J}}) \right) \cdot \nu \right] \frac{\psi}{\sqrt{\bar{u}_\alpha}} dS_y \end{aligned}$$

$$+ \int_{\Omega_{\mathbf{y}}} [\Delta_{\mathbf{y}} z - Vz] \frac{\psi}{\sqrt{u_{\alpha}}} d\mathbf{y}.$$

This justifies the interpretation of (3.14) as the weak formulation of (3.10) subject to the boundary condition (3.12).

We now define the operator $\mathcal{L} : D(\mathcal{L}) \subset \mathcal{H}_{\mathbf{y}} \rightarrow L^2(\Omega_{\mathbf{y}})$ by its action

$$\langle \mathcal{L}z, \psi \rangle := - \int_{\Omega_{\mathbf{y}}} \nabla_{\mathbf{y}} \left(\frac{z}{\sqrt{u_{\alpha}}} \right) \cdot \nabla_{\mathbf{y}} \left(\frac{\psi}{\sqrt{u_{\alpha}}} \right) d\bar{u}_{\alpha}(\mathbf{y}) \quad (3.17)$$

for all $z, \psi \in \mathcal{H}_{\mathbf{y}}$. We shall prove that the closure $\overline{\mathcal{L}}$ of $\mathcal{L}$ generates a contraction semigroup on $L^2(\Omega_{\mathbf{y}})$. For this sake, we study the resolvent problem

$$(-\mathcal{L} + \lambda)z = f \quad (3.18)$$

for (some) $\lambda > 0$ and $f \in L^2(\Omega_{\mathbf{y}})$.

Lemma 3. *Let $4N\mu \geq 1/2$. Then for every $f \in L^2(\Omega_{\mathbf{y}})$ the resolvent problem (3.18) has a unique solution $z \in \mathcal{H}_{\mathbf{y}}$.*

Proof: For a fixed $\lambda > 0$ we define the bilinear form $a : \mathcal{H}_{\mathbf{y}} \times \mathcal{H}_{\mathbf{y}} \rightarrow \mathbb{R}$,

$$a_{\lambda}(z, \psi) := \langle -\mathcal{L}z, \psi \rangle + \lambda(z, \psi),$$

where (z, ψ) denotes the standard scalar product on $L^2(\Omega_{\mathbf{y}})$. The resolvent problem (3.18) with the no-flux boundary conditions is equivalent to

$$a_{\lambda}(z, \psi) = (f, \psi) \quad \text{for all } \psi \in \mathcal{H}_{\mathbf{y}}.$$

A straightforward application of the Hölder inequality gives the continuity of a_{λ} ,

$$a_{\lambda}(z, \psi) \leq C \|z\|_{\mathcal{H}_{\mathbf{y}}} \|\psi\|_{\mathcal{H}_{\mathbf{y}}}$$

for a suitable constant $C > 0$; coercivity is straightforward. Finally, the mapping $\psi \mapsto (f, \psi)$ with $f \in L^2(\Omega_{\mathbf{y}})$ is an element of the dual space $(\mathcal{H}_{\mathbf{y}})'$. Consequently, an application of the Lax-Milgram theorem yields the existence and uniqueness of the solution $z \in \mathcal{H}_{\mathbf{y}}$. ■

Theorem 1. *Let $4N\mu \geq 1/2$. Then the closure $\overline{\mathcal{L}}$ of $\mathcal{L}$ generates a contraction semigroup on $L^2(\Omega_{\mathbf{y}})$.*

Proof: Since $\mathcal{H}_{\mathbf{y}}$ is densely embedded into $L^2(\Omega_{\mathbf{y}})$, the operator $\mathcal{L}$ is densely defined, and dissipative. Moreover, due to Lemma 3, the range of $-\mathcal{L} + \lambda$ is $L^2(\Omega_{\mathbf{y}})$ for all $\lambda > 0$. The claim then follows by an application of the Lumer-Phillips theorem [23]. ■

The contraction semigroup constructed in Theorem 1 provides the announced existence and uniqueness of weak solutions $z \in L^2((0, T); \mathcal{H}_{\mathbf{y}}) \cap C([0, T]; L^2(\Omega_{\mathbf{y}}))$ of (3.10) subject to the no-flux boundary condition (3.12) in the sense of Definition 1. The solutions are formally written as $z(t) =$

$e^{\mathcal{L}t}z_0$, where $z_0 \in L^2(\Omega_{\mathbf{y}})$ is the initial datum; see, e.g., [23]. Their nonnegativity follows from the well known fact [11] that if $\nabla_{\mathbf{y}}(z/\sqrt{u_{\alpha}}) \in L^2(\Omega_{\mathbf{y}})$, then $\nabla_{\mathbf{y}}(z^-/\sqrt{u_{\alpha}}) = \nabla_{\mathbf{y}}(z/\sqrt{u_{\alpha}})^- \in L^2(\Omega_{\mathbf{y}})$, where $z^- := -\min(0, z)$, and, formally,

$$\nabla_{\mathbf{y}}\left(\frac{z^-}{\sqrt{u_{\alpha}}}\right) = \chi_{\{z \leq 0\}} \nabla_{\mathbf{y}}\left(\frac{z}{\sqrt{u_{\alpha}}}\right),$$

where $\chi_{\{z \leq 0\}}$ denotes the characteristic function of the set where $z \leq 0$. Therefore, $\varphi := z^-$ is an admissible test function in (3.14) and we obtain

$$\frac{d}{dt} \int_{\Omega_{\mathbf{y}}} \left| \frac{z^-}{\sqrt{u_{\alpha}}} \right|^2 d\bar{u}_{\alpha}(\mathbf{y}) = - \int_{\Omega_{\mathbf{y}}} \left| \nabla_{\mathbf{y}} \left(\frac{z^-}{\sqrt{u_{\alpha}}} \right) \right|^2 d\bar{u}_{\alpha}(\mathbf{y}) \leq 0.$$

By the inverse coordinate transform to (3.6) we obtain nonnegative weak solutions of the original Fokker-Planck equation (1.1) subject to the non-negative initial datum (3.5) and the no-flux boundary condition (3.4).

4. SPECTRAL GAP - EXPONENTIAL CONVERGENCE TO EQUILIBRIUM

In this Section we shall perform a spectral analysis of the operator $(-\mathcal{L})$ and prove that boundedness below of the potential V (3.11) implies exponential convergence to equilibrium for (3.10). From the explicit expression (3.13) for V we see that V is bounded below if $4N\mu \geq 3/2$.

Lemma 4. *Let $4N\mu \geq 3/2$. Then the operator $(-\mathcal{L})$ defined in (3.17) has compact resolvent.*

Proof: We need to show that for some $\lambda > 0$ the operator $(-\mathcal{L} + \lambda)^{-1}$ is compact as a mapping from $L^2(\Omega_{\mathbf{y}})$ into itself. Let $f \in L^2(\Omega_{\mathbf{y}})$ and $z = (-\mathcal{L} + \lambda)^{-1}f$, constructed in Lemma 3. From Lemma 1 we have

$$((-\mathcal{L} + \lambda)z, z) \geq \int_{\Omega_{\mathbf{y}}} |\nabla_{\mathbf{y}} z|^2 + (V + \lambda)z^2 d\mathbf{y} \geq C \|z\|_{H^1(\Omega_{\mathbf{y}})}^2$$

for some constant $C > 0$ and λ chosen such that $\min_{\mathbf{y} \in \Omega_{\mathbf{y}}} (V(\mathbf{y}) + \lambda) > 0$. On the other hand, the Cauchy-Schwartz inequality gives

$$((-\mathcal{L} + \lambda)z, z) = (f, z) \leq \frac{1}{2\varepsilon} \|f\|_{L^2(\Omega_{\mathbf{y}})}^2 + \frac{\varepsilon}{2} \|z\|_{L^2(\Omega_{\mathbf{y}})}^2,$$

so for sufficiently small $\varepsilon > 0$ we conclude

$$\|(-\mathcal{L} + \lambda)^{-1}f\|_{H^1(\Omega_{\mathbf{y}})} = \|z\|_{H^1(\Omega_{\mathbf{y}})} \leq C \|f\|_{L^2(\Omega_{\mathbf{y}})}$$

and the claim follows by the compact embedding of the Sobolev space H^1 into L^2 . ■

Together with the obvious self-adjointness of $(-\mathcal{L})$, Lemma 4 implies that $(-\mathcal{L})$ has a discrete spectrum without finite accumulation points. Moreover, all its eigenvalues are nonnegative. This implies the existence of a positive

spectral gap and, consequently, exponential convergence to equilibrium as $t \rightarrow \infty$, see, e.g., [3].

Finally, let us remark that the calculations performed in Sections 3 and 4 are applicable to a more general class of linear Fokker-Planck equations than the particular form (1.1) considered here. The essential conditions for the applicability of the method are: (1) integrability of the steady state solution of the Fokker-Planck equation, (2) total mass conservation, and (3) boundedness from below of the potential V in the Hamiltonian formulation (3.10). We again refer to [3] for a comprehensive discussion.

5. THE DYNAMIC MAXIMUM ENTROPY APPROXIMATION

In typical applications in quantitative genetics the solution of the Fokker-Planck equation (1.1) is not the main object of interest. One is rather interested in the evolution of its certain (expectations of) observables (moments) that correspond to the macroscopic dynamics of observable quantitative traits. This naturally leads to the question whether one can derive a finite-dimensional system of differential equations that approximates the evolution of the observables of interest, avoiding the need of solving (1.1). This question has been studied previously by analogy with statistical mechanics: the allele frequency distribution is approximated by the stationary form, which maximizes the logarithmic relative entropy. Called Maximum Entropy Method, it has been applied to broad spectrum of problems ranging from the statistics of neural spiking [29, 31], bird flocking [5], protein structure [32], immunology [22] and more. For transient problems described by known dynamical equations (e.g., Fokker-Planck equation), the *Dynamic Maximum Entropy* (*DynMaxEnt*) method assumes quasi-stationarity at each time point. It has been applied, e.g., to modeling of cosmic ray transport [16], general Fokker-Planck equation [25], analysis of genetic algorithms [27], and population genetics [28, 4, 6].

In this section we focus on the DynMaxEnt method in the context of population genetics. In particular, in [6] it is observed that the "classical" DynMaxEnt method cannot be applied in the regime of small mutations, and the theory is extended for this regime to account for changes in mutation strength. Surprisingly, systematic numerical simulations [6] document superb approximation properties of the method even far from the quasi-stationary regime. However, derivation of analytic error estimates remains an open problem. Here we discuss several aspects of the DynMaxEnt method. First, in Section 5.1 we show that constrained maximization of a logarithmic entropy functional leads to a observables-matching condition. Then, in Section 5.2 we provide a simple and straightforward derivation of the DynMaxEnt method by adopting a quasi-stationary approximation. To our best knowledge, this derivation has not been known before. Finally, in Section 5.3 we consider the scalar case and derive a modified version

of the DynMaxEnt method, which is valid for the whole range of admissible parameters. Although the DynMaxEnt method, as a particular type of quasi-stationary approximation, is in principle applicable to a broad class of problems where the invariant measure is known in explicit form, we focus on the particular problem of dynamics of quantitative traits (1.1). This has two reasons: the DynMaxEnt method is frequently applied in this context; and, the modified version introduced in Section 5.3 is specific for the particular form (1.1)–(1.2).

5.1. Constrained entropy maximization. We shall call the vector $\alpha \in \mathbb{R}^d$ *admissible* if the corresponding normalization factor $\mathbb{Z}_\alpha$ (3.3) is finite. For any integrable function $u \in L^1(\Omega_{\mathbf{x}})$ with $\int_{\Omega_{\mathbf{x}}} u(\mathbf{x}) d\mathbf{x} = 1$ and any admissible $\alpha \in \mathbb{R}^d$ we define the logarithmic relative entropy

$$H(u|u_\alpha) := \int_{\Omega_{\mathbf{x}}} u \ln \frac{u}{u_\alpha} d\mathbf{x}, \quad (5.1)$$

where u_α is the normalized stationary solution of the Fokker-Planck equation (1.1), given by formula (3.2). Note that this is a different approach compared with [6], where the logarithmic entropy is taken relative to the neutral distribution of allele frequencies in the absence of mutation or selection, $\prod_{i=1}^L \xi_i^{-1}$, and the variational problem is complemented with normalization and observables constraints.

For a fixed $u \in L^1(\Omega_{\mathbf{x}})$ with finite $\mathbf{A}$ -observables, let us consider the maximization of the relative entropy (5.1) in terms of admissible $\alpha \in \mathbb{R}^d$, i.e., the task of maximizing the function $\alpha \mapsto H(u|u_\alpha)$. If a critical point exists, then for $i = 1, \dots, d$,

$$\begin{aligned} \frac{d}{d\alpha_i} H(u|u_\alpha) &= - \int_{\Omega_{\mathbf{x}}} \frac{u}{u_\alpha} \frac{d}{d\alpha_i} u_\alpha d\mathbf{x} \\ &= \langle A_i \rangle_{u_\alpha} - \langle A_i \rangle_u = 0. \end{aligned}$$

Consequently, if a maximizer α^* exists, then the $\mathbf{A}$ -observables corresponding to u_{α^*} must be matching the same observables of u . This naturally leads to the question of solvability of the nonlinear system of equations

$$\langle \mathbf{A} \rangle_{u_\alpha} = \langle \mathbf{A} \rangle_u$$

in terms of the admissible parameter vector $\alpha \in \mathbb{R}^d$, for a given, normalized $u \in L^1(\Omega_{\mathbf{x}})$ with finite $\mathbf{A}$ -observables. To address this question seems to be a very difficult task that we leave open. We merely remark that the Hessian matrix of $\alpha \mapsto H(u|u_\alpha)$,

$$\frac{d^2}{d\alpha_i d\alpha_j} H(u|u_\alpha) = \langle A_i A_j \rangle_{u_\alpha} - \langle A_i \rangle_{u_\alpha} \langle A_j \rangle_{u_\alpha},$$

is equal to the covariance matrix of the random variables $\mathbf{A}$ with the probability density u_α . Thus, the Hessian matrix is positive semidefinite. In the scalar case, solvability of the observables equation $\langle A \rangle_{u_\alpha} = \langle A \rangle_u$ can

be studied for particular choices of A . We shall give an example below in Section 5.3.1.

5.2. Derivation of the DynMaxEnt method. Let us consider $u = u(t)$ a solution of the Fokker-Planck equation (1.1) with admissible parameter vector α , subject to the initial datum $u(t = 0) = u_{\alpha^0}$ for some admissible α^0 . The DynMaxEnt method is derived in two steps: First, we multiply the equation in its form (3.1) by the vector $\mathbf{A}$ and integrate,

$$\begin{aligned} \frac{d}{dt} \langle \mathbf{A} \rangle_{u(t)} &= \int_{\Omega_{\mathbf{x}}} \mathbf{A} \nabla_{\mathbf{x}} \left(D u_{\alpha} \nabla_{\mathbf{x}} \left(\frac{u}{u_{\alpha}} \right) \right) d\mathbf{x} \\ &= - \int_{\Omega_{\mathbf{x}}} \nabla_{\mathbf{x}} \mathbf{A} D u_{\alpha} \nabla_{\mathbf{x}} \left(\frac{u}{u_{\alpha}} \right) d\mathbf{x}, \end{aligned}$$

where we assumed that the boundary term in the integration by parts vanishes (note that, in general, this does not necessarily follow from (3.4)). In the second step, we substitute $u(t)$ in the above expression by $u_{\alpha^*(t)}$ (3.2) with some time-dependent parameter vector $\alpha^* = \alpha^*(t)$, which leads to

$$\frac{d}{dt} \langle \mathbf{A} \rangle_{u_{\alpha^*(t)}} = - \int_{\Omega_{\mathbf{x}}} \nabla_{\mathbf{x}} \mathbf{A} D u_{\alpha} \nabla_{\mathbf{x}} \left(\frac{u_{\alpha^*(t)}}{u_{\alpha}} \right) d\mathbf{x} + \mathbf{R},$$

where $\mathbf{R}$ is a vector-valued residuum term. We now introduce an approximation by neglecting the residuum $\mathbf{R}$. Expanding the derivatives on both sides of the above equation leads then to

$$\begin{aligned} \left(\langle \mathbf{A} \otimes \mathbf{A} \rangle_{u_{\alpha^*(t)}} - \langle \mathbf{A} \rangle_{u_{\alpha^*(t)}} \otimes \langle \mathbf{A} \rangle_{u_{\alpha^*(t)}} \right) \frac{d\alpha^*(t)}{dt} &= \\ &= \frac{1}{2} \langle \xi \nabla_{\mathbf{x}} \mathbf{A} : \nabla_{\mathbf{x}} \mathbf{A} \rangle_{u_{\alpha^*(t)}} (\alpha - \alpha^*(t)), \end{aligned} \quad (5.2)$$

where $\nabla_{\mathbf{x}} \mathbf{A} : \nabla_{\mathbf{x}} \mathbf{A}$ is the symmetric $d \times d$ matrix with the (i, k) -component $\sum_{j=1}^d \partial_{x_j} A_i \partial_{x_j} A_k$. The nonlinear ODE system for $\alpha^* = \alpha^*(t)$ is called the *DynMaxEnt method* for approximation of the observables of (1.1). However, two comments have to be made: First, the matrix on the left-hand side,

$$\left(\langle \mathbf{A} \otimes \mathbf{A} \rangle_{u_{\alpha^*(t)}} - \langle \mathbf{A} \rangle_{u_{\alpha^*(t)}} \otimes \langle \mathbf{A} \rangle_{u_{\alpha^*(t)}} \right),$$

is positive semidefinite, since it is the covariance matrix of the observables $\mathbf{A}$ of the probability distribution $u_{\alpha^*(t)}$. However, in order (5.2) to be globally solvable, the covariance matrix must be uniformly (positive) definite, which in general may not be the case. Furthermore, the matrix $\langle \xi \nabla_{\mathbf{x}} \mathbf{A} : \nabla_{\mathbf{x}} \mathbf{A} \rangle_{u_{\alpha^*(t)}}$ may have infinite entries even for some admissible $\alpha^*(t)$, and if this is the case, then again the ODE system is not solvable. Since these two issues are very hard to resolve in general, we shall below resort to a simple case where α is a scalar.

5.3. Scalar case. To gain some more insight into the ODE (5.2), we consider the single locus case $x \in (0, 1)$ with $\mathbf{A}$ being a scalar function $A = A(x)$ and $\alpha \in \mathbb{R}$. The DynMaxEnt method (5.2) simplifies to the following ODE for $\alpha^* = \alpha^*(t)$,

$$\left(\langle A^2 \rangle_{u_{\alpha^*(t)}} - \langle A \rangle_{u_{\alpha^*(t)}}^2 \right) \frac{d\alpha^*(t)}{dt} = \frac{1}{2} \langle \xi(\partial_x A)^2 \rangle_{u_{\alpha^*(t)}} (\alpha - \alpha^*(t)). \quad (5.3)$$

An application of the Cauchy-Schwartz inequality implies that

$$\langle A^2 \rangle_{u_{\alpha^*(t)}} - \langle A \rangle_{u_{\alpha^*(t)}}^2 \geq 0,$$

and, moreover, equality holds if and only if A is a constant function. Consequently, for every nonconstant A the ODE (5.3) can be rewritten as

$$\frac{d\alpha^*(t)}{dt} = \frac{1}{2} \left(\langle A^2 \rangle_{u_{\alpha^*(t)}} - \langle A \rangle_{u_{\alpha^*(t)}}^2 \right)^{-1} \langle \xi(\partial_x A)^2 \rangle_{u_{\alpha^*(t)}} (\alpha - \alpha^*(t)). \quad (5.4)$$

However, the question of finiteness of the observable $\langle \xi(\partial_x A)^2 \rangle_{u_{\alpha^*(t)}}$ can be only answered by making a particular choice for $A = A(x)$.

As a toy model, let us choose $A = A(x)$ to be the scalar function $\ln(\xi(x))$. This corresponds to a population of individuals in a neutral environment ($\beta = h = 0$ in (1.2)) with the nonzero mutation rate $\alpha = 4N\mu$. With the singularities at $x \in \{0, 1\}$, the function $A(x) = \ln(\xi(x))$ well represents the issues that one encounters with the generic choice (1.2). The stationary solution u_{α^*} is then of the form

$$u_{\alpha^*} = \mathbb{Z}_{\alpha^*}^{-1} \xi^{\alpha^*-1}, \quad \mathbb{Z}_{\alpha^*} := \int_0^1 \xi^{\alpha^*-1} dx,$$

and the set of admissible values of α^* is the interval $(0, \infty)$. However, the observable $\langle \xi(\partial_x A)^2 \rangle_{u_{\alpha^*}}$ is only finite for $\alpha^* > 1$. Consequently, the DynMaxEnt method (5.4) is only applicable if both the initial value $\alpha^*(0) = \alpha^0$ and α are strictly larger than 1. Then, since obviously the solution $\alpha^*(t)$ of (5.4) is a monotone function of time, it will stay strictly larger than 1 for all $t \geq 0$ and asymptotically converge to α .

The issue of non-finiteness of the term $\langle \xi(\partial_x A)^2 \rangle_{u_{\alpha^*(t)}}$ was addressed in [6] by introducing a special treatment near the boundary (see Appendix E, equations E.10-E.13 of [6] for details of the derivation of the modified method). Here we propose an alternative way that treats the problem at least in the case $A(x) := \ln(\xi(x))$. It is based on the idea of multiplying the Fokker-Planck equation by a suitable function $B = B(x)$, instead of $A = A(x)$, and integrating on Ω_x . In the second step, one again approximates $u(t)$ by $u_{\alpha^*(t)}$ and neglects the residuum. This leads, in the scalar case, to the ODE

$$\left(\langle AB \rangle_{u_{\alpha^*(t)}} - \langle A \rangle_{u_{\alpha^*(t)}} \langle B \rangle_{u_{\alpha^*(t)}} \right) \frac{d\alpha^*(t)}{dt} = \frac{1}{2} \langle \xi(\partial_x B)^2 \rangle_{u_{\alpha^*(t)}} (\alpha - \alpha^*(t)) \quad (5.5)$$

Choosing $B(x) := \xi(x)$ leads then to finite $\langle \xi(\partial_x B)^2 \rangle_{u_{\alpha^*}}$ for all $\alpha^* > 0$, i.e., for all admissible values of α^* . Thus, our strategy is to obtain $\alpha^*(t)$ by

solving (5.5) for $t \geq 0$ and then calculate the observable $\langle \ln(\xi) \rangle_{u_{\alpha^*(t)}}$, which is expected to be a good approximation of the true observable $\langle \ln(\xi) \rangle_{u(t)}$. Clearly, one can use this strategy to obtain an approximation of any other observable of $u(t)$.

However, the method (5.5) suffers from a drawback, namely, it is only solvable if the covariance

$$\langle AB \rangle_{u_{\alpha^*(t)}} - \langle A \rangle_{u_{\alpha^*(t)}} \langle B \rangle_{u_{\alpha^*(t)}} \quad (5.6)$$

is nonvanishing for all $t \geq 0$, which in general may not be the case. Nonetheless, for the particular choice $A(x) = \ln(\xi(x))$ and $B(x) = \xi(x)$ we have:

Lemma 5. *The expression*

$$\langle \xi \ln(\xi) \rangle_{u_\alpha} - \langle \xi \rangle_{u_\alpha} \langle \ln(\xi) \rangle_{u_\alpha} \quad (5.7)$$

is strictly positive for all $\alpha > 0$.

Proof: For $\alpha > 0$ let us define the one-parameter family of probability measures on $(0, 1)$ with densities

$$\nu_\alpha(x) := \frac{\xi(x)^{\alpha-1}}{\int_0^1 \xi(s)^{\alpha-1} ds}, \quad (5.8)$$

where as before $\xi(x) = x(1-x)$. Moreover, let us define the function

$$G(\alpha) := \int_0^1 \ln(\xi(x)) d\nu_\alpha(x).$$

A simple calculation reveals that the claim is equivalent to $G(\alpha+1) > G(\alpha)$ for all $\alpha > 0$. We have

$$G'(\alpha) = \int_0^1 \ln(\xi(x))^2 d\nu_\alpha(x) - \left(\int_0^1 \ln(\xi(x)) d\nu_\alpha(x) \right)^2,$$

and thus $G'(\alpha) > 0$ by Jensen's inequality. ■

Consequently, the "modified" DynMaxEnt method (5.5) is safely applicable with $A(x) = \ln(\xi(x))$ and $B(x) = \xi(x)$. It even seems to provide better approximation results than the "original" method (5.3), as is documented by our numerical experiments in Section 6.

The plot of the expression (5.7) is presented in Fig. 1. For completeness, let us demonstrate a nice analytical proof that the limits of (5.7) vanish both as $\alpha \rightarrow 0$ and $\alpha \rightarrow +\infty$. The proof is based on the following Lemma.

Lemma 6. *For ν_α defined in (5.8) we have, in the sense of distributions,*

$$\begin{aligned} \nu_\alpha &\rightarrow \delta(\cdot - 1/2) & \text{as } \alpha \rightarrow \infty, \\ \nu_\alpha &\rightarrow \frac{1}{2}\delta(\cdot) + \frac{1}{2}\delta(\cdot - 1) & \text{as } \alpha \rightarrow 0, \end{aligned}$$

where $\delta(\cdot - x)$ denotes the Dirac-delta distribution concentrated at x .

FIGURE 1. Numerical calculation of the expression $\langle \xi \ln(\xi) \rangle_{u_\alpha} - \langle \xi \rangle_{u_\alpha} \langle \ln(\xi) \rangle_{u_\alpha}$ for $\alpha \in [0.01, 200]$. Compare with the analytical result of Lemma 5 and (5.9).

Proof: Let $\varphi \in C_c^\infty(0, 1)$ be any test function on the interval $(0, 1)$. We shall show that

$$\lim_{\alpha \rightarrow \infty} \int_0^1 \varphi(x) d\nu_\alpha(x) = \varphi(1/2).$$

The mean-value theorem gives

$$\left| \varphi(1/2) - \int_0^1 \varphi(x) d\nu_\alpha(x) \right| \leq \int_0^1 |\varphi'(\eta(x))| |x - 1/2| d\nu_\alpha(x) \leq C_\varphi \int_0^1 |x - 1/2| d\nu_\alpha(x).$$

Thus, our goal is to show that $\int_0^1 |x - 1/2| d\nu_\alpha(x)$ vanishes as $\alpha \rightarrow \infty$. For the numerator, we have

$$\int_0^1 (4\xi(x))^{\alpha-1} |x - 1/2| dx = 2 \int_0^{1/2} (4\xi(x))^{\alpha-1} (1/2 - x) dx,$$

and using the identity $1/2 - x = \xi'(x)/2$, we calculate

$$\int_0^1 (4\xi(x))^{\alpha-1} |x - 1/2| dx = \frac{1}{\alpha}.$$

The denominator is estimated from below using the elementary inequalities

$$\begin{aligned} 4\xi(x) &\geq 3x && \text{for } x \in [0, 1/4], \\ &\geq x + 1/2 && \text{for } x \in [1/4, 1/2], \end{aligned}$$

which give

$$\int_0^1 (4\xi(x))^{\alpha-1} dx \geq \frac{2}{\alpha} \left[\left(\frac{3}{2}\right)^\alpha - \frac{2}{3} \left(\frac{3}{4}\right)^\alpha \right].$$

Thus,

$$\int_0^1 |x - 1/2| d\nu_\alpha(x) \leq \frac{1}{2} \left[\left(\frac{3}{2}\right)^\alpha - \frac{2}{3} \left(\frac{3}{4}\right)^\alpha \right]^{-1} \rightarrow 0 \quad \text{as } \alpha \rightarrow \infty,$$

which proves the first claim.

To calculate the limit $\alpha \rightarrow 0$, due to the symmetry of $\xi(x) = x(1-x)$ with respect to $x = 1/2$, it is sufficient to prove that

$$\frac{\xi(x)^{\alpha-1}}{\int_0^{1/2} \xi(s)^{\alpha-1} ds} \rightarrow \delta(\cdot) \quad \text{as } \alpha \rightarrow 0.$$

Again, picking a test function $\varphi \in C_c^\infty[0, 1/2)$ and using the mean-value theorem, we have to show that

$$\left| \varphi(0) - \frac{\int_0^{1/2} \varphi(x) \xi(x)^{\alpha-1} dx}{\int_0^{1/2} \xi(s)^{\alpha-1} ds} \right| \leq C_\varphi \frac{\int_0^{1/2} |\varphi'(\eta(x))| x \xi(x)^{\alpha-1} dx}{\int_0^{1/2} \xi(s)^{\alpha-1} ds}$$

tends to zero as $\alpha \rightarrow 0$. However, this follows directly from the fact that the numerator is uniformly bounded for, say, $0 \leq \alpha < 1$, and that, obviously, the denominator tends to $+\infty$ as $\alpha \rightarrow 0$. ■

Then, the fact that

$$\langle \xi \ln(\xi) \rangle_{u_\alpha} - \langle \xi \rangle_{u_\alpha} \langle \ln(\xi) \rangle_{u_\alpha} \rightarrow 0 \quad (5.9)$$

follows directly from Lemma 6 since for $A(x) = \ln(\xi(x))$ we readily have $u_\alpha = \nu_\alpha$ with ν_α given by (5.8).

5.3.1. Solvability of the observables-equation. Finally, for a given $u \geq 0$ with $\int_0^1 u(x) dx = 1$, we study the solvability with respect to $\alpha > 0$ of the observables-equation

$$\langle A \rangle_{u_\alpha} = \langle A \rangle_u \quad (5.10)$$

with $A(x) = \ln(\xi(x))$, assuming that the right-hand side is finite. First of all, we note that the mapping $\alpha \mapsto \langle A \rangle_{u_\alpha}$ is strictly increasing for $\alpha > 0$. Indeed,

$$\frac{d}{d\alpha} \langle A \rangle_{u_\alpha} = \langle A^2 \rangle_{u_\alpha} - \langle A \rangle_{u_\alpha}^2 > 0,$$

where the strict positivity follows as before by the Cauchy-Schwartz inequality. Consequently, if a solution to the observables-equation (5.10) exists, it is unique. Next we claim that for any $u \geq 0$ with $\int_0^1 u(x) dx = 1$ we have $\langle \ln(\xi) \rangle_u \in [-\infty, \ln(1/4))$. Indeed, since $\xi(x) < 1/4$ on $(0, 1) \setminus \{1/2\}$,

$$\langle \ln(\xi) \rangle_u < \ln\left(\frac{1}{4}\right) \int_0^1 u(x) dx = \ln\left(\frac{1}{4}\right).$$

Thus, it remains to prove that the range of $\alpha \mapsto \langle A \rangle_{u_\alpha}$ is the interval $(-\infty, \ln(1/4))$. Since for $A(x) = \ln(\xi(x))$ we readily have $u_\alpha = \nu_\alpha$ with ν_α given by (5.8), Lemma 6 gives

$$\begin{aligned} \langle \ln(\xi) \rangle_{u_\alpha} &= \int_0^1 \ln(\xi(x)) d\nu_\alpha(x) \rightarrow \ln(1/4) \quad \text{as } \alpha \rightarrow +\infty, \\ &\rightarrow -\infty \quad \text{as } \alpha \rightarrow 0+. \end{aligned}$$

Indeed, the range of the mapping $\alpha \mapsto \langle \ln(\xi) \rangle_{u_\alpha}$ is the interval $(-\infty, \ln(1/4))$ and, therefore, the observables-equation (5.10) is uniquely solvable for every normalized $u \in L^1(0, 1)$ with finite $\langle \ln(\xi) \rangle_u$ -observable.

6. NUMERICAL EXPERIMENTS

6.1. Scalar case. We present results of several numerical experiments that aim to demonstrate the performance of the original (5.3) and modified (5.5) DynMaxEnt methods for the scalar (single locus) case $A(x) = \ln(\xi(x))$, as discussed in Section 5.3. Let us recall that this case corresponds to a population of individuals in a neutral environment ($\beta = h = 0$ in (1.2)) with the nonzero mutation rate $\alpha = 4N\mu$. For the modified method (5.5) we again choose $B(x) = \xi(x)$.

In all simulations we set $N = 1$ and start from the initial condition $\alpha^*(t = 0) = \alpha^0 := 2$ for the ODEs (5.3), (5.5), and the initial datum $u(t = 0) = u_{\alpha^0}$ for the Fokker-Planck equation (3.1). The ODEs (5.3), (5.5) are solved with simple forward Euler discretization on the time interval $[0, T]$ for different values of $T > 0$. We use $B(x) = \xi(x)$ for the modified DynMaxEnt method (5.5). The Fokker-Planck equation is discretized in space using the Chang-Cooper scheme [9] and forward Euler in time.

In Fig. 2 we plot the time evolution of the $\langle \ln(\xi) \rangle$ -observable of the Fokker-Planck solution $u(t)$ and its approximation obtained by the DynMaxEnt methods (5.3), (5.5) for the parameter values $\alpha \in \{1.1, 1.5, 2.5, 3\}$. Note that since $\alpha > 1$, both the methods (5.3), (5.5) are applicable. However, we observe that the modified method (5.5) gives better approximation results. To quantify the approximation error, we calculate the indicator

$$e := \frac{\int_0^T \left(\langle \ln(\xi) \rangle_{u(t)} - \langle \ln(\xi) \rangle_{u_{\alpha^*(t)}} \right)^2 dt}{\int_0^T \langle \ln(\xi) \rangle_{u(t)}^2 dt}, \quad (6.1)$$

where $\langle \ln(\xi) \rangle_{u_{\alpha^*(t)}}$ is the observable calculated by one of the DynMaxEnt methods (5.3), (5.5). The results for the values $\alpha \in \{1.1, 1.5, 2.5, 3\}$ given in Table 1 indeed suggest that the modified method (5.5) provides better approximation of the observable $\langle \ln(\xi) \rangle_{u(t)}$. Moreover, we observe that with increasing value of α the approximation properties of both methods seem to improve.

In Fig. 3 we plot the time evolution of the $\langle \ln(\xi) \rangle$ -observable of the Fokker-Planck solution $u(t)$ and its approximation obtained by the modified

FIGURE 2. Comparison of approximations of the observable $\langle \ln(\xi) \rangle$ obtained by the original (5.3), red dotted curves, and modified (5.5), green dashed curves, DynMaxEnt methods. The observable calculated from the solution of the Fokker-Planck equation is plotted with solid blue curves. Note the different vertical and horizontal axes scales for various values of α . The corresponding relative approximation errors are given in Table 1.

	$\alpha = 1.1$	$\alpha = 1.5$	$\alpha = 2.5$	$\alpha = 3.0$
method (5.3)	1.45×10^{-2}	4.09×10^{-3}	1.42×10^{-3}	1.37×10^{-3}
method (5.5)	3.78×10^{-3}	1.30×10^{-3}	3.65×10^{-4}	3.41×10^{-4}

TABLE 1. Relative errors of approximation (6.1) of the observable $\langle \ln(\xi) \rangle_{u(t)}$ by the original DynMaxEnt method (5.3), first row, and its modified version (5.5), second row. The corresponding plots are given in Fig. 2.

DynMaxEnt method (5.5) for the parameter values $\alpha \in \{0.7, 0.5, 0.3, 0.2\}$. Note that the original method (5.3) is no longer applicable since for $\alpha^* < 1$ the term $\langle \xi(\partial_x \ln(\xi))^2 \rangle_{u_{\alpha^*}}$ is not finite. Again, we calculate the approximation error (6.1) for the above mentioned values of α in Table 2. We observe that the approximation worsens for smaller values of α . This is presumably a consequence of the singularity of u_α at $x \in \{0, 1\}$ becoming stronger when

FIGURE 3. Approximations of the observable $\langle \ln(\xi) \rangle$ obtained by the modified (5.5) DynMaxEnt method, green dashed curves, for values of $\alpha < 1$. The observable calculated from the solution of the Fokker-Planck equation is plotted with solid blue curves. Note the different vertical and horizontal axes scales for various values of α . The corresponding relative approximation errors are given in Table 2.

	$\alpha = 0.7$	$\alpha = 0.5$	$\alpha = 0.3$	$\alpha = 0.2$
method (5.5)	1.42×10^{-2}	2.81×10^{-2}	5.79×10^{-2}	8.88×10^{-2}

TABLE 2. Relative errors of approximation (6.1) of the observable $\langle \ln(\xi) \rangle_{u(t)}$ by the modified DynMaxEnt method (5.5). The corresponding plots are given in Fig. 3.

α approaches zero. In fact, numerical solution of the Fokker-Planck equation (3.1) also becomes more difficult for small values of α . For $\alpha < 0.2$ our discrete scheme ceases to provide reliable results. That is why $\alpha = 0.2$ is the smallest value that we take into account.

6.2. Vector case. Finally, we consider the more general case with the function $\mathbf{A} = \mathbf{A}(\mathbf{x})$ being vector-valued, $\mathbf{A} : \Omega_{\mathbf{x}} \rightarrow \mathbb{R}^k$ with some $k \in \mathbb{N}$. It has been observed in [10, 6] that using more observables (i.e., higher k) in general

improves the approximation properties of the DynMaxEnt method. Inspired by the success of the modified DynMaxEnt method (5.5) demonstrated in Section 6.1, we consider an analogous approach also in the vector case. For this, we employ the idea of deriving a modified DynMaxEnt method as in Section 5.3: We multiply the Fokker-Planck equation (3.1) by a vector-valued function $\mathbf{B} : \Omega_{\mathbf{x}} \rightarrow \mathbb{R}^k$ to be chosen later and integrate by parts, assuming the boundary terms to vanish. Then, we approximate $u(t)$ by $u_{\alpha^*(t)}$ with the time-dependent vector $\alpha^* = \alpha^*(t)$ and neglect the residual term. This gives

$$\begin{aligned} & \left(\langle \mathbf{B} \otimes \mathbf{A} \rangle_{u_{\alpha^*(t)}} - \langle \mathbf{B} \rangle_{u_{\alpha^*(t)}} \otimes \langle \mathbf{A} \rangle_{u_{\alpha^*(t)}} \right) \frac{d\alpha^*(t)}{dt} \\ &= \frac{1}{2} \langle \xi \nabla_{\mathbf{x}} \mathbf{B} : \nabla_{\mathbf{x}} \mathbf{A} \rangle_{u_{\alpha^*(t)}} (\alpha - \alpha^*(t)), \end{aligned} \quad (6.2)$$

where $\mathbf{B} \otimes \mathbf{A}$ is the $d \times d$ matrix with the (i, k) -component $B_i A_k$ and $\nabla_{\mathbf{x}} \mathbf{B} : \nabla_{\mathbf{x}} \mathbf{A}$ is the $d \times d$ matrix with the (i, k) -component $\sum_{j=1}^d \partial_{x_j} B_i \partial_{x_j} A_k$. Clearly, uniform invertibility of the matrix

$$\left(\langle \mathbf{B} \otimes \mathbf{A} \rangle_{u_{\alpha^*(t)}} - \langle \mathbf{B} \rangle_{u_{\alpha^*(t)}} \otimes \langle \mathbf{A} \rangle_{u_{\alpha^*(t)}} \right)$$

is necessary for global solvability of the ODE system (6.2). This condition is satisfied in our numerical experiments below.

The goal of this Section is to illustrate the performance of the original (5.2) and modified (6.2) DynMaxEnt methods for the generic choice $\mathbf{A} = (\xi', \xi, \ln \xi)$. For simplicity, we shall still stick to the 1D (single locus) setting $x \in (0, 1)$. Choosing again $\beta = h = 1$ in (1.2), we have

$$\alpha \cdot \mathbf{A} = -\gamma \xi' + 2\eta \xi + 2\mu \ln \xi,$$

with $\alpha = (-\gamma, 2\eta, 2\mu)$. The parameters $\gamma, \eta \in \mathbb{R}$ represent the effects of loci on the traits and $\mu > 0$ is the mutation rate. For the modified DynMaxEnt method (6.2) we choose $\mathbf{B} = (\xi', \xi, \xi^2)$. Note that this choice prevents the issue of non-finiteness of the observable $\langle \xi \nabla_{\mathbf{x}} \mathbf{A} \otimes \nabla_{\mathbf{x}} \mathbf{A} \rangle_{u_{\alpha^*(t)}}$ for $4N\mu < 1$.

We carry out two numerical experiments. In both simulations we set $N = 1$ and the initial condition for the Fokker-Planck equation (3.1) to be the stationary distribution (3.2) with the parameters $4\mu_0 = 2$, $\eta_0 = -1$, $\gamma_0 = 2$. As before, the Fokker-Planck equation is discretized in space using the Chang-Cooper scheme [9] and forward Euler method in time. The ODE systems (5.2), (6.2) are discretized in time using the forward Euler scheme.

For the first experiment we use the parameter values $4\mu = 1.1$, $\eta = 1$, $\gamma = 0$. This corresponds to the abrupt change of parameters (evolutionary forces)

$$4\mu : 2 \mapsto 1.1, \quad \eta : -1 \mapsto 1, \quad \gamma : 2 \mapsto 0.$$

In Fig. 4 we plot the time evolution of the observables $\langle \ln(\xi) \rangle$, $\langle \xi \rangle$ and $\langle \xi' \rangle$ of the Fokker-Planck solution $u(t)$ and its approximation obtained by the original and, resp., modified DynMaxEnt methods (5.2), resp., (6.2).

FIGURE 4. Comparison of approximations of the observables $\langle \ln(\xi) \rangle$ (upper left panel), $\langle \xi \rangle$ (upper right panel) and $\langle \xi' \rangle$ (lower left panel), obtained by the original (5.2), red dotted curves, and modified (6.2), green dashed curves, DynMaxEnt methods. The observables calculated from the solution of the Fokker-Planck equation (3.1) is plotted with solid blue curves. The lower right panel shows the evolution of the parameters μ (solid blue), η (red dotted) and γ (green dashed) calculated by the original DynMaxEnt method (5.2). The initial and target parameters are $4\mu_0 = 2$, $\eta_0 = -1$, $\gamma_0 = 2$, $4\mu = 1.1$, $\eta = 1$, $\gamma = 0$. The corresponding relative approximation errors are given in Table 3.

Note that in this case both methods (5.2), (6.2) are applicable since the observable $\langle \xi \nabla_{\mathbf{x}} \mathbf{A} \otimes \nabla_{\mathbf{x}} \mathbf{A} \rangle_{u_{\alpha^*(t)}}$ is finite for all $t \geq 0$. Calculating the error of approximation (6.1) for the three observables, Table 3, we observe that the modified method (6.2) provides slightly more accurate results.

For our second experiment we use the parameter values $4\mu = 0.5$, $\eta = 1$, $\gamma = 0$. This corresponds to the rapid change of evolutionary forces

$$4\mu : 2 \mapsto 0.5, \quad \eta : -1 \mapsto 1, \quad \gamma : 2 \mapsto 0.$$

Note that in this case the original method (5.2) is not applicable any more since the observable $\langle \xi \nabla_{\mathbf{x}} \mathbf{A} \otimes \nabla_{\mathbf{x}} \mathbf{A} \rangle_{u_{\alpha^*(t)}}$ is not defined for $4\mu < 1$. In Fig. 5 we plot the time evolution of the observables $\langle \ln(\xi) \rangle$, $\langle \xi \rangle$ and $\langle \xi' \rangle$ of the

	$\langle \ln(\xi) \rangle$	$\langle \xi \rangle$	$\langle \xi' \rangle$
method (5.2)	9.24×10^{-3}	1.30×10^{-2}	5.03×10^{-2}
method (6.2)	6.79×10^{-3}	1.01×10^{-2}	4.14×10^{-2}

TABLE 3. Relative errors of approximation (6.1) of the observables $\langle \ln(\xi) \rangle$, $\langle \xi \rangle$ and $\langle \xi' \rangle$ by the original DynMaxEnt method (5.2), first row, and its modified version (6.2), second row. The initial and target parameters are $4\mu_0 = 2$, $\eta_0 = -1$, $\gamma_0 = 2$, $4\mu = 1.1$, $\eta = 1$, $\gamma = 0$. The corresponding plots are given in Fig. 4.

	$\langle \ln(\xi) \rangle$	$\langle \xi \rangle$	$\langle \xi' \rangle$
method (6.2)	2.45×10^{-2}	2.55×10^{-2}	1.24×10^{-1}

TABLE 4. Relative errors of approximation (6.1) of the observables $\langle \ln(\xi) \rangle$, $\langle \xi \rangle$ and $\langle \xi' \rangle$ by the modified DynMaxEnt method (6.2). The initial and target parameters are $4\mu_0 = 2$, $\eta_0 = -1$, $\gamma_0 = 2$, $4\mu = 0.5$, $\eta = 1$, $\gamma = 0$. The corresponding plots are given in Fig. 5.

Fokker-Planck solution $u(t)$ and its approximation obtained by the modified DynMaxEnt method (6.2). On the other hand, the results presented in Fig. 5 and Table 4 indicate that the modified DynMaxEnt method (6.2) provides a reasonably good approximation of the three observables.

Here we compared the solutions of the Fokker-Planck equation (1.1) with the DynMaxEnt approximation. In most of the typical biological situations mutation rate is small $4N\mu < 1$, leading to a U-shaped equilibrium allele frequency distribution. The singularities at the boundaries correspond to the fact that most of the alleles in the population are fixed, leading to rare polymorphic loci. These loci, however, contribute greatly to the changes in the trait mean and other important observables. Because the simple version of DynMaxEnt cannot be used in the regime $4N\mu < 1$, we tested an alternative, yet still simple approach: multiplying the Fokker-Planck equation by a function $\xi(x)$ that is not singular at $x = 0, 1$. The modified DynMaxEnt method does not only allow us to use the approximation for the biologically realistic mutation rates $4N\mu < 1$ but also leads to an overall increase in accuracy of the method in the regime $4N\mu > 1$, thus allowing us to obtain a good quantitative match for the trait mean dynamics, as well as for the dynamics of expected heterozygosity, which captures the amount of polymorphism in the population in time (Figure 5).

Acknowledgments. We thank Nicholas Barton (IST Austria) for his useful

FIGURE 5. Approximations of the observables $\langle \ln(\xi) \rangle$ (upper left panel), $\langle \xi \rangle$ (upper right panel) and $\langle \xi' \rangle$ (lower left panel), obtained by the modified (6.2) DynMaxEnt method (green dashed curves). The observables calculated from the solution of the Fokker-Planck equation (3.1) is plotted with solid blue curves. The lower right panel shows the evolution of the parameters μ (solid blue), η (red dotted) and γ (green dashed). The initial and target parameters are $4\mu_0 = 2$, $\eta_0 = -1$, $\gamma_0 = 2$, $4\mu = 0.5$, $\eta = 1$, $\gamma = 0$. The corresponding relative approximation errors are given in Table 4.

comments and suggestions. JH and PM are funded by KAUST baseline funds and grant no. 1000000193.

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- We study a Fokker-Planck equation with degenerate diffusion coefficients.
- The equation describes the dynamics of quantitative traits.
- We construct mass and positivity preserving weak solutions of the equation.
- Existence of a positive spectral gap implies exponential convergence to equilibrium.
- We study analytically and numerically the „Dynamic Maximum Entropy“ approximation.