Algorithmic Verification of Linearizability for Ordinary Differential Equations

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ABSTRACT

For a nonlinear ordinary differential equation solved with respect to the highest order derivative and rational in the other derivatives and in the independent variable, we devise two algorithms to check if the equation can be reduced to a linear one by a point transformation of the dependent and independent variables. The first algorithm is based on a construction of the Lie point symmetry algebra and on the computation of its derived algebra. The second algorithm exploits the differential Thomas decomposition and allows not only to test the linearizability, but also to generate a system of nonlinear partial differential equations that determines the point transformation and the coefficients of the linearized equation. The implementation of both algorithms is discussed and their application is illustrated using several examples.

CCS CONCEPTS

• **Computing methodologies** → **Algebraic algorithms**; • **Mathematics of computing** → *Ordinary differential equations*; *Nonlinear equations*;

KEYWORDS

Algorithmic linearization test, determining equations, differential Thomas decomposition, Lie symmetry algebra, ordinary differential equations, point transformation, power series solutions

1 INTRODUCTION

Solving nonlinear ordinary differential equations (ODEs) is one of the classical and practically important research areas in applied mathematics. In practice, such equations are mostly solved numerically or by approximate analytical methods since obtaining their explicit solution is usually very difficult or even impossible. One of the important approaches for solving a nonlinear ODE explicitly considers the existence of an invertible linearizing transformation of the variables and its construction. The reduction of a nonlinear ODE to a linear one makes its explicit integration much easier and often allows for obtaining an exact solution.

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The linearization problem for a second-order ODE

$$y'' + f(x, y, y') = 0 \quad (1)$$

was solved by Lie ([18], Sect. 1), who applied his general theory of integration of ODEs by means of a group of point transformations. He proved that f is at most cubic in y' for a linearizable equation and derived the necessary and sufficient conditions of linearizability. These conditions have the form of two explicit and easily verifiable equalities (21) containing differential polynomials in the coefficients of f as a polynomial in y' :

$$f = F_3(x, y)(y')^3 + F_2(x, y)(y')^2 + F_1(x, y)y' + F_0(x, y). \quad (2)$$

Lie's ideas and methods were extended and applied to third-order equations $y''' = f(x, y, y', y'')$ [12] and later to fourth-order equations $y'''' = f(x, y, y', y'', y''')$ [13]. In these contributions, all possible structures of the candidates for the linearization were found and the explicit form of necessary and sufficient linearizability conditions of the coefficients of those structures were derived. Therefore, given an ODE of second or third order, to check whether it is linearizable by a point transformation or not, it is sufficient to verify whether the relevant explicit linearizability conditions are satisfied or not.

In practice, such a verification typically needs a computer-based symbolic algebraic computation for the simplification of the resulting expressions. An additional point to emphasize is that if the ODE contains parameters and/or arbitrary functions, then the linearizability conditions imply the algebraic and/or differential constraints on these parameters and/or functions that provide the linearization. Generally, however, these constraints may include the point transformation functions, and it may be highly conjectural to solve the constraints and to find a linearizing point transformation.

In the present paper we suggest two algorithmic linearization tests applicable to a quasi-linear ODE (solved for the highest derivative) of any order greater or equal to two with a rational dependence on the other derivatives and the independent variable. The first linearization test is applicable to ODEs which do not contain parameters and arbitrary functions. This test is based on the construction of the Lie point symmetry algebra for the input ODE. The relevant mathematical methods are described in several textbooks (see, for example, [5, 14, 21–23, 27]). To detect linearizability we compute the maximal abelian dimension of the Lie symmetry algebra and make use of the results of Mahomed and Leach [20]. Unlike the first test, our second test exploits the differential Thomas decomposition ([3, 10, 17, 26, 31, 32, 34]), an universal algorithmic tool for the algebraic analysis of polynomially-nonlinear systems of partial differential equations (PDEs), and allows not only for the detection of

linearizability but also for the derivation of necessary and sufficient linearizability conditions for arbitrary functions or parameters occurring in these equations. An example of such a problem is given by Eq. (1) whose linearizability conditions are given by Eq. (2). Therefore, the second test can reproduce the above mentioned results of [12, 13, 18]. Besides, via the second linearization test one can generate differential equations for a linearizing point transformation and for the coefficients of the linearized equation that are suitable for finding the transformation and the coefficients. However, the first linearization test is computationally more efficient and it is therefore advisable to apply it first when considering higher-order equations, and then, in the case of linearizability, apply the second test in order to construct the linearizing point transformation and the reduced linear form of the ODE.

This paper is organized as follows. In Sect. 2 we briefly describe the mathematical objects we deal with before presenting our algorithms in Sect. 3. The implementation of these algorithms in Maple is then described in Sect. 4 and its application is illustrated in Sect. 5 using several examples. Finally, we provide a conclusion in Sect. 6.

2 UNDERLYING EQUATIONS

In this paper we consider ODEs of the form

$$y^{(n)} + f(x, y, y', \dots, y^{(n-1)}) = 0, \quad y^{(k)} := \frac{d^k y}{dx^k} \quad (3)$$

with $f \in \mathbb{C}(x, y, y', \dots, y^{(n-1)})^1$ solved with respect to the highest order derivative. As additional arguments, the function f may also include parameters and/or arbitrary functions in x and/or y . Given an ODE of the form (3), our aim is to check the existence of an invertible transformation²

$$u = \phi(x, y), \quad t = \psi(x, y) \quad (4)$$

which maps (3) into a linear n -th order homogeneous equation

$$u^{(n)}(t) + \sum_{k=0}^{n-1} a_k(t) u^{(k)}(t) = 0, \quad u^{(k)} := \frac{d^k u}{dt^k}. \quad (5)$$

The invertibility of (4) is provided by the inequation

$$J := \phi_x \psi_y - \phi_y \psi_x \neq 0. \quad (6)$$

If such a transformation exists for $n \geq 3$, then it can always be chosen (cf. [22], Thm. 6.54; [14], Thm. 6.6.3) in a way that (5) takes the Laguerre-Forsyth normal form

$$u^{(n)}(t) + \sum_{k=0}^{n-3} a_k(t) u^{(k)}(t) = 0. \quad (7)$$

A first-order ODE $y' = f(x, y)$ is always linearizable, but its linearization procedure is as hard as the integration of the equation (cf. [1], Ch. 2, Thm. 1). For $n = 2$ any homogeneous linear equation

$$y''(x) + a(x)y'(x) + b(x)y(x) = 0$$

can be transformed by a substitution

$$t = \varphi(x), \quad \varphi'(x) \neq 0, \quad u = \sigma(x)y, \quad \sigma(x) \neq 0$$

¹In the subsequent, everywhere where it is necessary from the computational point of view, the field $\mathbb{Q}$ is assumed to be considered instead of the field $\mathbb{C}$.

²Hereafter, all functions we deal with are assumed to be smooth.

to the simplest second order equation ([14], Thm. 3.3.1)

$$u''(t) = 0. \quad (8)$$

One way to check the linearizability of Eq. (3) is to follow the classical approach by Lie [18] to study the symmetry properties of Eq. (3) under the *infinitesimal* transformation

$$\tilde{x} = x + \varepsilon \xi(x, y) + O(\varepsilon^2), \quad \tilde{y} = y + \varepsilon \eta(x, y) + O(\varepsilon^2). \quad (9)$$

The *invariance condition* for Eq. (3) under the transformation (9) is given by the equality

$$\mathcal{X}(y^{(n)} + f(x, y, \dots, y^{(n-1)}))|_{y^{(n)}+f(x, y, \dots, y^{(n-1)})=0} = 0, \quad (10)$$

where the *symmetry operator* reads

$$\mathcal{X} := \xi \partial_x + \sum_{k=0}^n \eta^{(k)} \partial_{y^{(k)}}, \quad \eta^{(k)} := D_x \eta^{(k-1)} - y^{(k)} D_x \xi, \quad (11)$$

$\eta^{(0)} := \eta$ and $D_x := \partial_x + \sum_{k \geq 0} y^{(k+1)} \partial_{y^{(k)}}$ is the total derivative operator with respect to x .

The invariance condition (10) means that its left-hand side vanishes when Eq. (3) holds. Then the application of (11) to the left-hand side of Eq. (3) and the substitution of

$$y^{(n)} = -f(x, y, \dots, y^{(n-1)})$$

in the resulting expression leads to the equality $g = 0$ with the polynomial dependence of g on the derivatives $y', \dots, y^{(n-1)}$. Since, by Def. (9), the functions ξ and η do not depend on these derivatives, the equality $g = 0$ holds if and only if all coefficients in $y', \dots, y^{(n-1)}$ are equal to zero. This leads to an overdetermined system of linear PDEs in ξ and η called *determining system*. Its solution yields a set of symmetry operators whose cardinality we denote by m . This set forms a basis of the m -dimensional *Lie symmetry algebra*

$$[\mathcal{X}_i, \mathcal{X}_j] = \sum_{k=1}^m C_{i,j}^k \mathcal{X}_k, \quad 1 \leq i < j \leq m. \quad (12)$$

We denote the Lie symmetry algebra by L and $m = \dim(L)$. Its *derived algebra* $L' \subset L$ is a subalgebra that consists of all commutators of pairs of elements in L .

Lie showed ([19], Ch. 12, p. 298, "Satz" 3) that the Lie point symmetry algebra of an n -order ODE has a dimension m satisfying

$$n = 1, m = \infty; \quad n = 2, m \leq 8; \quad n \geq 3, m \leq n + 4.$$

Later, the interrelations between n and m were established that provide the linearizability of (3) by a point transformation (4) in the absence of parameters and arbitrary functions. Here we present the two theorems that describe such interrelations and form the basis of our first linearization test.

THEOREM 2.1. ([20], Thm. 1) *A necessary and sufficient condition for the linearization of (3) with $n \geq 3$ via a point transformation is the existence of an abelian n -dimensional subalgebra of (12).*

THEOREM 2.2. ([20], Sect. 2-4 and Thms. 6.8; [27], Thm. 5.19)³ *Eq. (3) with $n \geq 2$ is linearizable by a point transformation if and only if one of the following conditions is fulfilled:*

- (1) $n = 2, m = 8$;
- (2) $n \geq 3, m = n + 4$;

³Cf. Thm. 6.39 in [22] regarding $n = 2$.

- (3) $n \geq 3$, $m \in \{n+1, n+2\}$ and (12) admits an abelian subalgebra of dimension n .

These theorems show that the verification of the third condition requires, in addition to the determination of m , a computation to check the existence of an abelian Lie symmetry subalgebra of dimension n . To our knowledge, there is only an algorithm described in the literature [7] for the computation of the *maximal abelian dimension*, i.e. dimension of the maximal abelian subalgebra of a finitely-dimensional Lie algebra given by its structure constants. The algorithm is reduced to solving the quadratically nonlinear system of multivariate polynomial equations providing vanishing of the Lie bracket between two arbitrary vectors in the Lie algebra. Clearly, the computing time of the algorithm is at least exponential in the dimension m of the algebra.

Instead, to verify the third condition in Thm. 2.2 we devise a much more efficient algorithm. Our algorithm relies on the following statement which is a corollary to Thms. 2.1 and 2.2.

COROLLARY 2.3. *The third condition is equivalent to*

- 3'. $n \geq 3$, $m \in \{n+1, n+2\}$ and the derived algebra of (12) is abelian of dimension n .

PROOF. Under the third condition, since $L' \subset L$, Eq. (3) is linearizable by Thm. 1. Let Eq. (3) be linearizable. The symmetry Lie algebra of (5) is *similar* and hence *isomorphic* to that of (3) (cf. [23], Ch. 2, §7.9). It is easy to see that a linear n -th order equation (5) with variable coefficients admits the Lie point symmetry group

$$\{\bar{t} = t, \bar{u} = u + c_i \cdot v_i(t) \ (i = 1, \dots, n), \bar{u} = c_{n+1} \cdot u\}, \quad (13)$$

where c_i, c_{n+1} are constants (the group parameters) and $v_i(t)$ are the fundamental solutions of (5). The Lie group (13) has the $(n+1)$ -dimensional Lie algebra (cf. [27], Thm. 5.19)

$$L_{n+1} := \{X_i := v_i(t) \partial_u \ (i = 1, \dots, n), X_{n+1} := u \partial_u\}. \quad (14)$$

If a linear n -th order Eq. (5) has constant coefficients, then in addition to (14) the Lie point symmetry group (13) includes the translation $\bar{t} = t + c_{n+2}$ and, hence, its Lie algebra, in addition to (14), has one more element:

$$L_{n+2} := L_{n+1} \cup \{X_{n+2} := \partial_t\}. \quad (15)$$

Furthermore, $[X_{n+1}, X_{n+2}] = 0$, and for all $i \in \{1, \dots, n\}$:

$$[X_i, X_{n+1}] = X_i, \ [X_i, X_{n+2}] = -v'_i(t) \partial_u = \sum_{j=1}^n \alpha_j X_j,$$

where α_i are constants. Therefore, both Lie algebras (14) and (15) have abelian derived algebras of dimension n . $\square$

It is important to emphasize that m can be algorithmically computed without solving the determining system what is generally impossible. It suffices to complete the last system to involution (for the theory of completion to involution we refer to [28]) and to construct power series solutions to the involutive system [24, 25]. For instance, as we do in our implementation (Sect. 4) of the algorithm **LinearizationTest I** described in Sect. 3.1, one can apply to the determining system the *differential Thomas decomposition* [3, 26] for a *degree-reverse lexicographical ranking* and then compute the *differential dimension polynomial* [15] for the output Janet basis.

The differential Thomas decomposition was suggested in [31, 32] as a generalization of the Riquier-Janet theory of *passive* linear and *orthonomic* PDE systems (see also [28] and the references therein) to polynomially-nonlinear systems of general form. The Thomas decomposition provides a universal algorithmic tool [3, 26] to study a *differential system*, which is defined as follows.

Definition 2.4. ([3, 10, 17, 26, 31, 32, 34]) A *differential system* is a system $S := \{S^-, S^\# \}$ of differential equations and (possibly) inequations of the form

$$S^- := \{g_1 = 0, \dots, g_s = 0\}, \ S^\# := \{h_1 \neq 0, \dots, h_t \neq 0\},$$

where s is a positive integer as well as t if $S^\# \neq \emptyset$, and q_i, h_j are elements in the differential polynomial ring in finitely many differential indeterminates (dependent variables) over the differential field of characteristics zero.

The Thomas decomposition applied to a differential system S yields a finite set of passive (involutive) and *differentially triangular* differential systems called *simple* (see [3, 10, 17, 26, 31, 32, 34]) that partition the solution set of the input differential system. Algebraically, this provides a characterizable decomposition [11] of the radical differential ideal $\sqrt{I}$ where I is the differential ideal generated by the polynomials in S^- .

Unlike the **LinearizationTest I** where one can use, due to the linearity of determining systems, any procedure of completion to involution (e.g. the standard form algorithm [24]), our second algorithm **LinearizationTest II** (Sect. 3.2) is oriented to the Thomas decomposition.

To apply it, we need to formulate the conditions for the functions $\phi(x, y)$, $\psi(x, y)$ in (4) and for the coefficients $a_k(t)$ in (7) (if $n \geq 3$) such that these conditions hold if and only if (3) is linearizable. In addition to the input differential system, the Thomas decomposition is determined by a *ranking*, that is, a linear ordering on the partial derivatives compatible with derivations ([3, 10, 17, 26, 31, 32, 34]) (in our case with ∂_x and ∂_y).

By differentiating the equality $u(\psi(x, y(x))) = \phi(x, y(x))$, that follows from (4), n times with respect to x , we obtain the following equalities:

$$\begin{aligned} u'(t) &= \frac{\phi_x + \phi_y y'}{\psi_x + \psi_y y'}, \\ u''(t) &= \frac{J}{(\psi_x + \psi_y y')^3} y'' + \frac{P_2(y')}{(\psi_x + \psi_y y')^3}, \\ &\vdots \\ u^{(n)}(t) &= \frac{J}{(\psi_x + \psi_y y')^{n+1}} y^{(n)} + \frac{P_n(y', \dots, y^{(n-1)})}{(\psi_x + \psi_y y')^{2n-1}}. \end{aligned} \quad (16)$$

Here J is the Jacobian (6), P_k ($k = (2, \dots, n)$) are polynomials in their arguments whose coefficients are differential polynomials in ϕ and ψ , for example,

$$\begin{aligned} P_2(y') &= (\psi_x + \psi_y y') (\phi_{xx} + \phi_{xy} y' + \phi_{yy} (y')^2) \\ &\quad - (\phi_x + \phi_y y') (\psi_{xx} + \psi_{xy} y' + \psi_{yy} (y')^2). \end{aligned}$$

Now we replace the derivatives $u^{(k)}$ occurring in (7) (or the second-order derivative in (8) if $n = 2$) with the appropriate right-hand sides in Eqs. (16) and solve the obtained equality with respect

to $y^{(n)}$ (or y''). As a result, we obtain the equality

$$y^{(n)} + \frac{R(y', \dots, y^{(n-1)})}{J \cdot (\psi_x + \psi_y y')^{n-2}} = 0, \quad (17)$$

where R is a polynomial in $y', \dots, y^{(n-1)}$ whose coefficients for $n \geq 3$ are the differential polynomials not only in ϕ and ψ but also in $a_k(t) = a_k(\psi(x, y(x)))$, the coefficients in Eq. (7).

Denote by M and N the numerator and denominator of the function f in Eq. (3). Then, after elimination of $y^{(n)}$ from the equation system (3), (17) and clearing denominators in the rational functions of the obtained equality we obtain equation

$$R \cdot N - J \cdot M \cdot (\psi_x + \psi_y y')^{n-2} = 0. \quad (18)$$

This equation is a polynomial in $y', y'', \dots, y^{(n-1)}$, and there are no constraints on these variables. Therefore, the equation holds if and only if all coefficients of the polynomial in the left-hand side vanish. This condition gives a partial differential equation system in ϕ, ψ and a_k . If the function f in Eq. (3) depends on parameters and/or undetermined functions in (x, y) , then Eq. (18) contains these parameters/functions.⁴

Let $S^=$ be the set of equations obtained from Eq. (18) by equating the coefficients of the polynomial (in $y', \dots, y^{(n-1)}$) in the left-hand side to zero. If $n \geq 3$ we enlarge $S^=$ with the set of equations

$$S^= = S^= \cup_{k=0}^{n-3} \{\psi_y(a_k)_x - \psi_x(a_k)_y = 0\}. \quad (19)$$

The equation $\psi_y(a_k)_x - \psi_x(a_k)_y = 0$ means that a_k is a function of t in accordance to (7). It is easy to see by differentiating the equality $a_k(t) = a_k(\psi(x, y))$ as follows:

$$(a_k)_x = a_t \psi_x, (a_k)_y = a_t \psi_y, \iff \psi_y(a_k)_x - \psi_x(a_k)_y = 0.$$

Since we admit the invertible transformations (4) only, one has to add to the enlarged equation set $S^=$ the inequation

$$S^\neq := \{J \neq 0\} \quad (20)$$

where J is the Jacobian (6).

Thereby, the main object of our construction and the statements on its relation to the linearization are given as follows.

Definition 2.5. The differential system (see Def. 2.4) made up of the above constructed PDE set $S^=$ and of the inequation set $S^\neq = \{J \neq 0\}$ will be called *linearizing differential system*.

THEOREM 2.6. *Eq. (3) is linearizable via a point transformation (4) if and only if the linearizing differential system is consistent, i.e. has a solution.*

COROLLARY 2.7. *Eq. (3) is nonlinearizable via a point transformations (4) if and only if the result of the Thomas decomposition algorithm ([3], Alg. 2.25; [26], Alg. 2.2.56) applied to the linearizing differential system is the empty set.*

PROOF. [3], Remark 2.3 and [26], Remark 2.2.58. $\square$

⁴One can always consider parameters as functions in x and y with zero derivatives.

3 LINEARIZATION TESTS

In this section we present our algorithms **LinearizationTest I** and **LinearizationTest II**. These algorithms, given an input equation (3), verify its linearizability by the point transformation (4). In so doing, the first test is applicable only to an ODE without parameters and undetermined functions in the variables x and y . The second algorithm admits a rational dependence of the function f in Eq. (3) on such parameters and functions.

3.1 Linearization Test I

Our first test, presented below, is based on the computation of the Lie symmetry algebra and its analysis. In line 2 we compute the determining system for (3). It is the straightforward procedure outlined in the preceding section and described in most textbooks on Lie symmetry analysis, in particular in [5, 14, 21–23, 27]. As a routine, this procedure is present in most computer algebra packages specialized to such an analysis, for example, in the Maple packages DESOLV [6], DESOLVII [33], and SADE [9].

Since the determining system is linear, one can use any algorithm for its completion to involution in line 3, (cf. [24] and [28], Sect. 10.7). However, we prefer to use the differential Thomas algorithm here ([3], Sect. 3 and [26] Sect. 2.2).

The dimension m of the Lie algebra (12) (line 4) is the dimension of the solution space of the determining system and can be computed in several ways (cf. [28], Sect. 8.2 and 9.3). Having computed the Janet involutive form of the determining system, it is easy to compute the dimension of its solution via an algorithmic construction of the *differential dimension polynomial* [15].

LinearizationTest I (q)

Input: q , a nonlinear differential equation of form (3)

Output: True, if q is linearizable and False, otherwise

```

1:  $n := \text{order}(q)$ ;
2:  $DS := \text{DeterminingSystem}(q)$ ;
3:  $IDS := \text{InvolutiveDeterminingSystem}(DS)$ ;
4:  $m := \dim(\text{LieSymmetryAlgebra}(IDS))$ ;
5: if  $n = 1 \vee (n = 2 \wedge m = 8) \vee (n > 2 \wedge m = n + 4)$  then
6:   return True;
7: elif  $n > 2 \wedge (m = n + 1 \vee m = n + 2)$  then
8:    $L := \text{LieSymmetryAlgebra}(IDS)$ ;
9:    $DA := \text{DerivedAlgebra}(L)$ ;
10:  if  $DA$  is abelian and  $\dim(DA) = n$  then
11:    return True;
12:  fi
13: fi
14: return False;
```

We refer to [25] for the subalgorithm providing computation of the Lie symmetry algebra (line 8), i.e. for the computation of the structure constants $C_{i,j}^k$ in Eq. (12). The last subalgorithm **DerivedAlgebra** in line 9 does the straightforward computation of the derived algebra via the structure constants.

Correctness and termination. For the subalgorithms both these properties are either obvious (as for **DerivedAlgebra**) or shown in the papers we referred to in the description of the subalgorithms

above. Therefore, the whole algorithm **LinearizationTest I** terminates, and its correctness is provided by Thms. 2.1 and 2.2, and Cor. 2.3.

3.2 Linearization Test II

Our second test is based on the differential Thomas decomposition [3, 26]. It admits the rational dependence of Eq. (3) on a finite set of parameters (constants) and/or undetermined functions in (x, y) . In the absence of parameters/functions the corresponding sets are inputted as the empty ones.

LinearizationTest II (q, P, H)

Input: q , a nonlinear differential equation of form (3) of order ≥ 2 ;
 P , a set of parameters; H , a set of undetermined functions in (x, y)

Output: Set G of differential systems for functions ϕ and ψ in (4) and (possibly) in elements of P and H if (3) is linearizable, and the empty set, otherwise

```

1:  $n := \text{order}(q)$ ;
2:  $G := \emptyset$ ;
3:  $M := \text{numerator}(f)$ ;  $N := \text{denominator}(f)$ ;
4:  $J := \phi_x \psi_y - \phi_y \psi_x$ ; Jacobian (6)
5: if  $n = 2$  then
6:    $r := u''(t) = 0$ ; ODE (8)
7:    $A := \emptyset$ ;
8: else
9:    $r := u^{(n)}(t) + \sum_{k=0}^{n-3} a_k(t)u^{(k)}(t) = 0$ ; ODE (7)
10:   $A := \{a_0, \dots, a_{n-3}\}$ ;
11: fi
12:  $r \xrightarrow{\text{by (4)}} y^{(n)} + \frac{R(y', \dots, y^{(n-1)})}{J \cdot (\psi_x + \psi_y y')} = 0$ ; Eq. (17)
13:  $T := R \cdot N - M \cdot J \cdot (\psi_x + \psi_y y')^{(n-2)} = 0$ ; Eq. (18)
14:  $S^- := \{c = 0 \mid c \in \text{coeffs}(T, \{y', \dots, y^{(n-1)}\})\}$ ;
15:  $S^- := S^- \cup_{p \in P} \{p_x = 0, p_y = 0\}$ ;
16:  $S^- := S^- \cup_{a \in A} \{a_x \psi_y - a_y \psi_x = 0\}$ ; Eq. (19)
17:  $S^\# := \{J \neq 0\}$ ; Ineq. (20)
18:  $G := \text{ThomasDecomposition}(S^-, S^\#)$ ;
19: return  $G$ ;
```

In lines 3–17 of the algorithm **LinearizationTest II** the input linearizing differential system (Def. 2.5) is constructed for the Thomas decomposition computed in line 18. This construction is done in correspondence with the formulas (6)–(7), (8), and (17)–(20). Furthermore, if the output set of the Thomas decomposition is nonempty, then Eq. (3) is linearizable by Thm. 2.6. In this case the simple systems in the decomposition provide a partition of the solution space of the linearizing differential system and their solutions determine the invertible point transformation (4) and the coefficients $a_k(t)$ of the linearized form (7) or (8). In addition, if there are parameters and/or undetermined functions in (3), then the output differential systems of the Thomas decomposition provide the compatibility conditions to these parameters/functions imposed by the linearization.

Correctness and termination are provided by those of the Thomas decomposition ([3], Sect. 3.4; [26], Thr. 2.2.57).

4 IMPLEMENTATION

We implemented both linearization tests in Maple. Our implementation runs on Version 16 and the subsequent ones.

First, we describe our implementation of the algorithm **LinearizationTest I**. Given an ordinary differential equation of the form (3), to generate the determining system, denoted by DS in line 3, we use the routine *gendef* of the Maple package DESOLV [6, 33]. Then, to complete the system DS to involution (line 3), we choose the *orderly* (“*DegRevLex*”) ranking (cf. [26], Def. A.3.2) such that

$$\partial_x > \partial_y, \quad \xi > \eta,$$

and apply the routine *DifferentialThomasDecomposition* of the package *DifferentialThomas*. This package is freely available [16]. To compute the dimension of the Lie symmetry algebra (line 4), we invoke the routine *DifferentialSystemDimensionPolynomial*. Since in our case the solution space of the determining system is finitely dimensional, the last routine outputs just the dimension of the solution space.

The subalgorithm **LieSymmetryAlgebra** of line 8 was implemented in Maple (see [25], Sect. 6). The implementation is based on the one of two other algorithms: the standard form algorithm for completion of the determining system to involution and on the algorithm of calculating power series solutions [24]. Since that implementation done in Maple V has not been adopted to the subsequent versions of Maple, we decided to make our own implementation of the algorithmic approach suggested in [25] to compute the structure constants in (12). Our implementation takes the Janet involutive form of the determining system outputted by the package *DifferentialThomas* and exploits its routine *PowerSeriesSolution*.

To construct the derived algebra (line 9) we invoke the routine *DerivedAlgebra* which is a part of the built-in package *DifferentialGeometry:-LieAlgebras*.

In our implementation of **LinearizationTest II** we compute the expressions (16) to obtain the left-hand side in (18) (line 13) that is a polynomial in $y', y'', \dots, y^{(n)}$. Then equating of all coefficients in the polynomial to zero (line 14) and enlarging it with additional equations (lines 15 and 16) and the Jacobian inequation (line 17) yields the input linearizing differential system for the subroutine *differentialThomasDecomposition* (line 18). By default, we choose the orderly ranking on the partial derivatives of the functions ϕ and ψ , and of those in the sets A (line 10):

$$\partial_x > \partial_y, \quad \xi > \eta > a_0 > \dots > a_{n-2}.$$

If the input ODE (3) contains (nonempty) sets of parameters and/or undetermined functions, then their rankings are less than those of a_{n-2} in order to derive the compatibility conditions for parameters and functions.

5 EXAMPLES

In this section we demonstrate our algorithmic linearization tests using several examples. All timings given below were obtained with Maple 16 running on a desktop computer with an Intel(R)Xeon(R) X5680 CPU clocked at 3.33 GHz and 8 GB RAM.

Example 5.1. [18] Consider the second-order Eq. (1) in which f is given by (2) with undetermined functions F_k , $0 \leq k \leq 3$. Algorithm **LinearizationTest I** is not applicable to this case, so we apply the

algorithm **LinearizationTest II** with an orderly ranking such that

$$\partial_x > \partial_y, \quad \phi > \psi > F_3 > F_2 > F_1 > F_0.$$

Then the routine *DifferentialThomasDecomposition* of the package *DifferentialThomas* [16] outputs three differential systems with disjoint solutions space in about 0.4 sec.:

$$S_1 := \{S_1^-, S_1^\# \}, \quad S_2 := \{S_2^-, S_2^\# \}, \quad S_3 := \{S_3^-, S_3^\# \}.$$

Cor. 2.7 guarantees that there are linearizable equations among the equations in family (1)–(2). One of the output differential systems, namely S_1 , is a *generic simple system* (see [26], Def. 2.2.67). It has eight equations, and the last two of them that contain solely functions F_0, F_1, F_2, F_3 are the *compatibility conditions* for these functions whose solutions admit linearization. These conditions have the following form:

$$\begin{aligned} 3(F_3)_{xx} - 2(F_2)_{xy} + (F_1)_{yy} - 3F_1(F_3)_x + 2F_2(F_2)_x \\ - 3F_3(F_1)_x + 3F_0(F_3)_y + 6F_3(F_0)_y - F_2(F_1)_y &= 0, \\ (F_2)_{xx} - 2(F_1)_{xy} + 3(F_0)_{yy} - 6F_0(F_3)_x + F_1(F_2)_x \\ - 3F_3(F_0)_x + 3F_0(F_2)_y + 3F_2(F_0)_y - 2F_1(F_1)_y &= 0. \end{aligned} \quad (21)$$

These equations are exactly the linearizability conditions for (1)–(2) obtained by Lie in [18] (cf. [14], Thm. 6.5.2). The inequations in the S_1 system are

$$S_1^\# = \{J \neq 0, \psi_x \neq 0, \psi_y \neq 0\}.$$

The two other differential systems S_2 and S_3 have the following inequations:

$$S_2^\# = \{\phi_x \neq 0, \psi_y \neq 0\}, \quad S_3^\# = \{\phi_y \neq 0, \psi_x \neq 0\}. \quad (22)$$

Each of these systems has eight equations as S_1 . Every equation in S_2 as well as in S_3 is valid on all common solutions to the equations in S_1 (cf. [26], Cor. 2.2.66). In doing so,

$$\psi_x = 0 \in S_2^-, \quad \psi_y = 0 \in S_3^-, \quad (23)$$

and hence each of (22) and (23) implies $J \neq 0$. Therefore, algorithm **LinearizationTest II** reproduces Lie's classical results on the necessary and sufficient conditions for the linearization of the second-order ODEs from family (1)–(2).

Example 5.2. ([8], Eq. 2.50) We consider the third order ODE

$$y''' - 6 \frac{y'}{x^2} + 3 \frac{(y')^2}{x} - \frac{1}{2} (y')^3 = 0. \quad (24)$$

This equation is linearizable by the generalized Sundman transformation⁵ [8]. Here we check its linearizability via the point transformation (4). Eq. (24) admits both our tests since it does not contain parameters and/or undetermined functions. Our implementation of algorithm **LinearizationTest I** returns *false* in 0.05 sec. and that of **LinearizationTest II** returns the empty set in 0.4 sec.

Example 5.3. We consider the fourth-order ODE

$$\begin{aligned} 2x^2 y y'''' + x^2 y^2 + h(x, y) y' y''' + 16x y y'' \\ + 6x^2 (y'')^2 + 48x y' y'' + 24y y'' + 24(y')^2 = 0, \end{aligned} \quad (25)$$

where $h(x, y)$ is an undetermined function. To find all values of this function providing linearization, we again apply algorithm

⁵A kind of a nonlocal transformation, which is in general not a point transformation.

LinearizationTest II. The package *DifferentialThomas* for the orderly ranking satisfying

$$\partial_x > \partial_y, \quad \phi > \psi > a_0 > a_1 > b$$

outputs in 3.3 sec. two differential systems S_1 and S_2 (see (27) and (29)). Each system has only one equation containing $h(x, y)$:

$$h(x, y) - 8x^2 = 0. \quad (26)$$

The linearizability of (25) under condition (26) was established in [13], and our computation shows that there are no other linearizable equations of family (25). Moreover, the simple systems S_1 and S_2 allow for the explicit construction of the linearizing point transformation (4) and the coefficients $a_0(t)$ and $a_1(t)$ in the Lagerre-Forsyth form (7) of the image of (25) under mapping (4):

$$u^{(4)}(t) + a(t)u(t) + b(t)u'(t) = 0.$$

To show this, consider first the equations in S_1^- :

$$\begin{aligned} y \phi_y - 2 \phi - 2 \phi_{xxx} &= 0, \quad \phi_{xxx} = 0, \\ x^2 \phi_{xy} - 2 \phi_y &= 0, \quad x \phi_{xy} - 2 \phi_y = 0, \\ y \phi_{yy} - \phi_y &= 0, \quad a \psi_x^4 - 1 = 0, \quad \psi_y = 0, \\ a_x &= 0, \quad a_y = 0, \quad b = 0, \quad h - 8x^2 = 0, \end{aligned} \quad (27)$$

and its inequations

$$S_1^\# = \{a \neq 0, \phi_y \neq 0\}. \quad (28)$$

The equation system (27) can easily be integrated by hand or using the Maple routine *pdsolve*. The general solution to (27), in addition to (26), reads

$$\begin{aligned} \phi_1 := c_1 x^2 y^2 + \sin\left(\frac{x}{\sqrt{2}}\right) \left(c_2 \exp\left(-\frac{x}{\sqrt{2}}\right) + c_3 \exp\left(\frac{x}{\sqrt{2}}\right) \right) \\ + \cos\left(\frac{x}{\sqrt{2}}\right) \left(c_4 \exp\left(-\frac{x}{\sqrt{2}}\right) + c_5 \exp\left(\frac{x}{\sqrt{2}}\right) \right), \\ \psi_1 := \frac{x}{c_6^{1/4}} + c_7, \quad a_1 := c_6, \quad b_1 := 0, \end{aligned}$$

where c_k ($k \in \{1, \dots, 7\}$) are arbitrary constants and the subscript 1 represents the obtained solution to the differential system S_1 . Ineq. (28) imply $c_1 \neq 0$ and $c_6 \neq 0$.

The second differential system S_2 is generic, and its set of equations is given by

$$\begin{aligned} 32 y a \phi_y - 64 a^3 (\phi + \phi_{xxx}) - 96 a^2 a_x \phi_{xxx} \\ - 36 a a_x^2 \phi_{xx} - 3 a_x^3 \phi_x &= 0, \\ 128 x^2 a^3 \phi_{xxy} + 15 x^2 a_x^3 \psi_y - 144 x a a_x^2 \phi_y \\ + 288 a^2 a_x \phi_y &= 0, \\ 16 x^2 a^2 \phi_{xy} - 3 x^2 a_x^2 \phi_y + 24 x a a_x \phi_y \\ - 32 a^2 \phi_y &= 0, \\ x a \phi_{xy} + 3 x a_x \phi_y - 16 a \phi_y &= 0, \\ y \phi_{yy} - \phi_y = 0, \quad a \psi_x^4 - 1 = 0, \quad \psi_y = 0, \\ 8 a a_{xx} - 7 a_x^2 = 0, \quad a_y = 0, \\ b = 0, \quad h - 8x^2 = 0. \end{aligned} \quad (29)$$

The set of inequations in S_2 consists three elements:

$$S_2^\# = \{a \neq 0, \phi_y \neq 0, a_x \neq 0\}. \quad (30)$$

Table 1: CPU times (sec.)

Test	Order n of the ODE (32)												
	3	4	5	6	7	8	9	10	11	12	13	14	15
I	0.20	0.61	1.27	2.54	4.18	6.49	10.20	23.21	39.79	63.38	91.54	119.42	150.13
I _A	0.28	0.83	1.51	3.01	5.28	9.52	16.83	45.40	80.72	150.19	291.13	484.35	751.20
II	0.65	2.33	13.28	80.76	376.40	1525.1	7512.9	OOM	OOM	OOM	OOM	OOM	OOM

The differential system (29) is also easily solvable, and its general solution reads

$$\begin{aligned} \phi_2 &:= \frac{\phi_1}{(c_6x+c_7)^3}, & \psi_2 &:= c_8 - \frac{1}{c_6(c_6x+c_7)}, \\ a_2 &:= (c_6x+c_7)^8, & b_2 &:= 0, & h &:= 8x^2. \end{aligned} \quad (31)$$

Here ϕ_1 is the above presented solution to (27) with arbitrary constants

$$\phi \text{ and } c_i \ (1 \leq i \leq 8).$$

The constraints that follow from (30) are those in S_1 , $c_1 \neq 0$, $c_6 \neq 0$, and the additional inequation $c_6x+c_7 \neq 0$ what rules out singularity in (31).

The obtained explicit solutions to S_1 and S_2 form disjointed sets, since $a_x = 0$ for a solution to S_1 and $a_x \neq 0$ for that to S_2 . The disjointness of solution sets for the output simple systems is guaranteed by the Thomas decomposition algorithm ([3, 10, 17, 26, 31, 32]). In the given case a solution to S_1 provides a mapping of (25) into the linear ODE

$$u^{(4)}(t) + c_6u(t) = 0$$

with constant coefficients, whereas a solution to S_2 maps (25) into an equation with variable coefficients

$$u^{(4)}(t) + (c_6x+c_7)^8u(t) = 0.$$

In [13], the simplest form of the linearizing transformation (4) was found:

$$t = x, \quad u = x^2y^2,$$

which maps (25) and (26) into $u^{(4)} + u = 0$ and corresponds to the solution of S_1 with

$$c_1 = 1, \quad c_2 = c_3 = c_4 = c_5 = c_7 = 0.$$

Example 5.4. As a serial example, we consider

$$(y^2)^{(n)} + y^2 = 0, \quad n \in \mathbb{N}_{\geq 3}. \quad (32)$$

Obviously, Eq. (32) becomes $u^{(n)} + u = 0$ via transformation (4) of the form $t = x$ and $u = y^2$.

We use this example as a benchmark for a comparative experimental analysis of the time behavior of our algorithms when the order of the ODE grows. Additionally, we measure the CPU time for the algorithm **LinearizationTest I** whose subalgorithm **DerivedAlgebra** (in line 9) is replaced with the Maple implementation [7] for the detection of an n -dimensional abelian subalgebra of the Lie symmetry algebra (line 8). By Thm. 2.1, the existence of such a subalgebra yields the criterion of linearization.

Table 1 presents the CPU times, where “OOM” is an acronym for runs “Out Of Memory”. The timings in the table correspond to **LinearizationTest I** (upper row), to its above described modification denoted by I_A (middle row), and to **LinearizationTest II** (bottom

row). As one can see, our first test (I) is the fastest and the second test (II) is the slowest. However, the last one, unlike the other two, outputs much more information on the linearization. This fact was illustrated by Example 5.3.

6 CONCLUSIONS

For the first time, the problem of the linearization test for a wide class of ordinary differential equation of arbitrary order was algorithmically solved. In doing so, we have restricted ourselves to the quasi-linear equations with a rational dependence on the other variables and to point transformations, and designed two algorithmic tests in order to check linearizability.

The main benefits of these restrictions are (i) the algorithmic construction of the Lie symmetry algebra for the input equation and (ii) the reduction of the number of coefficients in the linearized equation due to the Lagerre-Forsyth canonical form (7). The benefit (i) allowed us to design an efficient algorithm **LinearizationTest I** which checks the linearizability of the equations. The second benefit (ii) provides the feasibility of the algorithm **LinearizationTest II** because of the overdeterminacy (cf. [28], Sect. 7.5) of a linearizing differential system. This overdeterminacy simplifies the consistency analysis of the linearizing system answering the same question as the first test.

Moreover, due to finite-dimensionality of the solution space (cf. [22], Prop. 6.57) of a linearizing system, the Thomas decomposition algorithm outputs overdetermined subsystems, as those in Example 5.3. In practice, the overdeterminacy of the outputted simple systems of the Thomas decomposition of a linearizing differential system makes them easily solvable, much like the determining systems in the Lie symmetry analysis. Thereby, with the algorithm **LinearizationTest II** one can not only detect linearizability, but also find the linearizing transformation (4) and the coefficients in the linear form of the equation.

The Thomas decomposition for linearizing differential systems, even in the case of its inconsistency, may be time and space consuming, especially for higher-order ODEs. That is why, in practice, it is advisable to check the linearizability of the equation under consideration by the first algorithm before applying the second one. In the case when Eq. (3) contains parameters and/or arbitrary functions, there is no choice and one has to use the second algorithm.

In doing so, according to our computer experiments, most of the computing time in algorithm **LinearizationTest I** is spent for the construction of a Lie symmetry algebra (line 8) for the involutive determining system obtained in line 3. This construction takes more than 90% of the total computing time and its optimization is a further source of acceleration of the algorithm.

The second algorithm may also improve the built-in Maple solver *dsolve* of differential equations. For example, *dsolve* applied to the equation

$$y''' + \frac{3y'}{y}(y'' - y') - 3y'' + 2y' - y = 0 \quad (33)$$

outputs its solution implicitly in the complicated form of double integrals including Maple's symbolic presentation *RootOf* for the roots of expressions. On the other hand, Eq. (33) admits the linearization (cf. [14], Eqs. 6.6.57–6.6.59)

$$u''' - \frac{2}{t^3}u = 0, \quad t = \exp(x), \quad u = y^2$$

which is easily obtained by our algorithm **LinearizationTest II** and provides the explicit form of the solution to (33).

Some obvious cases of linearizability like maximality of symmetry algebra could be effectively verified by implementing them as a preprocessing procedure for the standard ODE solvers. Computationally, such a verification can be rather fast by checking the dimension of a symmetry algebra without finding the structure constants.

Our algorithms admit a generalization to the case of contact and higher symmetries for scalar ordinary differential equations. And it would be beneficial to generalize it for the case of second-order ordinary differential system, which is of high interest in natural sciences and engineering. Some results in this regards were recently obtained for particular cases of systems of two second-order ordinary differential equations [4, 29, 30], but the algorithmicity of linearizability for such systems as well as the structure of relevant computations is still an open question (cf. [2], Sect. 4).

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