Effect of corner radius in stabilizing the low-Re flow past a cylinder

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ABSTRACT

We perform global linear stability analysis on low- Re flow past an isolated cylinder with rounded corners. The objective of the present work is to investigate the effect of cylinder geometry (corner radius) on the stability characteristics of the flow. Our investigation sheds light on new physics that the flow can be stabilized by partially rounding the cylinder in the critical and weakly super-critical flow regimes. The flow is first stabilized and then gradually destabilized as the cylinder varies from square to circular geometry. The sensitivity analysis reveals that the variation of stability is attributed to the different spatial variation trends of the backflow velocity in the near- and far-wake regions for various cylinder geometries. The results from the stability analysis are also verified with those of the direct simulations and very good agreement is achieved.

1 Introduction

The flow past a circular cylinder is undoubtedly one of the most classical fluid mechanics problems in the context of both academic research and industrial applications. The flow patterns are categorized into a sequence of stages entirely dependent on the Reynolds number, which is defined based on the cylinder diameter and incoming flow velocity. The flow is fully attached to the cylinder surface for $Re \lesssim 6.0$ [1]. As the Reynolds number increases, two steady symmetric recirculation bubbles form downstream of the cylinder. The steadiness and symmetry of the recirculation bubbles break at $Re \approx 47$ due to the onset of first instability [2], with the appearance of time-periodic self-sustained shedding vortices, and the wake flow becomes unsteady but remains two-dimensional. This two-dimensional unsteady flow persists until $Re \approx 189$ where the three-dimensional mode-A small-scale structures are observed [3]. The flow separation and transition can also be caused by the cylinder oscillation or rotation [4–7]. Defining the critical Reynolds number $Re_{cr} \approx 47$, the flow is considered stable for $Re < Re_{cr}$ since any perturbation will decay in time and the flow recovers to the original steady-state, while it is unstable for $Re > Re_{cr}$ in that the infinitesimal perturbation will grow in time and the steadiness of the flow is destroyed. To better understand the onset of first instability and the initial development of the perturbation, a number of linear stability analyses have been performed for flow past the circular cylinder with Reynolds number at or slightly above Re_{cr} , i.e., critical or weakly super-critical flows. Considering that the wake flow is essentially non-parallel within the recirculation bubbles, the global linear stability analysis approach is employed to take into account all possible spatial variations of both the base flow and perturbation [8–12]. The results reveal that for the super-critical flow above Re_{cr} , the growth rate of perturbation

monotonically increases with the Reynolds number in the two-dimensional flow regime. A sensitivity study aids in the development and implementation of control techniques to stabilize the flow and delay the occurrence of vortex shedding [11, 13, 14].

The occurrence of first instability is also observed for the square cylinder at $Re_{cr} \approx 45$ [15]. The primary difference between the two flow configurations is that the attached flow experiences forced separation at the sharp leading corners of the square cylinder due to geometrical discontinuity, while for the circular cylinder the separation naturally appears on the cylinder surface due to the adverse pressure gradient. For the super-critical two- and three-dimensional flows at $Re \sim O(10^2 - 10^3)$, Sohankar *et al.* [16] found that the aerodynamic quantities, such as the mean and fluctuating lift and drag coefficients, are substantially affected by the cylinder geometry. Recent numerical and experimental studies [17–19] confirmed that the flow pattern is determined by the corner radius of a partially rounded cylinder. By rounding the corners of a square cylinder with a small radius of curvature, the formation of the shedding vortices are initialized further downstream away from the cylinder compared with both the circular and square geometries, and results in the streamwise expansion of the recirculation bubbles. Similar numerical and experimental studies are also performed for the fully developed turbulent wake flow downstream of a partially rounded cylinder at $Re \sim O(10^4 - 10^6)$ [20–22]. It can be concluded from these studies that the aerodynamic characteristics of the cylinder and wake flow pattern are significantly affected by the cylinder geometry, more specifically, the corner radius. Acknowledging the importance of the cylinder geometry to the flow pattern, we would like to know if the stability of the flow be (substantially) affected by the cylinder geometry.

The first issue in the investigation is the clarification and quantification of the flow stability. We use two indicators to quantify the flow stability in the present work. The first indicator is the critical Reynolds number of the targeted flow above which the flow transits from stable to unstable; the flow with a higher Re_{cr} is regarded more stable. This indicator is only physically significant for flow in the vicinity of the critical point. The second indicator is the temporal growth rate of the perturbation for the super-critical flow; a higher growth rate implies rapidly growing perturbations, and thus the flow is considered more unstable. It is easily concluded that the variation of cylinder geometry changes the (base) flow pattern, which further alters the stability characteristics, while we would like to know how sensitive the perturbed flow is with respect to the different flow patterns as resulted from the geometry variation. In the present study, we perform global linear stability analysis on low- Re flow past an isolated cylinder. The Reynolds number is varied in the range $[Re_{cr}, 110]$ to account for both the critical flow and the two-dimensional weakly super-critical flow, in which the critical Reynolds number is around that of the circular cylinder, i.e., $Re_{cr} \approx 47$, and will be identified and discussed in this work. The cylinder geometry is characterized by the radius of the rounded corners, including the circular, square and partially rounded ones. The objective of our work is to investigate the effect of cylinder geometry on the stability characteristics of critical and weakly super-critical flows. The stability analysis is performed to identify the critical Reynolds number as well as the growth rate of perturbation for the weakly super-critical flow under different corner radii. The sensitivity of the flow is studied based on the adjoint mode analysis to explore the source of the variation of flow stability.

2 Numerical setup

2.1 Problem description

The schematic configuration of the physical problem is depicted in Fig. 1. An isolated cylinder of diameter D is placed at the origin of the coordinate system. The corners of the cylinder are rounded at a radius of curvature R , which is non-dimensionalized as $R^+ = R/D$ in the present work. The geometry of the cylinder is characterized by R^+ , which can be a square cylinder with sharp corners ($R^+ = 0.00$), a partially rounded cylinder ($0.00 < R^+ < 0.50$) or a circular cylinder ($R^+ = 0.50$). The outer radius of the computational domain is large as $W = 200D$ to model an infinite medium, which is comparable or larger than a number of works [9, 11, 12, 23]. The velocity of the incoming flow is $(u, v) = (U_0, 0)$. The Reynolds number is defined as $Re = \rho U_0 D / \mu$ where $\rho = 1$ is the fluid density of the incompressible flow. The computational domain is discretized by an O-type grid. The size of the first grid layer at the cylinder surface is about $0.005D$ in the wall-normal direction to guarantee the well resolving of the boundary layer. Since the flow is primarily governed by the wake instability, up to 70% of the grid is clustered in the downstream half of the domain

2.2 Governing equations

The flow is governed by the two-dimensional incompressible continuity and momentum equations in the non-dimensional form:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \frac{1}{Re} \nabla^2 \mathbf{u}, \quad \nabla \cdot \mathbf{u} = 0. \quad (1)$$

The governing equations are non-dimensionalized by length D , velocity U_0 , pressure ρU_0^2 and time D/U_0 . Let $(\mathbf{U}, P)$ denote the variables of the base flow and $(\mathbf{u}', p')$ the infinitesimal perturbation. The primitive variables are expressed as:

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{U}(\mathbf{x}) + \mathbf{u}'(\mathbf{x}, t), \quad p(\mathbf{x}, t) = P(\mathbf{x}) + p'(\mathbf{x}, t). \quad (2)$$

Substituting the decomposition into the governing equations (1) and omitting the high-order infinitesimal terms, we have the governing equations for the steady-state base flow:

$$(\mathbf{U} \cdot \nabla) \mathbf{U} = -\nabla P + \frac{1}{Re} \nabla^2 \mathbf{U}, \quad \nabla \cdot \mathbf{U} = 0, \quad (3)$$

and the governing equations for the perturbed flow:

$$\frac{\partial \mathbf{u}'}{\partial t} + (\mathbf{U} \cdot \nabla) \mathbf{u}' + (\mathbf{u}' \cdot \nabla) \mathbf{U} = -\nabla p' + \frac{1}{Re} \nabla^2 \mathbf{u}', \quad \nabla \cdot \mathbf{u}' = 0. \quad (4)$$

2.3 Direct simulation

The two-dimensional direct simulation is performed by our in-house code used in previous studies [19, 24–26]. The governing equations (1) are discretized on the grid using a second-order central difference scheme. The discretized equations are solved by a semi-implicit fractional step method [27], in which the convective terms are treated explicitly by the Adams-Bashforth scheme and the viscous terms treated implicitly. The non-dimensional time step size is about $0.003D/U_0$. The flow is assumed uniform at the inflow boundary, and fully developed ($\partial \mathbf{u}/\partial x = 0$) at the outflow boundary. No-slip and no-penetrating velocity boundary condition is prescribed at the cylinder surface.

2.4 Base flow

The base flow is the equilibrium state about which linear stability analysis is performed and stability characteristics are investigated. For flow past the cylinder beyond Re_{cr} , the base flow cannot be obtained by directly time-integrating the governing equations (3) since the unstable modes will grow and induce vortex shedding, thus never attaining the steady-state. In this work, the selective frequency damping method proposed by Åkervik *et al.* [28] is used to obtain the steady-state solution. A time derivative term and an extra forcing term are added to the original momentum equation (3), and the system is augmented with an ODE as follows:

$$\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U} \cdot \nabla) \mathbf{U} = -\nabla P + \frac{1}{Re} \nabla^2 \mathbf{U} - \chi(\mathbf{U} - \mathbf{q}), \quad \nabla \cdot \mathbf{U} = 0, \quad (5)$$

$$\frac{\partial \mathbf{q}}{\partial t} = \frac{\mathbf{U} - \mathbf{q}}{\Delta}, \quad (6)$$

where $\mathbf{q}=(q_u, q_v)$ is a field vector that is physically the same as the velocity vector $\mathbf{U}$ of the base flow. χ and Δ are positive user-defined constants chosen based on experiences of similar physical problems (respectively 1.0 and 5.0 in this work), whose values will not affect the solution $(\mathbf{U}, P)$ after the iteration converges [28]. The nonlinear equation system (5)-(6) is iterated over time t to obtain the solution $(\mathbf{U}, P)$ of the base flow, while the additional vector $\mathbf{q}$ is simultaneously solved during the iteration and equals to $\mathbf{U}$ after convergence. The additions of the terms and the ODE provide a feedback mechanism to damp the unstable modes. The governing equations (5)-(6) are solved using our finite difference solver. The integration over time continues until the convergence criterion $(\partial_t \mathbf{U})_{\max} < 10^{-10}$ is achieved. It is found that a more strict convergence criterion does not have noticeable effect on the stability analysis results.

2.5 Direct mode analysis

In the present global linear stability analysis, the perturbations are assumed of the form:

$$\mathbf{u}'(\mathbf{x}, t) = \hat{\mathbf{u}}(\mathbf{x}) e^{\lambda t} + \text{c.c.}, \quad p'(\mathbf{x}, t) = \hat{p}(\mathbf{x}) e^{\lambda t} + \text{c.c.}, \quad (7)$$

where c.c. denotes the complex conjugate. $\lambda = \sigma + i2\pi f$ is the complex circular frequency, in which the real part (σ) is the growth rate and the imaginary part (f) is the frequency of the temporal growth of perturbation. The positive σ indicates the exponential growth of perturbation of the corresponding eigenmode, while a negative value reflects the temporal damping of perturbation. The system is considered unstable if there is at least one eigenmode with positive growth rate. The equations for the amplitude function $(\hat{\mathbf{u}}, \hat{p})$ are obtained by

substituting Eq.(7) into Eq.(4), which results in a generalized eigenvalue equation (GEV):

$$A\hat{\phi} = \lambda M\hat{\phi}, \quad (8)$$

$$A = \begin{bmatrix} -(\mathbf{U} \cdot \nabla) - (\nabla \mathbf{U} \cdot) + \nabla^2 / Re & -\nabla \\ \nabla \cdot & 0 \end{bmatrix}, \quad M = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{\phi} = \begin{bmatrix} \hat{\mathbf{u}} \\ \hat{p} \end{bmatrix}, \quad (9)$$

where the terms $-(\mathbf{U} \cdot \nabla)$ and $-(\nabla \mathbf{U} \cdot)$ denote the advection of perturbation and production of perturbation by strain, respectively. The same mesh and spatial discretization scheme used for direct simulation are utilized in the solution of the GEV. The perturbed velocity $\hat{\mathbf{u}}$ is zero at the inflow boundary and the cylinder surface, and follows homogeneous Neumann condition at the outflow boundary. The GEV is solved for the unstable mode using our in-house code [29, 30]. It is noted that since the coefficient matrix of the GEV is real, conjugate complex eigenpairs are expected representing the same growth rate but opposite phase angle. To ensure correctness of the computed eigenpair, we use the following criterion for both the real and imaginary parts of the GEV to retain only the converged results:

$$\left| (A\hat{\phi} - \lambda M\hat{\phi}) / \hat{\phi} \right|_{\infty} < 10^{-7}. \quad (10)$$

2.6 Adjoint mode analysis and sensitivity analysis

Denoting $\hat{\phi}^+ = (\hat{\mathbf{u}}^+, \hat{p}^+)$ as the variables of the adjoint mode and λ^+ the corresponding complex circular frequency, the adjoint GEV is formed:

$$A^+ \hat{\phi}^+ = \lambda^+ M^+ \hat{\phi}^+, \quad (11)$$

$$A^+ = \begin{bmatrix} (\mathbf{U} \cdot \nabla) - [(\nabla \mathbf{U})^T \cdot] + \nabla^2 / Re & -\nabla \\ \nabla \cdot & 0 \end{bmatrix}, \quad M^+ = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{\phi}^+ = \begin{bmatrix} \hat{\mathbf{u}}^+ \\ \hat{p}^+ \end{bmatrix}, \quad (12)$$

where the superscript T designates the transpose of a matrix. The above GEV is solved with similar boundary conditions demonstrated by Giannetti & Luchini [9] and Marquet *et al.* [11].

The computed variables are normalized to satisfy:

$$|\hat{u}|_{\infty} = 1, \quad \int_{\Omega} [(\hat{u}^+)^* \cdot \hat{u}] d\Omega = 1, \quad (13)$$

where the superscript $*$ designates the complex conjugate and Ω the computational domain.

Marquet *et al.* [11] studied the flow past a circular cylinder and derived the explicit formulation for the sensitivity of the eigenvalue to base flow modification:

$$\nabla_u \lambda = -(\nabla \hat{u})^H \cdot \hat{u}^+ + \nabla \hat{u}^+ \cdot \hat{u}^*, \quad (14)$$

where the superscript H designates transconjugate. The first and second terms on the RHS of the equation denote the transportation and production of perturbation by the base flow, respectively [11, 31]. This sensitivity quantity determines the region in the computational domain where the eigenvalue is the most sensitive to arbitrary base flow modification, and therefore the local base flow is crucial to the perturbation growth. The drift of growth rate is the real part of the above sensitivity quantity:

$$\nabla_u \sigma = \text{Re}(\nabla_u \lambda). \quad (15)$$

2.7 Code verification

The code is verified through a grid sensitivity study to determine the resolution sufficient for the accurate prediction of unstable mode, and the results of the mesh sensitivity study are listed in Tab. 1. The results are presented for two extreme geometries, i.e., square and circular cylinders, to facilitate comparison with other studies. Four resolutions are employed with doubling the number of grid in each coordinate direction until changes of $O(10^{-3})$ or less occur in the magnitudes of growth rate and frequency of the unstable mode. It is evident from the table that the present results well agree with other studies in literatures. The 256×512 grid generally gives good predictions, while a further refinement in the radial direction (η) only slightly affects the results in the fourth significant digit or even smaller for both cylinders.

However, the refinement in the azimuthal direction from 256 to 512 grids has a pronounced effect on the result since the grid is refined in the near-wake where the strain is significant due to flow recirculation. The largest difference of growth rate appearing in the third significant digit is at most 3% for the square cylinder, and is smaller for the circular cylinder. We deem that the 512×512 grid is adequate for the accurate computation of the unstable mode in all cases, and will be employed in all following computations.

3 Results and discussion

3.1 Stabilization of critical flow

In this section we assess the effect of corner radius on the onset of first instability where the flow past the cylinder transits from steady to unsteady. For each corner radius, a number of computations are carried out around a guess value of Re_{cr} using the bisection method until two Reynolds numbers, the weakly sub- and super-critical ones, are obtained with $\Delta Re=0.1$. The critical Reynolds number and corresponding critical frequency f_{cr} are then linearly interpolated. Since the solution of the base flow is rather computationally intensive, the effect of corner radius on the critical Reynolds number is only quantified at several selected corner radii. The critical Reynolds number and frequency are quantitatively tabulated in Tab. 2. For the critical flow, the frequency of perturbation growth is same in magnitude as the Strouhal number (see e.g., [8, 10]), thus the frequency listed in the table can also be interpreted as the critical Strouhal number. As R^+ increases, Re_{cr} first increases and then decreases although the variation magnitude is tiny compared with its magnitude. The flow past the partially rounded cylinder is more stable with the maximum Re_{cr} appearing at $R^+=0.30$ for the several radii listed, i.e., the flow is stabilized by partially rounding the cylinder compared with the square and circular

geometries. The critical frequency monotonically increases with R^+ . The spatial mode structure of the eigenfunction is exemplified in Fig. 2 at $R^+=0.30$ and $Re=47.0$. It is seen that the unstable mode is antisymmetric about the wake centerline and propagates downstream. The eigenfunction contours at other (Re, R^+) combinations are quite similar and hence are not presented here for brevity.

The critical Reynolds number $Re_{cr}=46.59$ for the circular cylinder case in our work is well in consistent with the benchmark values obtained from other global stability analysis (46.7 [9] and 46.8 [11]), direct simulation (46.88 [32]) and experimental (47.4 [33]) studies. The critical frequency $f_{cr}=0.11596$ is also believed accurately predicted (0.118 [9], 0.116 [11] and 0.1168 [32]). There is still minor difference between the results in various works, which is mainly attributed to the different mesh, numerical methods or experimental techniques used in the researches. For the critical flow, various approaches ought to give the same prediction regarding the frequency of perturbation.

The variation of critical Reynolds number with respect to corner radius is analyzed by the sensitivity to base flow modification. Since the flow stability is quantified by the growth rate of perturbation, the growth rate sensitivity $(\nabla_U \sigma)$ is computed and given in Fig. 3. Based on Eqs.(14)-(15), this quantity is a real vector field, thus its orientation is exhibited by the streamlines in addition to the magnitude in contour. For the critical flow, its stability is the most sensitive to the base flow modification in the wake region approximately at $x/D < 4.0$, while the downstream modification has negligible effect due to the spatial separation of direct and adjoint modes. The orientation of growth rate sensitivity is generally quite similar for flows at all corner radii, with a backflow direction that is especially notable on the wake centerline. This

type of orientation indicates that the flow can be destabilized ($\delta\sigma>0$) if the base flow modification (δU) is oriented in the same direction anywhere in the flow, e.g., intensification of backflow velocity in the wake increases the destabilizing effect [11]. It is seen in the figure that the spatial structure of the $\nabla_U \sigma$ contour varies significantly with the corner radius. The growth rate sensitivity is pronounced in the wake and at the rear corners at $R^+=0.00$. As R^+ increases, the sensitivity gets weaker at the rear corners but remains notably significant in the wake. For $R^+=0.40$ and 0.50 , the sensitivity is the most pronounced in the near-wake ($x/D \approx 1.0$) but gets weaker further downstream, reflecting that it is only determined by the near-wake base flow modification. Acknowledging the importance of the wake flow on the growth rate sensitivity, the distribution of the streamwise base flow velocity is plotted in Fig. 4 for different corner radii. The variation trend of the streamwise velocity with corner radius is different in the near- and far-wake regions. As the corner radius increases, the magnitude of negative backflow velocity increases in the near-wake region but decreases in the far-wake region, with the reversal of the variation trend occurring around $x/D=1.8$. Referring to Fig. 3 and the above analysis, we conclude from Fig. 4 that as the corner radius increases from $R^+=0.00$ to $R^+=0.30$, the flow gets stabilized due to the weakened backflow in the far-wake region $x/D>1.8$ where the growth rate sensitivity is significant for this R^+ range, thus Re_{cr} increases accordingly. However, the flow is destabilized as the corner radius further increases to $R^+=0.40$ and 0.50 since the near-wake ($x/D<1.8$) backflow is getting stronger where the growth rate sensitivity is greatly determined. This analysis reveals that the stabilizing and destabilizing of the flow are attributed to the spatial variation trend of backflow velocity; the effect is determined by both the backflow variation (enhancing or weakening) and its spatial distribution in near- and

far-wake regions.

3.2 Stabilization of super-critical flow

3.2.1 Characteristics of base flow

The steady-state base flow is characterized by two symmetric recirculation bubbles downstream of the cylinder that are elongated in the streamwise direction. The size of the recirculation bubbles is measured by the streamwise length from cylinder rear to reattachment point at the wake centerline, which is termed as L_r . The characteristic length L_r depends on both the Reynolds number and corner radius. Previous studies have found that the length increases linearly with the Reynolds number for the circular cylinder for both sub- and super-critical flows. Zielinska *et al.* [34] provided a linear fit of $L_r=0.0670Re-0.405$ in the regime $Re=[6, 100]$, which is supported by the result of Giannetti & Luchini [9]. In our work the length for the circular cylinder case is linearly fitted as $L_r=0.0648Re-0.303\pm0.4\%$, where the minor difference is attributed to the different domain size and resolution used in various simulations. We also find that L_r linearly increases with Reynolds number for other corner radii (not shown here). The variation of L_r with R^+ is quite nonlinear and dependent on the Reynolds number, as shown in Fig. 5. Here we employ the exponential fitting and very good linearity is observed based on the quantity $\exp(-R^+/c)$, with a maximum relative error of about 1%. The characteristic length monotonically decreases with the corner radius and the decreasing is more significant as Reynolds number increases, as revealed by the increasing k and decreasing c . The smaller recirculation bubbles for the rounded cylinders are resulted from the gradual delay of separation at the rear corners as R^+ increases.

3.2.2 Direct mode analysis

The flow is unstable to two-dimensional perturbation as characterized by the wake instability. The variation of growth rate of the unstable mode with Reynolds number and corner radius is given in Fig. 6. The growth rate monotonically increases with the Reynolds number for the square cylinder case as listed in the figure. For the rounded cylinders, the growth rate is presented by its relative difference to that of the square cylinder case for clarity. It is noted that *the super-critical flow can be stabilized by (partially) rounding the cylinder*. At $Re=50$, the growth rate first decreases with R^+ and reaches its minimum at $R^+=0.26$, which is about 37% smaller than that of the square cylinder case. As R^+ further increases, the growth rate monotonically increases until $R^+=0.50$ where the magnitude is still about 20% smaller than that of the square cylinder case, indicating that the weakly super-critical flow past the circular cylinder is more stable than the square cylinder at this Reynolds number. This variation pattern is expected and is consistent with the findings of critical flow since the Reynolds number of 50 is only slightly above Re_{cr} for all corner radii. The stabilization of the flow is also observed at $Re=60$. The minimum growth rate occurs at $R^+=0.18$ where the magnitude is about 10% lower than that of the square cylinder case; the flow is destabilized as R^+ further increases. As the Reynolds number further increases, the flow could be stabilized until $Re=100$ but the stabilization is less effective as measured by the reduced magnitude of negative $\Delta\sigma$; the flow could only be weakly stabilized by slightly rounding the corners within a small range of R^+ . At $Re=110$ the growth rate monotonically increases with R^+ and the relative difference $\Delta\sigma$ is always positive, indicating that the flow past the square cylinder is the least unstable and the rounding the corners only makes the flow more unstable.

The effect of Reynolds number on the spatial structure of unstable mode is shown in Fig. 7

where the real part of perturbed streamwise velocity is plotted. The mode is antisymmetric and travels downstream similar to that of the critical flow. At $Re=50$, the magnitude of perturbed velocity first increases and then decreases in the streamwise direction and reaches its maximum at around $x/D=18$. Similar spatial structure is also observed for higher Reynolds numbers but with mode structures reduced in size in both directions. The effect of corner radius on the unstable mode is presented in Fig. 8 at $Re=80$, and we note (without additional plots) that similar behavior occurs for other Reynolds numbers. The increasing corner radius leads to the strong perturbation in the wake close to the cylinder. By comparing Fig. 7 and Fig. 8, it is noticed that the effect of corner radius is less significant than the Reynolds number.

3.2.3 Growth rate sensitivity to base flow modification

In this section the stabilizing effect of the rounded corners on the super-critical flow is analyzed by examining the growth rate sensitivity to base flow modification. Fig. 9 exhibits the spatial structure of $\nabla_U \sigma$ at several corner radii at $Re=50$. Similar to the findings for the critical flow (Fig. 3), the growth rate sensitivity at $Re=50$ is determined by the backflow in the wake, roughly at $x/D < 4$, for all corner radii, where the flow could be destabilized by the intensification of local backflow velocity. It is seen in Fig. 9 that for $R^+ \leq 0.20$, the magnitude of $\nabla_U \sigma$ is the most pronounced on the wake centerline at $x/D = [1.0, 3.5]$, reflecting that the growth rate can be increased by an intensified backflow in both near- and far-wake regions. For $R^+ = 0.26$ where the growth rate is the minimum at this Reynolds number, there are two regions where the magnitude of $\nabla_U \sigma$ is the most significant: one is the near-wake region at $x/D = [0.8, 1.4]$, while the second is the far-wake region relatively further away from the cylinder at $x/D = [1.6, 3.5]$; the region in-between has a weaker sensitivity magnitude. For corner radii of

$R^+=0.40$ and 0.50 , the magnitude of $\nabla_U \sigma$ is still significant in the near-wake region but is weakened in the far-wake region, and the growth rate is most sensitive to the near-wake base flow. The spatial variation trend of growth rate sensitivity with respect to corner radius for this weakly super-critical flow is the same as that of critical flow in Fig. 3.

The streamwise distribution of the backflow velocity on the wake centerline is shown in Fig. 10 to explain the flow stabilization by the rounded corners. At $Re=50$, the magnitude of backflow velocity monotonically decreases with R^+ in the far-wake region $x/D=2.0$. The variation of the backflow velocity is opposite in orientation to the growth rate sensitivity (see Fig. 9a), thus the flow instability is weakened and the flow is getting more stable in the range $R^+=[0.00, 0.26]$. For $R^+=[0.27, 0.50]$, the growth rate is more sensitive to the near-wake backflow; the increased magnitude of backflow velocity with R^+ in the near-wake region $x/D<1.8$ destabilizes the flow, resulting in the increasing growth rate.

The variations of growth rate sensitivity and backflow velocity distribution with the corner radius at other Reynolds numbers are similar to those shown in Fig. 9 and 10a. For small R^+ , the magnitude of the sensitivity is first significant in a large region in the wake. As the corner radius increases, the sensitivity gets significantly weakened in the far-wake but remains pronounced in the near-wake. The stabilization of flow over partially rounded cylinders is correlated with *the weakening of backflow in the far-wake region where the growth rate is determined*. However, as the Reynolds number increases, the intensified backflow brought on by increasing R^+ is notably observed for a larger region in the near-wake, i.e., from the cylinder rear to about $x/D=2.2$ at $Re=60$ and to $x/D=3.0$ at $Re=100$, indicating that the near-wake backflow governing the flow stability is continuously intensified. Consequently, the flow is

prone to be more unstable as R^+ increases at higher Reynolds numbers. This explains the observation in Fig. 6 that for high Reynolds numbers, the stabilizing effect is somewhat small and the flow is destabilized as the corner radius increases for a large range of R^+ .

3.2.4 Comparison with direct simulation

The present stability analysis is an attempt to investigate the effect of cylinder geometry on the temporal growth of perturbation for the weakly super-critical flow. The findings are somewhat counterintuitive in that the flow may be stabilized (with smaller growth rate) by rounding the corners of the cylinder compared with the square cylinder case for $Re \leq 100$, and the stabilization is the most effective for the partially rounded cylinders with the minimum growth rate observed at certain corner radius. In order to verify the above findings, we perform comparative two-dimensional direct simulations to study the same physical problem as an initial value problem. The original governing equations (1) are directly integrated in time using our in-house code. The direct simulation is initialized with the base flow, and the flow gets perturbed by the round-off error and numerical residual during the time integration. The time history of the perturbed transverse velocity is recorded at probes in the wake and shown in Fig. 11. The transverse velocity fluctuates in time and its maximum magnitude is indicative of the growth rate of the perturbed velocity (v'). The perturbation grows linearly in the natural logarithmic scale at all probes in near- and far-wake regions, and the measured growth rates at all probes are identical. In Fig. 11b we present the time history of the local maximum magnitude measured by the near-wake probe. Perfect linear pattern is observed for all cases during the initial stage of perturbation growth. Due to the small growth rate of this type of flow ($\sigma \sim O(10^{-2}-10^{-1})$), the temporal growth of perturbation is rather slow, thus the linearity

assumption is valid for a sufficiently long duration after the perturbation is stimulated. The growth rate of perturbation in the direct simulation is measured by the slope of the linear fit in Fig. 11b, and the frequency is measured by the periodic fluctuation in Fig. 11a (and also the horizontal distance between neighboring symbols in Fig. 11b). The growth rate and frequency computed from the direct simulation are listed in Tab. 3 in comparison with those from the global stability analysis. Very good agreement is achieved between the results obtained by these two approaches, which further verifies the reliability of the stability analysis code. The large relative error for the growth rate at $Re=50$ is mainly attributed to its small magnitude that the prediction by direct simulation is affected by the sampling error as in Fig. 11.

4 Conclusions

We perform global linear stability analysis on flow past an isolated cylinder with rounded corners. The geometry of the cylinder is characterized by the non-dimensional corner radius R^+ varied from a square to a circular geometry. The objective of the present work is to investigate the effect of cylinder geometry on the stability characteristics of the flow. The study is performed using our in-house finite difference code, and the results are well verified against previous similar works, and also with direct simulations. The effect of rounded corners on the flow stability is quantified in a twofold manner: by the critical Reynolds number for the onset of first instability, and by the temporal growth rate of the perturbation for the weakly super-critical flow at $Re=[50, 110]$.

We first reveal the physics that *the flow can be stabilized by partially rounding the cylinder corners* in the critical and weakly super-critical flow regimes. The critical Reynolds number for the onset of first instability is 44.81 for the square cylinder case, while it increases with the

corner radius and reaches the maximum 46.96 at $R^+=0.30$, and then decreases to 46.59 for the circular cylinder. Although the variation of Re_{cr} is tiny compared with its magnitude, the stabilizing effect is confirmed as the onset of first instability is delayed when the cylinder corners are partially rounded. The super-critical flow is also stabilized by the partially rounded corners, as indicated by the reduced growth rate of perturbation. For the super-critical flow at $Re=50$, the growth rate reaches minimum at $R^+=0.26$, which is about 37% smaller than that of the square cylinder case. The stabilization persists as the Reynolds number increases until $Re=100$ but is gradually less effective in the smaller growth rate reduction. As the Reynolds number or corner radius increases, the perturbed velocity is more dominant in the near-wake and rapidly damped in the far-wake, and the spatial extents in both directions are reduced.

The physical mechanism leading to the flow stabilization is investigated by analyzing the sensitivity of perturbation growth rate to the base flow modification. It is shown that for both critical and weakly super-critical flows, the variation of growth rate with respect to corner radius is determined by *the variation of the backflow velocity in different regions in the wake*, i.e., in the near- and far-wake regions. For square-like cylinders, the growth rate is the most sensitive to the backflow in both near- and far-wake regions, with the latter being more crucial; the rounding of the cylinder corners weakens the backflow velocity in the far-wake and thus stabilizes the flow. As the corner radius keeps increasing that the cylinder becomes circular-like, the growth rate is only sensitive to the near-wake backflow, which is enhanced as the cylinder corners are rounded and the flow becomes more unstable.

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Nomenclature

D	cylinder diameter	U, V, P	base flow velocity and pressure
f	frequency	$\hat{u}, \hat{v}, \hat{p}$	amplitude function
R	corner radius	$\hat{u}^+, \hat{v}^+, \hat{p}^+$	adjoint mode amplitude function
R^+	nondimensional corner radius	U_0	incoming flow velocity
Re	Reynolds number	x, y	Cartesian coordinate
t	time	λ	complex circular frequency
u, v, p	velocity and pressure	σ	growth rate
u', v', p'	perturbed velocity and pressure	ω	vorticity

Figure and table captions

Fig. 1. Left: schematic of physical domain, cylinder geometry and coordinate system. Right: enlarged view of a typical mesh with every fourth gridline shown in each direction for clarity.

Fig. 2. Contour of the eigenfunction at $Re=47.0$ and $R^+=0.30$. Real part of (a) streamwise velocity $Re(\hat{u})$ and (b) transverse velocity $Re(\hat{v})$.

Fig. 3. Growth rate sensitivity to base flow modification $\nabla_u \sigma$ of the unstable mode at Reynolds number slightly above the critical value: (a) $R^+=0.00$, $Re=44.9$; (b) $R^+=0.10$, $Re=46.2$; (c) $R^+=0.20$, $Re=46.8$; (d) $R^+=0.30$, $Re=47.0$; (e) $R^+=0.40$, $Re=46.9$; (f) $R^+=0.50$, $Re=46.6$. The magnitude of $\nabla_u \sigma$ is visualized by the contour and its orientation by arrows.

Fig. 4. Distribution of streamwise velocity of base flow along the wake centerline.

Fig. 5. Fitting of L_r with corner radius as the function $L_r/D = k \exp(-R^+/c) + b$. The hollow symbols are the computed results and dashed lines are the fitting curves.

Fig. 6. Dependency of growth rate on Reynolds number and corner radius with respect to that of the square cylinder case $\Delta\sigma = \sigma - \sigma_{R^+=0.00}$, the latter is listed in the figure; the plus symbol marks the minimum value. The Reynolds number increases in the direction of the arrow.

Fig. 7. Eigenfunction $Re(\hat{u})$ at $R^+=0.50$.

Fig. 8. Eigenfunction $Re(\hat{u})$ at $Re=80$.

Fig. 9. Growth rate sensitivity to base flow modification $\nabla_u \sigma$ of the unstable mode at $Re=50$: (from a to f) $R^+=0.00, 0.10, 0.20, 0.26, 0.40, 0.50$. The magnitude of $\nabla_u \sigma$ is visualized by the contour and its orientation by arrows.

Fig. 10. Distribution of streamwise velocity of base flow along the wake centerline.

Fig. 11. Temporal growth of perturbed transverse velocity $|v - V| / U_0$ at $Re=50$ for the

two-dimensional direct simulation: (a) growth history of perturbation probed at different locations on the wake centerline; the dashed lines approximate the local maxima in each curve; (b) temporal history of local maxima for different corner radii; the dashed lines are the linear fitting curves.

Table 1. Mesh dependency of growth rate and frequency of the primary wake instability for flow past a square or circular cylinder. In each element the first value is growth rate and the second is frequency. All the reference results listed in the table are extracted from the figures in literatures, as denoted by the superscript asterisk for the estimated value.

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R^+	Source	Mesh (ξ, η)	$Re=50$	$Re=110$
0.00	[23]	419×181	(0.01699, -)*	(0.11539, -)*
	Present	256×256	(0.01782, 0.10252)	(0.11586, 0.09333)
		256×512	(0.01761, 0.10246)	(0.11521, 0.09322)
		512×512	(0.01709, 0.10280)	(0.11654, 0.09465)
		256×1024	(0.01763, 0.10229)	(0.11498, 0.09276)
0.50	[11]	≈375,000	(0.01306, 0.11709)*	-
	[9]	350×600	(0.01327, 0.11883)*	-
	Present	256×256	(0.01446, 0.11648)	(0.13197, 0.11242)
		256×512	(0.01377, 0.11656)	(0.13126, 0.11263)
		512×512	(0.01359, 0.11668)	(0.13223, 0.11283)
		256×1024	(0.01331, 0.11662)	(0.13107, 0.11277)

Table 1. Mesh dependency of growth rate and frequency of the primary wake instability for flow past a square or circular cylinder. In each element the first value is growth rate and the second is frequency. All the reference results listed in the table are extracted from the figures in literatures, as denoted by the superscript asterisk for the estimated value.

R^+	Re_{cr}	f_{cr}
0.00	44.81	0.10197
0.10	46.10	0.10536
0.20	46.77	0.10829
0.30	46.96	0.11082
0.40	46.83	0.11327
0.50	46.59	0.11596

Table 2. Critical Reynolds number and frequency.

Re	Item	Source	$R^+=0.00$	0.10	0.20	0.30	0.40	0.50
50	σ	Stability analysis	0.01709	0.01287	0.01100	0.01081	0.01189	0.01359
		Direct simulation	0.01681	0.01308	0.01104	0.01103	0.01214	0.01369
		Error(%)	1.67	1.61	0.36	1.99	2.06	0.73
	f	Stability analysis	0.10280	0.10622	0.10905	0.11156	0.11400	0.11668
		Direct simulation	0.10283	0.10621	0.10911	0.11153	0.11395	0.11661
		Error(%)	0.03	0.01	0.05	0.02	0.04	0.06
	σ	Stability analysis	0.11654	0.11754	0.11999	0.12351	0.12770	0.13223
		Direct simulation	0.11544	0.11739	0.11998	0.12354	0.12773	0.13232
		Error(%)	0.96	0.13	0.01	0.02	0.02	0.07
110	f	Stability analysis	0.09465	0.10357	0.10807	0.11058	0.11195	0.11283
		Direct simulation	0.09467	0.10359	0.10807	0.11061	0.11197	0.11294
		Error(%)	0.02	0.02	0.00	0.03	0.02	0.10

Table 3. Comparison of growth rate and frequency of perturbation predicted by two-dimensional direct simulation and global linear stability analysis.