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Bridging Asymptotic Independence and Dependence in Spatial Extremes Using Gaussian Scale Mixtures

Raphaël Huser¹, Thomas Opitz² and Emeric Thibaud³

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Abstract

Gaussian scale mixtures are constructed as Gaussian processes with a random variance. They have non-Gaussian marginals and can exhibit asymptotic dependence unlike Gaussian processes, which are asymptotically independent except in the case of perfect dependence. In this paper, we study the extremal dependence properties of Gaussian scale mixtures and we unify and extend general results on their joint tail decay rates in both asymptotic dependence and independence cases. Motivated by the analysis of spatial extremes, we propose flexible yet parsimonious parametric copula models that smoothly interpolate from asymptotic dependence to independence and include the Gaussian dependence as a special case. We show how these new models can be fitted to high threshold exceedances using a censored likelihood approach, and we demonstrate that they provide valuable information about tail characteristics. In particular, by borrowing strength across locations, our parametric model-based approach can also be used to provide evidence for or against either asymptotic dependence class, hence complementing information given at an exploratory stage by the widely used nonparametric or parametric estimates of the χ and $\bar{\chi}$ coefficients. We demonstrate the capacity of our methodology by adequately capturing the extremal properties of wind speed data collected in the Pacific Northwest, US.

Keywords: asymptotic dependence and independence; censored likelihood inference; spatial copula; extreme event; random scale model; threshold exceedance.

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1 Introduction

Gaussian processes have been used extensively in classical spatial statistics thanks to their appealing theoretical properties, tractability in high dimensions, explicit conditional distributions and ease of simulation ([Gelfand and Schliep, 2016](#)). However, as far as the modeling of extremes is concerned, they have been heavily criticized ([Davison et al., 2013](#)) as being unable to capture asymptotic dependence; Gaussian processes are asymptotically independent, meaning that the dependence strength between events observed at two distinct spatial locations vanishes as their extremeness increases. Without firm knowledge about the tail properties of the data, it is more conservative in terms of probability of concurrent extremes to assume asymptotic dependence. Using stochastic processes that lack flexibility in the joint tail may lead to severe under- or overestimation of probabilities associated to simultaneous extreme events. This lack of solid theoretical foundations for extrapolation beyond the range of the observations has been a catalyst for extensive research in extreme-value theory (EVT).

Classical EVT provides support for the use of max-stable models for block-maxima (e.g., annual maxima of daily temperature or precipitation), because they are the only possible limits of renormalized pointwise maxima of spatial processes ([de Haan and Ferreira, 2006](#)). Their strong asymptotic justification is both a blessing and a curse: max-stability provides a robust modeling framework when few extreme data are available, but this strong assumption may be far from satisfied at subasymptotic levels arising with finite samples. An instructive example is asymptotic independence, where the limiting max-stable distribution is the product of independent margins and cannot capture the potentially strong dependence that remains at extreme subasymptotic levels. In addition to this possibly large gap between the theory and the data, inference for max-stable models is tricky. Full likelihoods can only be calculated in small dimensions, which led to the use of less efficient inference techniques, such as composite likelihoods ([Padoan et al., 2010](#); [Huser and Davison, 2013](#); [Castruccio et al., 2016](#)). [Thibaud et al. \(2016\)](#) and [Dombry et al. \(2016\)](#) recently showed how the full likelihood may be approximated in Bayesian or frequentist settings by integrating out a random partition using Monte Carlo techniques; however, these approaches remain computer-intensive in large dimensions. Alternatively, models for threshold exceedances based on the limit-

ing Poisson ([Wadsworth and Tawn, 2014](#); [Engelke et al., 2015](#)) or the generalized Pareto ([Ferreira and de Haan, 2014](#); [Thibaud and Opitz, 2015](#)) process are the counterparts of max-stable models for threshold exceedances and have become increasingly popular because they circumvent many of the computational bottlenecks of max-stable processes. However, analogous to max-stable processes, Pareto processes are threshold-stable and thus lack tail flexibility, especially when fitted to asymptotically independent data. Very extreme joint risks tend to be overestimated by these models if the data exhibit decreasing dependence strength at more extreme levels.

Because of the practical limitations of ultimate models, such as max-stable or Pareto processes, it is natural to seek penultimate (i.e., subasymptotic) models for spatial extremes, which combine tail flexibility with computational tractability and have known tail characteristics, in the same vein as penultimate approximations in univariate EVT ([Kaufmann, 2000](#)). In the case of asymptotic independence, Gaussian models might be reasonable. Alternatively, [Wadsworth and Tawn \(2012\)](#) proposed inverted max-stable models, which were found to be more flexible than Gaussian processes in some applications ([Thibaud et al., 2013](#); [Davison et al., 2013](#)), but are as difficult to fit as max-stable models. A more complex Bayesian nonparametric copula model was proposed by [Fuentes et al. \(2013\)](#). Recently, [Opitz \(2016\)](#) advocated a specific Gaussian scale mixture model designed for asymptotic independence, constructed from the product of a standard Gaussian process with a random variance following the exponential distribution, yielding Laplace random fields. In the case of asymptotic dependence, subasymptotic models were also proposed. [Wadsworth and Tawn \(2012\)](#) developed max-mixtures involving inverted max-stable and max-stable models, which add flexibility over the latter at the price of a relatively large number of parameters to be estimated. [Morris et al. \(2017\)](#) proposed a Bayesian space-time skew- t model for threshold exceedances. [Krupskii et al. \(2017\)](#) proposed factor copula models constructed from Gaussian processes with a random mean, which can capture asymptotic dependence or independence.

The above spatial models are useful in many respects by introducing more flexibility or by improving computation over max-stable models, but they focus on modeling exclusively either asymptotically independent or asymptotically dependent data. In contrast, the pseudo-polar representations of multivariate limit distributions have motivated [Wadsworth](#)

[et al. \(2017\)](#) to explore how more flexible transitions between dependence classes can be achieved in the bivariate case through a common random scaling applied to a random vector on the unit sphere, the latter being defined from a norm on $\mathbb{R}^2$. Using the pseudo-polar representation of multivariate elliptical distributions such as the multivariate Gaussian, we argue in this paper that a flexible and natural extension of this approach to spatial modeling consists in using the wide class of randomly scaled Gaussian processes, also known as Gaussian scale mixtures, which comprise all infinite-domain processes with nondegenerate finite-dimensional elliptical distributions ([Huang and Cambanis, 1979](#)).

A main theoretical contribution of this paper is that we give detailed results on the joint tail decay rates of Gaussian scale mixture processes under general assumptions on their random scale, defining conditions to capture asymptotic dependence or asymptotic independence. This tail characterization then leads to our main methodological novelty: we propose new spatial subasymptotic copula models, which smoothly bridge the two asymptotic dependence regimes and allow estimating the latter from the data. The model type is usually chosen a priori using the coefficients χ and $\bar{\chi}$ ([Coles et al., 1999](#)), see §4.2, whose nonparametric estimation entails large uncertainties and does not yield a spatially coherent model. In contrast, our model-based estimators, which extend the parametric estimators of [Coles et al. \(1999\)](#) to the spatial setting, are more reliable and allow borrowing strength across locations for better estimation and discrimination between the two asymptotic classes. We also show how to exploit the underlying Gaussian structure of our new models to make inference based on a full likelihood with partial censoring and to efficiently perform conditional simulation, which is more tricky when max-stable models are involved ([Dombry et al., 2013](#)).

For modeling and estimation, we treat marginal distributions and dependence structure separately in a copula framework. By Sklar's Theorem ([Sklar, 1959](#)), any continuous joint distribution $G(\mathbf{x})$, $\mathbf{x} = (x_1, \dots, x_D)^T \in \text{Supp}(G) \subset \mathbb{R}^D$, with univariate margins $G_1, \dots, G_D$ may be uniquely represented as

$$G(\mathbf{x}) = C\{G_1(x_1), \dots, G_D(x_D)\} \iff C(\mathbf{u}) = G\{G_1^{-1}(u_1), \dots, G_D^{-1}(u_D)\}, \quad (1)$$

where $C(\mathbf{u})$, $\mathbf{u} = (u_1, \dots, u_D)^T \in [0, 1]^D$, is the copula, also called the dependence function. In §2 we define Gaussian scale mixture models that are elliptic extensions of Gaussian

processes from which flexible elliptical copula families can be derived.

The definition and properties of Gaussian scale mixtures are detailed in §2. In §3.1, we unify and extend theoretical results on joint tail decay rates and detail the conditions leading to asymptotic dependence or asymptotic independence of Gaussian scale mixtures. Parametric modeling is discussed in §3.2, and we propose several new models able to transition from one asymptotic regime to the other. Full likelihood inference is discussed in §4.1, followed by a simulation study in §4.2. We illustrate our modeling approach in §5 by analyzing wind speed extremes in the Pacific Northwest, US. §6 concludes with some discussion. Proofs are deferred to the Appendix.

2 Gaussian scale mixture processes

2.1 Construction

To create flexible spatial models, we define a Gaussian scale mixture process (i.e., a Gaussian process with random variance) as follows:

$$X(\mathbf{s}) = RW(\mathbf{s}), \quad \mathbf{s} \in \mathcal{S} \subset \mathbb{R}^d, \quad (2)$$

where $W(\mathbf{s})$ is a standard Gaussian process with correlation function $\rho(\mathbf{s}_1, \mathbf{s}_2)$, and $R \sim F(r)$ is a positive random variable, independent of $W(\mathbf{s})$, with distribution $F(r)$ and density $f(r)$ if the latter exists, which we will assume in the remainder of the paper if not stated otherwise. Conditional on R , the random process $X(\mathbf{s})$ is Gaussian with zero mean and variance R^2 . Gaussian processes arise as a special case when $R = r_0$ almost surely for some $r_0 > 0$. In this paper, we use the copula associated to (2) through (1) as a model for extremal dependence.

2.2 Finite dimensional distributions

When the process (2) is observed at D spatial locations, $\mathbf{s}_1, \dots, \mathbf{s}_D \in \mathcal{S}$, we write $X_j = X(\mathbf{s}_j)$ and $W_j = W(\mathbf{s}_j)$, $j = 1, \dots, D$, yielding the random vectors $\mathbf{X} = (X_1, \dots, X_D)^T$ and $\mathbf{W} = (W_1, \dots, W_D)^T$. From (2), one has the representation

$$\mathbf{X} = R\mathbf{W}, \quad R \sim F(r) \perp\!\!\!\perp \mathbf{W} \sim \mathcal{N}_D(\mathbf{0}, \Sigma), \quad (3)$$

where Σ is a correlation matrix determined by the spatial configuration of sites, and $\mathbf{0} \in \mathbb{R}^D$ is the zero vector. By conditioning on R , the distribution G and density g of $\mathbf{X}$ are

$$G(\mathbf{x}) = \int_0^\infty \Phi_D(\mathbf{x}/r; \Sigma) f(r) dr, \quad g(\mathbf{x}) = \int_0^\infty \phi_D(\mathbf{x}/r; \Sigma) r^{-D} f(r) dr, \quad \mathbf{x} \in \mathbb{R}^D, \quad (4)$$

where $\Phi_D(\cdot; \Sigma)$ and $\phi_D(\cdot; \Sigma)$, respectively, denote the D -variate Gaussian distribution and density with zero mean and covariance matrix Σ . Some non-trivial choices of the mixing density $f(r)$ lead to a closed-form expression of the density $g(\mathbf{x})$, including the Student- t , Laplace and slash models (Kotz et al., 2004). In general, the unidimensional integrals in (4) can be accurately approximated using numerical integration. Marginal distributions G_k and their corresponding densities g_k , $k = 1, \dots, D$, are

$$G_k(x_k) = \int_0^\infty \Phi(x_k/r) f(r) dr, \quad g_k(x_k) = \int_0^\infty \phi(x_k/r) r^{-1} f(r) dr, \quad x_k \in \mathbb{R}, \quad (5)$$

where $\Phi(\cdot) = \Phi_1(\cdot; 1)$ and $\phi(\cdot) = \phi_1(\cdot; 1)$ denote the univariate standard Gaussian distribution and density, respectively. The censored likelihood defined in §4.1 requires the partial derivatives of the distribution G in (4). Let $I \subset \{1, \dots, D\}$ be a set of indices of cardinality $|I|$ corresponding to components exceeding a high threshold in §4.1, with its complement in $\{1, \dots, D\}$ denoted by I^c , and let $\mathbf{x} = (x_1, \dots, x_D)^T \in \mathbb{R}^D$. For any sets of indices $A, B \subset \{1, \dots, D\}$, let $\mathbf{x}_A$ denote the subvector of $\mathbf{x}$ obtained by retaining the components indexed by A , let $\Sigma_{A;B}$ denote the matrix Σ restricted to the rows in A and the columns in B , and let $\Sigma_{A|B} = \Sigma_{A;A} - \Sigma_{A;B} \Sigma_{B;B}^{-1} \Sigma_{B;A}$. Differentiation of G in (4) yields

$$\begin{aligned} G_I(\mathbf{x}) &:= \frac{\partial^{|I|}}{\partial \mathbf{x}_I} G(\mathbf{x}) = \int_0^\infty \frac{\partial^{|I|}}{\partial \mathbf{x}_I} \Phi_D(\mathbf{x}/r; \Sigma) f(r) dr \\ &= \int_0^\infty \Phi_{|I^c|} \{(\mathbf{x}_{I^c} - \Sigma_{I^c;I} \Sigma_{I;I}^{-1} \mathbf{x}_I)/r; \Sigma_{I^c|I}\} \phi_{|I|}(\mathbf{x}_I/r; \Sigma_{I;I}) r^{-|I|} f(r) dr, \end{aligned} \quad (6)$$

which involves the conditional Gaussian distribution of $\mathbf{X}_{I^c}$ given $\mathbf{X}_I = \mathbf{x}_I/r$. The computation of G in (4) and G_I in (6) relies on the Gaussian distribution function in dimension D and $|I^c|$ respectively, which can be estimated without bias (Genz and Bretz, 2009).

2.3 Interpretation as elliptic processes

Gaussian scale mixtures (3) are elliptically contoured distributions (Cambanis et al., 1981), which may be written in pseudo-polar representation as

$$\mathbf{X} \stackrel{D}{=} R^* \Sigma^{1/2} \mathbf{U}, \quad (7)$$

where $\Sigma = \Sigma^{1/2} \Sigma^{T/2}$ is a covariance matrix and $R^* \geq 0$ is a positive random variable, called radius, that is independent of a random vector $\mathbf{U} = (U_1, \dots, U_D)^T$ uniformly distributed on the Euclidean unit sphere in $\mathbb{R}^D$ (i.e., $\|\mathbf{U}\|_2 = 1$). We assume that Σ is invertible if not stated otherwise. Elliptical distributions can be viewed as a random scaling of a uniform random vector residing on the unit sphere defined with respect to the Mahalanobis norm $\|\mathbf{x}\|_\Sigma = \sqrt{\mathbf{x}^T \Sigma^{-1} \mathbf{x}}$. Using elliptic theory (Cambanis et al., 1981), one can equivalently rewrite the multivariate density in (4) as

$$g(\mathbf{x}) = |\Sigma|^{-1/2} h_D(\|\mathbf{x}\|_\Sigma^2), \quad \mathbf{x} \in \mathbb{R}^D,$$

for some function $h_D : [0, \infty) \rightarrow [0, \infty)$. The D -variate Gaussian distribution is characterized by $h_D(t) = (2\pi)^{-D/2} \exp(-t/2)$, $t \geq 0$. A change of variables gives the density f_{R^*} of the radial component R^* in (7) as $f_{R^*}(r) = A_D r^{D-1} h_D(r^2)$, $r > 0$, where A_D denotes the surface area of the unit ball in $\mathbb{R}^D$ (i.e., $A_1 = 1$ and $A_D = 2\pi^{D/2} \{\Gamma(D/2)\}^{-1}$, $D > 1$, with the gamma function $\Gamma(\cdot)$). Because the Gaussian vector $\mathbf{W}$ in (3) is itself elliptic, the radius of $\mathbf{X} = R\mathbf{W}$ is $R^* = RR_{\mathbf{W}}$ with $R_{\mathbf{W}}$ distributed according to the chi-distribution with D degrees of freedom has density $f_{R_{\mathbf{W}}}(r) = 2^{1-D/2} \{\Gamma(D/2)\}^{-1} r^{D-1} \exp(-r^2/2)$, $r > 0$.

2.4 Conditional distributions and simulation algorithm

We now develop an efficient algorithm for conditional simulation of Gaussian scale mixtures, which is crucial for spatial prediction and estimation of complex spatial functionals. For $\mathbf{X} = (\mathbf{X}_1^T, \mathbf{X}_2^T)^T = R(\mathbf{W}_1^T, \mathbf{W}_2^T)^T$ a D -dimensional Gaussian scale mixture partitioned into subvectors $\mathbf{X}_1$ and $\mathbf{X}_2$ of dimensions $D_1 \in \{1, \dots, D-1\}$ and $D_2 = D - D_1$, respectively, we first derive the conditional distributions of $\mathbf{X}_2$ given $\mathbf{X}_1$ and of the latent variable R given $\mathbf{X}_1$. Let $\Sigma_{i;j}$, $i, j \in \{1, 2\}$, denote the corresponding blocks of the covariance matrix Σ of $\mathbf{W} = (\mathbf{W}_1^T, \mathbf{W}_2^T)^T$ and $\Sigma_{i|j} = \Sigma_{i;i} - \Sigma_{i;j} \Sigma_{j;j}^{-1} \Sigma_{j;i}$.

Theorem 1 (Conditional distributions). *The conditional distribution of $\mathbf{X}_2$ given $\mathbf{X}_1 = \mathbf{x}_1$ is elliptic with density*

$$f_{\mathbf{X}_2|\mathbf{X}_1=\mathbf{x}_1}(\mathbf{x}_2) = c_0^{-1} |\Sigma_{2|1}|^{-1/2} h_D \left\{ (\mathbf{x}_2 - \boldsymbol{\mu}_{2|1})^T \Sigma_{2|1}^{-1} (\mathbf{x}_2 - \boldsymbol{\mu}_{2|1}) + c_1 \right\}, \quad \mathbf{x}_2 \in \mathbb{R}^{D_2}, \quad (8)$$

Figure 1: Conditional simulations from an (asymptotically independent) Gaussian copula (left), an (asymptotically dependent) Student- t copula with 3 degrees of freedom (middle) and one of our new proposed Gaussian scale mixtures defined in (14) with $\beta = \gamma = 1$ leading to asymptotic independence (right), displayed on exponential margins. The underlying Gaussian field has exponential correlation function $\rho(\mathbf{s}_1, \mathbf{s}_2) = \exp(-\|\mathbf{s}_1 - \mathbf{s}_2\|/0.1)$. Simulation is conditional to the central grid cell at $(0.5, 0.5)$ being equal to the 99.99%-quantile. The conditional simulation of the scale variable $\tilde{R}$ is based on a Metropolis-Hastings algorithm. The three conditional simulations of the Gaussian component $\tilde{\mathbf{W}}$ use the same random seed.

where $\boldsymbol{\mu}_{2|1} = \boldsymbol{\Sigma}_{2;1}\boldsymbol{\Sigma}_{1;1}^{-1}\mathbf{x}_1$, $c_0 = A_{D_2} \int_0^\infty h_D(r^2 + c_1)r^{D_2-1}dr$, $c_1 = \mathbf{x}_1^T \boldsymbol{\Sigma}_{1;1}^{-1}\mathbf{x}_1$ and

$$h_D(t) = A_D^{-1}t^{(1-D)/2}f_{RRW}(\sqrt{t}), \quad f_{RRW}(r) = \int_0^\infty F(s^{-1}) \{f_{RW}(rs) + f'_{RW}(rs)rs\} ds.$$

It has pseudo-polar representation $\boldsymbol{\mu}_{2|1} + R_{2|1}\boldsymbol{\Sigma}_{2|1}^{1/2}\mathbf{U}$ with $\boldsymbol{\Sigma}_{2|1} = \boldsymbol{\Sigma}_{2|1}^{1/2}\boldsymbol{\Sigma}_{2|1}^{T/2}$, $\|\mathbf{U}\|_2 = 1$ and radius $R_{2|1}$ whose density is $f_{R_{2|1}}(r) = A_{D_2}c_0^{-1}r^{D_2-1}h_D(r^2 + c_1)$, $r > 0$. The conditional density of R given $\mathbf{X}_1 = \mathbf{x}_1 \in \mathbb{R}^{D_1}$, with $1 \leq D_1 \leq D$, is

$$f_{R|\mathbf{X}_1=\mathbf{x}_1}(r) = r^{-D_1}f(r)\phi_{D_1}(\mathbf{x}_1/r; \boldsymbol{\Sigma}_{1;1})/g(\mathbf{x}_1), \quad r \geq 0. \quad (9)$$

Simulation of $\mathbf{X}_2$ conditional on $\mathbf{X}_1 = \mathbf{x}_1$ can be done either by directly simulating the elements of the pseudo-polar representation $\boldsymbol{\mu}_{2|1} + R_{2|1}\boldsymbol{\Sigma}_{2|1}^{1/2}\mathbf{U}$ or by exploiting the latent Gaussian structure in a two-step procedure: to simulate $R\mathbf{W}_2$ conditional on $R\mathbf{W}_1 = \mathbf{x}_1$, first generate a realization $\tilde{r}$ of the conditional scale variable $\tilde{R}$ according to its density $f_{R|\mathbf{X}_1=\mathbf{x}_1}$ in (9), then sample a realization $\tilde{\mathbf{w}}_2$ of $\mathbf{W}_2$ conditional on $\tilde{R} = \tilde{r}$ and $\mathbf{X}_1 = \mathbf{x}_1$, i.e., sample $\tilde{\mathbf{w}}_2$ according to the conditional Gaussian distribution $\mathbf{W}_2 | \mathbf{W}_1 = \mathbf{x}_1/\tilde{r}$ with mean $\boldsymbol{\mu}_{2|1}/\tilde{r}$ and covariance matrix $\boldsymbol{\Sigma}_{2|1}$. Then, $\tilde{r}\tilde{\mathbf{w}}_2$ is a realization of the conditional vector $\mathbf{X}_2$ given $\mathbf{X}_1 = \mathbf{x}_1$.

To illustrate our conditional simulation algorithm based on the latent Gaussian structure, Figure 1 displays realizations of three Gaussian scale mixture models on $[0, 1]^2$, conditioned on being equal to the 99.99%-quantile at the point $(0.5, 0.5)$. By using the same random seed for the conditional Gaussian sample, the impact of choosing different random scale distributions emerges clearly. Large contiguous areas with high values arise for the asymptotically dependent Student- t model, while the Gaussian one generates spatially isolated peaks of highly variable size. The third simulation exemplifies one of the new models that we propose in §3.2, providing a continuous transition from Gaussian to asymptotic dependence, here with parameters leading to asymptotic independence. Simulation times are largely dominated by the conditional Gaussian simulation.

3 Tail behavior of Gaussian scale mixture models

3.1 Asymptotic independence and dependence

We now characterize the bivariate joint tail decay of Gaussian scale mixtures with two commonly used coefficients, χ and $\bar{\chi}$ (Coles et al., 1999), defined as $\chi := \lim_{u \rightarrow 1} \chi(u)$ and $\bar{\chi} := \lim_{u \rightarrow 1} \bar{\chi}(u)$, where

$$\chi(u) = 2 - \frac{\log\{C(u, u)\}}{\log(u)}, \quad \bar{\chi}(u) = \frac{2\log(1-u)}{\log\{\bar{C}(u, u)\}} - 1, \quad (10)$$

and $C(u_1, u_2)$ is the bivariate copula stemming from the pair of variables $(X_1, X_2)^T$ through (1) with survival copula $\bar{C}(u_1, u_2) = 1 - u_1 - u_2 + C(u_1, u_2)$. To get a deeper understanding of what these coefficients mean, assume that the random vector $(X_1, X_2)^T$ follows the bivariate distribution G with margins G_1 and G_2 , and that the joint tail may be expanded as $\Pr\{X_1 > G_1^{-1}(u), X_2 > G_2^{-1}(u)\} \sim \mathcal{L}\{(1-u)^{-1}\}(1-u)^{1/\eta}$, as $u \rightarrow 1$, where $\mathcal{L}$ is a slowly varying function (i.e., $\mathcal{L}(ax)/\mathcal{L}(x) \rightarrow 1$, as $x \rightarrow \infty$, with $a > 0$) and $\eta \in (0, 1]$ is the coefficient of tail dependence (Ledford and Tawn, 1996). From the definition (10), it can be deduced that $\chi(u) \sim \Pr\{X_1 > G_1^{-1}(u) \mid X_2 > G_2^{-1}(u)\} = \mathcal{L}\{(1-u)^{-1}\}(1-u)^{1/\eta-1}$, and that $\bar{\chi}(u) \sim 2\eta - 1$, as $u \rightarrow 1$. Two distinct asymptotic regimes can occur: the variables X_1 and X_2 are either asymptotically dependent when $\chi \in (0, 1]$ and $\bar{\chi} = 1$ (i.e., $\eta = 1$ and $\mathcal{L}\{(1-u)^{-1}\} \rightarrow 0$, as $u \rightarrow 1$), or they are asymptotically independent when $\chi = 0$ and

$\bar{\chi} \in (-1, 1]$ (i.e., $\eta \in (0, 1)$ or $\eta = 1$ and $\mathcal{L}\{(1-u)^{-1}\} \rightarrow 0$, as $u \rightarrow 1$). In practice, the pair of coefficients $\{\chi(u), \bar{\chi}(u)\}$ evaluated for increasingly large thresholds $u \in [0, 1]$ may help assess the asymptotic dependence class (Coles et al., 1999), and the bivariate joint tail decay rate is essentially governed by the parameter η (or equivalently, $\bar{\chi}$).

General results on joint tail decay rates can be derived from tail dependence characterizations of elliptic distributions (Berman, 1992) by taking into account the specific structure of the radial variable in Gaussian scale mixtures. We fix $D = 2$ and study the asymptotic properties of $\mathbf{X} = (X_1, X_2)^T = R(W_1, W_2)^T$, where the Gaussian vector $\mathbf{W} = (W_1, W_2)^T$ has correlation $\rho \in (-1, 1)$. We first recall results on asymptotic dependence and then provide new detailed results on asymptotic independence, which is obtained for the wide class of Weibull-like tail decay in R (see Theorem 2). Parametric modeling is discussed in §3.2.

Asymptotic dependence occurs when R is regularly varying at infinity, i.e.,

$$\frac{\Pr(R \geq tr)}{\Pr(R \geq t)} = \frac{1 - F(tr)}{1 - F(t)} = r^{-\gamma}, \quad r > 0, \quad t \rightarrow \infty, \quad (11)$$

for some $\gamma > 0$, i.e., when R has Pareto-like tail behavior. Then $\bar{\chi} = 1$ and $\chi = 2[1 - T\{(1+\gamma)^{1/2}(1-\rho)(1-\rho^2)^{-1/2}; \gamma+1\}]$, where $T(\cdot; \text{Df}) = T_1(\cdot; 1, \text{Df})$ is the univariate Student- t distribution with $\text{Df} > 0$ degrees of freedom. The joint tail can be written as $\bar{C}(1-1/x, 1-1/x) \sim \chi/x$ if $x \rightarrow \infty$, see Opitz (2013) and the references therein.

Theorem 2 (Asymptotic independence for Gaussian scale mixtures). *Suppose that*

$$\Pr(R \geq r) = 1 - F(r) \sim \alpha r^\gamma \exp(-\delta r^\beta), \quad r \rightarrow \infty, \quad (12)$$

for some constants $\alpha > 0$, $\beta > 0$, $\gamma \in \mathbb{R}$ and $\delta > 0$. Then $\chi = 0$ and $\bar{\chi} = 2\{(1+\rho)/2\}^{\beta/(\beta+2)} - 1$. The joint tail can be written as $\bar{C}\{1-1/x, 1-1/x\} = \mathcal{L}(x)x^{-2/(1+\bar{\chi})}$ for $x \rightarrow \infty$, where $\mathcal{L}(x) \sim K \log(x)^{(1-1/\eta)\frac{2\gamma+\beta}{2\beta}+1/(2\eta)-1}$ is a slowly varying function as $x \rightarrow \infty$ and K is a positive constant depending on α , β , γ and δ ; see the proof in the appendix.

We use the terms Pareto-type and Weibull-type distributions for variables R with tail representation (11) and (12), respectively. The case where R is deterministic or upper-bounded almost surely can be interpreted as a limit of (12) as $\beta \rightarrow \infty$ and in this case $\bar{\chi} = \rho$. Results on off-diagonal decay rates are in the Supplementary Material.

To better understand the extremal dependence structure of Gaussian scale mixture processes, we shortly recall related max-stable limits; technical details are given in the Supplementary Material. For a regularly varying scale distribution F as in (11) we get extremal- t limit processes (Opitz, 2013). By contrast, if F has a Weibull-type tail as defined by (12) and provided that the Gaussian correlation is not perfect between distinct sites, the asymptotic independent structure yields a “white noise” max-stable limit. In this case, more insight can be obtained by considering triangular arrays of Gaussian scale mixtures with correlation increasing to one at a certain speed (Hashorva, 2013), similar to standard results for Gaussian triangular arrays (Hüsler and Reiss, 1989); in this case, the Brown–Resnick process (Kablichko et al., 2009) results in the limit. Brown–Resnick processes further arise as certain limits of extremal- t processes, in such a way that appropriately defined triangular arrays of Gaussian scale mixtures with regularly varying scale variable converge to Brown–Resnick limits when the regular variation index γ in (11) tends to ∞ .

3.2 Models bridging asymptotic dependence and independence

When modeling dependence in spatial extremes, imposing a specific type of asymptotic behavior has important consequences on the estimation of return levels for spatial functionals. This model choice may be guided by a preliminary assessment of asymptotic dependence using the coefficients $\chi(u)$ and $\bar{\chi}(u)$ defined in (10), although nonparametric estimates may be highly variable especially with small sample sizes as uncertainties become increasingly large as $u \rightarrow 1$; see the simulation results reported in §4.2. Thus, preliminary selection of the asymptotic dependence regime may be awkward, and it is highly valuable to fit flexible models encompassing both regimes in order to borrow strength across all stations for efficient estimation of tail properties from the data. As Gaussian processes have been widely applied and advocated in spatial statistics, it is useful to have the Gaussian copula as a special case. Gaussian models may provide an appropriate description of the dependence in some applications, and it is instructive to assess how far the dependence structure is from Gaussianity. Based on these arguments, a useful distribution for R has to be chosen when fitting a Gaussian mixture model to observed extremes. We first describe some known parametric families that may be used to capture either asymptotic independence or asymptotic dependence,

then we propose novel parsimonious models that encompass the two dependence classes.

In the asymptotic independence case, a flexible distribution satisfying Weibull-type tail behavior as in (12) is the generalized gamma distribution; it covers the full range of Weibull coefficients $\beta > 0$ in (12) and encompasses several well-known simpler parametric families, including the gamma distribution, the Weibull, exponential, Nakagami, Rayleigh, chi, chi-squared, half-normal and log-normal ones; see the Supplementary Material for details. However, its three parameters may lead to issues of parameter parsimony and identifiability in practice. A Rayleigh-distributed random scale R yields Laplace random fields $X(\mathbf{s})$ with explicit joint densities $g(\mathbf{x})$ in (4) and a certain type of univariate generalized Pareto tails (Opitz, 2016). Asymptotic dependence models are obtained from a regularly varying random scale R ; see §3.1. One possibility is to choose the distribution of R^2 as the inverse-gamma distribution with rate and shape parameters equal to $\text{Df}/2$, yielding Student- t random fields with $\text{Df} > 0$ degrees of freedom (Røislien and Omre, 2006; Demarta and McNeil, 2005) and converging to the Gaussian limit as $\text{Df} \rightarrow \infty$. When R is Pareto distributed, the Gaussian scale mixture has multivariate slash densities available in closed form, albeit in terms of the incomplete gamma function (Wang and Genton, 2006).

We now explore models able to bridge the two asymptotic regimes while keeping flexibility in both cases. To assess the bivariate tail flexibility of Gaussian scale mixtures, we consider the range of χ and $\bar{\chi}$ coefficients that may be generated for fixed correlation $-1 < \rho < 1$ in the underlying bivariate Gaussian variable. Among the aforementioned models, only the (asymptotically dependent) Student- t model covers both regimes by considering its extension to the (asymptotically independent) Gaussian model on the boundary of its parameter space when $\text{Df} = \infty$. Any value of $\chi \in [0, 1]$ may be obtained by varying the degrees of freedom Df , with $\chi \downarrow 0$ as $\text{Df} \rightarrow \infty$, which implies that the Student- t model is fairly flexible in the asymptotic dependence case. However, using Theorem 2, only two values of $\bar{\chi}$ can be obtained, namely $\bar{\chi} = \rho$ when $\text{Df} = \infty$ or $\bar{\chi} = 1$ when $\text{Df} < \infty$. The jump in the $\bar{\chi}$ value when moving from asymptotic dependence to asymptotic independence implies a lack of flexibility to capture joint tail decay rates in the asymptotic independence regime due to the relatively rigid Gaussian tail structure. We now propose two new random scale models linking the two asymptotic dependence classes in a more flexible way.

For Model 1, we transpose the arguments of [Wadsworth et al. \(2017\)](#) concerning the choice of the random scaling in pseudo-polar representations to our framework, and we model the random scale using a generalized Pareto distribution,

$$F(r) = \begin{cases} 1 - (1 + \xi r)_+^{-1/\xi}, & \xi \neq 0, \\ 1 - \exp(-r), & \xi = 0, \end{cases}, \quad 0 \leq r < r^*, \quad (13)$$

with upper endpoint $r^* = \infty$ when $\xi \geq 0$ and $r^* = -1/\xi$ otherwise, and where $x_+ = \max(x, 0)$. The resulting Gaussian scale mixture model adapts the idea of [Wadsworth et al. \(2017\)](#) to infinite-dimensional domains by replacing the underlying random vector restricted to the sphere (with respect to a norm) by a Gaussian process. Model 1 is Pareto-like with tail parameter $\gamma := 1/\xi$ in (11) when $\xi > 0$, yielding asymptotic dependence with $\bar{\chi} = 1$ and $\chi \downarrow 0$ as $\xi \downarrow 0$. It is exponential (i.e., Weibull-like with tail parameter $\beta = 1$ in (12)) when $\xi = 0$ or $\xi \rightarrow 0$, yielding asymptotic independence with $\bar{\chi} = 2\{(1 + \rho)/2\}^{1/3} - 1$ thanks to Theorem 2. Finally, it is upper-bounded when $\xi < 0$, which gives asymptotic independence with $\bar{\chi} = \rho$. This model is parsimonious and rather flexible in the asymptotic dependence case, but contrary to the Student- t model, the transition between asymptotic dependence regimes takes place in the interior of the parameter space. However, it still lacks flexibility for asymptotic independence as only two specific values of $\bar{\chi} < 1$ can arise for fixed ρ . Therefore, both the Student- t model and Model 1 should be used when there is a dominating yet still uncertain suspicion that the data are asymptotically dependent.

Since we seek to obtain more flexibility and a smoother transition to asymptotic dependence within asymptotic independence, we propose a new model (Model 2) that can generate any value of $\bar{\chi} \in [\rho, 1]$, for fixed Gaussian correlation $\rho < 1$. This novel two-parameter Weibull-type distribution with support $[1, \infty)$ contains the Dirac mass at 1 as limiting case, yielding asymptotically independent standard Gaussian processes, and the Pareto distribution as boundary case, yielding asymptotic dependence. Its distribution, with parameters $\beta \geq 0$ and $\gamma > 0$, is defined as

$$F(r) = \begin{cases} 1 - \exp\{-\gamma(r^\beta - 1)/\beta\}, & \beta > 0, \\ 1 - r^{-\gamma}, & \beta = 0, \end{cases} \quad r \geq 1. \quad (14)$$

The distribution (14) forms a continuous parametric family with respect to β as the term $(r^\beta - 1)/\beta$ converges to $\log r$ as $\beta \downarrow 0$. The type of asymptotic dependence is determined by

the value of β . When $\beta > 0$, (14) coincides with the tail representation (12) with the same tail parameter β , yielding asymptotic independence. When $\beta = 0$ or $\beta \downarrow 0$, the variable R is Pareto distributed with $F(r) = 1 - r^{-\gamma}$, $r \geq 1$, yielding asymptotic dependence. The Dirac mass at 1 is obtained as $\beta \rightarrow \infty$ or as $\gamma \rightarrow \infty$. The benefit of Model 2 is to provide a smooth transition from asymptotic independence to asymptotic dependence with $\bar{\chi} \uparrow 1$ as $\beta \downarrow 0$ and $\gamma > 0, \rho < 1$ fixed; moreover, it still keeps a smooth transition from asymptotic dependence to asymptotic independence with $\chi \downarrow 0$ as $\gamma \uparrow \infty$ and $\beta = 0, \rho < 1$ fixed, leading to a Gaussian limit in analogy to the Student- t model. An intuitive interpretation of Model 2 is that the spatial correlation function gives control over the decrease in spatial dependence with distance (at a certain quantile level) while the parameters β, γ allow for a separate adjustment of the rate of joint tail decay when moving towards more and more extreme quantiles. A noteworthy link between Models 1 and 2 appears through the Box-Cox transform: the random scale in Model 1 arises from applying the Box-Cox transform with power ξ to a standard Pareto random variable, while Model 2 is the result of applying the inverse Box-Cox transform with power β to an exponential variable with rate γ .

Figure 2 shows the coefficients $\chi(u)$ and $\bar{\chi}(u)$ with respect to the parameters β and γ of Model 2 in (14) as a function of the threshold $u \in [0.9, 1]$ for various parameters and correlation ρ of $(W_1, W_2)^T$. The plots illustrate the ability of Model 2 to flexibly capture both asymptotic independence and asymptotic dependence. To contrast the flexibility of Model 2 with respect to the Gaussian copula, we also display $\chi(u)$ and $\bar{\chi}(u)$ for the Gaussian copula, whose correlation coefficient is chosen such that these coefficients for the two models match at the level $u = 0.95$. Significant differences appear between the Gaussian copula and Model 2 when $\beta \leq 1$, especially for $u \approx 1$. When β or γ increases, our model approaches the Gaussian copula although the limit quantities $\bar{\chi}$ may still be quite different for moderate values of β or γ . The extra tail flexibility of our model compared to the Gaussian copula is apparent when we vary the parameters β and γ . Different joint tail decay rates compared to the Gaussian copula are even more evident in higher dimensions.

Figure 2: Coefficients $\chi(u)$ (top) and $\bar{\chi}(u)$ (bottom), $u \in [0.9, 1]$, for Model 2 defined in (14) (solid black) and for the Gaussian copula matching at $u = 0.95$ (dashed red). Parameter configurations are $\beta = 0, \gamma = 1$ (left; asymptotically equivalent to $\xi = 1$ in Model 1), $\beta = 1, \gamma = 1$ (middle; asymptotically equivalent to $\xi = 0$ in Model 1), $\beta = 5, \gamma = 1$ (right). Thin to thick curves correspond to increasing correlations $\rho = 0, 0.3, 0.6, 0.9$ for model (14). Limit quantities χ and $\bar{\chi}$ are represented by dots at $u = 1$. The left column displays an asymptotic dependence scenario; the middle and right columns display asymptotic independence cases.

4 Likelihood inference

4.1 Maximum likelihood approach with partial censoring

To estimate the extremal dependence structure from the observed high spatial threshold exceedances, we advocate a two-step procedure: marginal distributions are estimated non-parametrically based on ranks and then copula parameters are estimated using a full pseudo-likelihood with partial censoring preventing estimates from being influenced by low and moderate values. Although it has never been applied to Gaussian scale mixture models, this censored approach is now quite popular in statistics of extremes (see, e.g., [Thibaud et al., 2013](#); [Huser and Davison, 2014](#); [Wadsworth and Tawn, 2014](#)), and results from [Thibaud and Opitz \(2015\)](#) and [Huser et al. \(2016\)](#) suggest it provides a reasonable compromise between bias and variance compared to alternative approaches to fit threshold exceedances.

Let $\mathbf{Y}_i = (Y_{1i}, \dots, Y_{Di})^T$, $i = 1, \dots, n$, denote n independent and identically distributed

observations from a process $Y(\mathbf{s})$ at the stations $\mathbf{s}_1, \dots, \mathbf{s}_D \in \mathcal{S}$. We assume that in the joint tail region corresponding to large values of $Y_{1i}, \dots, Y_{Di}$, the vectors $\mathbf{Y}_i$, $i = 1, \dots, n$, are well described by a continuous joint distribution H with margins $H_1, \dots, H_D$ and copula C stemming from a Gaussian scale mixture (2). Since we focus on the extremal dependence structure, we estimate marginal distributions $H_1, \dots, H_D$ in a first step using empirical distribution functions. Defining $\hat{H}_k(y) = (n+1)^{-1} \sum_{i=1}^n \mathbb{I}(Y_{ki} \leq y)$ with the indicator function $\mathbb{I}(\cdot)$, we transform the data to a pseudo-uniform scale as

$$U_{ki} := \hat{H}_k(Y_{ki}) = \frac{\text{rank}(Y_{ki})}{n+1}, \quad k = 1, \dots, D, \quad i = 1, \dots, n, \quad (15)$$

where $\text{rank}(Y_{ki})$ is the rank of Y_{ki} among the variables $Y_{k1}, \dots, Y_{kn}$. The denominator $(n+1)$ in (15) ensures that transformed variables U_{ki} , $k = 1, \dots, D, i = 1, \dots, n$, are within $(0, 1)$. Since $\hat{H}_k$ is a consistent estimator of H_k for each component $k = 1, \dots, D$ as $n \rightarrow \infty$, the variables $U_{k1}, \dots, U_{kn}$ form an approximate uniform $\text{Unif}(0, 1)$ random sample for large n . The nonparametric estimator (15) may not be very good for the most extreme values very close to 1, but it is very robust and its bias is of the order $\mathcal{O}(n^{-1})$. We have verified through simulations that it yields reasonable results, even for moderate values of n ; see §4.2.

In the second step, we assume that the transformed variables $U_{k1}, \dots, U_{kn}$, $k = 1, \dots, D$, are perfect random samples from the $\text{Unif}(0, 1)$ distribution, and we fit the copula (1) stemming from (2) (e.g., based on our model (14)) by maximum likelihood, censoring values not exceeding marginal thresholds $v_1, \dots, v_D \in (0, 1)$. Let $C(\cdot; \boldsymbol{\psi})$ and $c(\cdot; \boldsymbol{\psi})$ respectively denote the chosen parametric copula family and its density, where $\boldsymbol{\psi} \in \Psi \subset \mathbb{R}^p$ is the unknown vector of parameters to be estimated. The thresholds v_k , $k = 1, \dots, D$, are usually chosen such that $1 - v_k$ is a low exceedance probability like 0.05 or 0.01. For each independent temporal replicate $i = 1, \dots, n$, the vector $\mathbf{U}_i = (U_{1i}, \dots, U_{Di})^T$ is either below, above or partially above the threshold $\mathbf{v} = (v_1, \dots, v_D)^T$, leading to different censored likelihood contributions. We adopt the lowercase notation $\mathbf{u}_i = (u_{1i}, \dots, u_{Di})^T$, $i = 1, \dots, n$, for the realized values of $\mathbf{U}_i$ and write $\mathbf{u}_i^* = \{\max(u_{1i}, v_1), \dots, \max(u_{Di}, v_D)\}^T$. Three distinct scenarios can occur:

1. if all components of the vector $\mathbf{u}_i$ are below the threshold $\mathbf{v}$ (i.e., $\mathbf{u}_i^* = \mathbf{v}$), we use the fully censored likelihood contribution

$$L(\mathbf{u}_i; \boldsymbol{\psi}) = C(\mathbf{u}_i^*; \boldsymbol{\psi}) = C(\mathbf{v}; \boldsymbol{\psi}) = G\{G_1^{-1}(v_1), \dots, G_D^{-1}(v_D)\}.$$

2. if all components of the vector $\mathbf{u}_i$ are above the threshold $\mathbf{v}$ (i.e., $\mathbf{u}_i^* = \mathbf{u}_i$), we use the uncensored contribution

$$L(\mathbf{u}_i; \boldsymbol{\psi}) = c(\mathbf{u}_i^*; \boldsymbol{\psi}) = c(\mathbf{u}_i; \boldsymbol{\psi}) = \frac{g\{G_1^{-1}(u_{1i}), \dots, G_D^{-1}(u_{Di})\}}{\prod_{k=1}^D g_k\{G_k^{-1}(u_{ki})\}}.$$

3. if some but not all components of the vector $\mathbf{u}_i$ are above the threshold $\mathbf{v}$ (indexed by the set $I_i \subset \{1, \dots, D\}$ with complement $I_i^c \subset \{1, \dots, D\}$), we use the partially censored likelihood contribution

$$L(\mathbf{u}_i; \boldsymbol{\psi}) = C_{I_i}(\mathbf{u}_i^*; \boldsymbol{\psi}) = \int_0^{v_{I_i^c}} c(\mathbf{u}_i; \boldsymbol{\psi}) d\mathbf{u}_{i_{I_i^c}} = \frac{G_{I_i}\{G_1^{-1}(u_{1i}^*), \dots, G_D^{-1}(u_{Di}^*)\}}{\prod_{k \in I_i} g_k\{G_k^{-1}(u_{ki}^*)\}}.$$

In the above expressions, the joint distribution G and density g , the marginal distributions G_k and densities g_k , $k = 1, \dots, D$ and the partial derivatives G_{I_i} are given by (4), (5) and (6) respectively. Our censored log likelihood is defined as the sum of all individual log contributions, that is,

$$\ell(\boldsymbol{\psi}) = \sum_{i=1}^n \log\{L(\mathbf{u}_i; \boldsymbol{\psi})\}. \quad (16)$$

The maximum likelihood estimator (MLE) $\hat{\boldsymbol{\psi}}$, obtained by maximizing (16) over Ψ , is a full likelihood estimator for the censored observations $\mathbf{U}_i^* = \max(\mathbf{U}_i, \mathbf{v})$ augmented by the indicator variables $\mathbb{I}(U_{ki} \leq v_k)$, $k = 1, \dots, D$, $i = 1, \dots, n$. If the copula C is well specified and if the marginal estimation performed in the first step is perfect such that the transformed observations $\mathbf{U}_i$ are perfectly uniform, the estimator $\hat{\boldsymbol{\psi}}$ obeys classical likelihood theory: as $n \rightarrow \infty$, it is strongly consistent, asymptotically normal, attains the Cramér–Rao bound and converges at rate $\mathcal{O}(n^{1/2})$ under well-known regularity conditions. Notice that with Model 2 in (14), the case $\beta = 0$ is nonstandard as it lies on the boundary of the parameter space and must be treated separately; this issue, however, does not arise with Model 1 in (13). Moreover, the nonparametric transformation in (15) results in a slight asymptotically vanishing marginal misspecification for finite n . Despite this issue, Genest et al. (1995) show that under mild conditions, the maximum pseudo-likelihood estimator has similar asymptotic properties to the MLE, although with a slight loss in efficiency.

Table 1: Bias (B), standard deviation (SD) and root mean squared error (RMSE) of estimated parameters $\hat{\psi} = (\hat{\lambda}, \hat{\nu}, \hat{\beta}, \hat{\gamma})^T$ for the Gaussian scale mixture model (2) using (14) with correlation function $\rho(\mathbf{s}_1, \mathbf{s}_2) = \exp\{-(\|\mathbf{s}_1 - \mathbf{s}_2\|/\lambda)^\nu\}$ and parameters $\beta = 0, 0.5, 1$, $\gamma = 1$, $\lambda = 0.5, 1$ and $\nu = 1$. The values for the four estimated parameters are separated by slash symbols. Simulations are based on $n = 1000$ independent replicates observed at $D = 15$ uniform locations in $[0, 1]^2$. Maximum likelihood estimation is based on (16) with threshold $\mathbf{v} = (v, \dots, v)^T$, $v = 0.95$. Results stem from 500 independent experiments.

	100 × B		100 × SD		100 × RMSE	
	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 0.5$	$\lambda = 1$
$\beta = 0$	4/5/8/11	12/7/11/19	9/7/8/23	19/6/10/27	10/9/12/25	22/9/15/33
$\beta = 0.5$	4/2/8/25	15/4/3/24	10/7/23/51	25/5/31/73	11/7/24/57	29/7/31/77
$\beta = 1$	5/2/27/64	14/3/18/58	10/6/44/97	23/5/53/108	11/7/51/116	27/6/56/123

4.2 Simulation study

To assess the performance of the maximum pseudo-likelihood estimator $\hat{\psi}$ defined through (16), we simulate $n = 1000$ independent copies of the Gaussian scale mixture $X(\mathbf{s}) = RW(\mathbf{s})$ defined in (2) at $D = 5, 10, 15$ locations $\mathbf{s}_1, \dots, \mathbf{s}_D$ uniformly generated in $\mathcal{S} = [0, 1]^2$. Here, we focus on our random scale model (14) (Model 2), which we choose to work with in our application in §5 for the reasons explained below. For the Gaussian process $W(\mathbf{s})$, we choose a stationary isotropic correlation function $\rho(\mathbf{s}_1, \mathbf{s}_2) = \exp\{-(\|\mathbf{s}_1 - \mathbf{s}_2\|/\lambda)^\nu\}$ with range $\lambda = 0.5, 1$ and smoothness $\nu = 1$. Sample paths of $X(\mathbf{s})$ are thus continuous but nondifferentiable. For the random scale variable R , we consider three scenarios: setting $\gamma = 1$, we choose $\beta = 0$ (asymptotic dependence) or $\beta = 0.5, 1$ (asymptotic independence). We estimate all copula parameters $\psi = (\lambda, \nu, \beta, \gamma)^T \in \Psi = (0, \infty) \times (0, 2] \times [0, \infty) \times (0, \infty)$ by maximizing the censored pseudo-likelihood (16) using marginal thresholds $\mathbf{v} = (v, \dots, v)^T$ with $v = 0.95$. This yields $n(1 - v) = 50$ exceedances at each location, occurring simultaneously or not depending on the extremal dependence strength. We repeat this experiment 500 times to obtain boxplots of estimated parameters and compute performance metrics. Computational details are reported in the Supplementary Material.

Table 1 reports the absolute bias (B), standard deviation (SD) and root mean squared error ($\text{RMSE} = (B^2 + \text{SD}^2)^{1/2}$) of estimated parameters for $D = 15$ based on the 500 experiments. Figure 3 displays boxplots of estimated parameters for $\lambda = \nu = \beta = \gamma = 1$ as a

Figure 3: Boxplots of estimated parameters $\hat{\psi} = (\hat{\lambda}, \hat{\nu}, \hat{\beta}, \hat{\gamma})^T$ for the Gaussian scale mixture model (2) using (14) with correlation function $\rho(\mathbf{s}_1, \mathbf{s}_2) = \exp\{-(\|\mathbf{s}_1 - \mathbf{s}_2\|/\lambda)^\nu\}$ and parameters $\lambda = \nu = \beta = \gamma = 1$. Simulations are based on $n = 1000$ independent replicates observed at $D = 5, 10, 15$ uniform locations in $[0, 1]^2$. Maximum likelihood estimation is based on (16) with threshold $\mathbf{v} = (v, \dots, v)^T$, $v = 0.95$. Boxplots are produced from 500 independent experiments. Red points/horizontal lines show the estimated means/true values.

function of the dimension $D = 5, 10, 15$. Generally, the estimation procedure appears to work well although the variability of estimates is relatively large owing to the fairly low number of exceedances considered in this simulation setting. The bias is almost always smaller than the standard deviation but not by much. The root mean squared error indicates that parameters seem overall easier to estimate for lower values of β and λ . Indeed, when β increases, the parameters β and γ are more complicated to identify because the resulting copula converges to the Gaussian copula as $\beta \rightarrow \infty$ or $\gamma \rightarrow \infty$, resulting in higher standard deviations and biases. Overall, standard deviations seem to decrease slightly as the dimension D increases, even though ψ is not a consistent estimator for model (2) under infill asymptotics. This improvement is difficult to discern for γ , but is striking for the smoothness parameter ν , which is hard to estimate with few scattered locations. The slightly increasing bias for larger D is due to the nonparametric marginal estimation in (15) as n is kept fixed here, but it vanishes as $n \rightarrow \infty$. Figure 4 displays estimated $\chi(u)$ and $\bar{\chi}(u)$ coefficients in (10) when $\lambda = \nu = \beta = \gamma = 1$ based either on simple nonparametric estimators or on our parametric model-based approach with $D = 5, 10, 15$ locations. Visual inspection of these plots confirms that the extremal dependence structure is well estimated, and that the rate of tail decay is relatively well captured. In contrast with nonparametric estimators, the model-based estimates have largely reduced uncertainties especially for large thresholds v and borrow strength across locations for better tail estimation; hence they provide higher confidence

Figure 4: Dependence structure estimation under well-specified setting. Estimated coefficients $\chi(u)$ (left) and $\bar{\chi}(u)$ (right), $u \in [0.9, 1]$, for model (2) using (14) with $\beta = \gamma = 1$ (asymptotic independent case with $\chi = 0$) and correlation function $\rho(\mathbf{s}_1, \mathbf{s}_2) = \exp\{-(\|\mathbf{s}_2 - \mathbf{s}_1\|/\lambda)^\nu\}$ with $\lambda = \nu = 1$ for two points at distance $\|\mathbf{s}_2 - \mathbf{s}_1\| = 0.5$. Estimation is either nonparametric (yellow) or parametric based on $D = 5$ (red), 10 (green) or 15 (blue) uniform locations in $[0, 1]^2$. The number of replicates is $n = 1000$. Solid lines show means of 500 simulations, while shaded areas are 95% overall confidence envelopes. True curves are in black, and the threshold $v = 0.95$ used in (16) is the vertical dashed line.

about the asymptotic dependence class. Similar results were obtained with other parameter combinations under asymptotic independence ($\beta > 0$) or dependence ($\beta = 0$).

To investigate the tail flexibility of our model, we consider a misspecified setting. We simulate Student- t random processes with Df = 1, 2, 5, 10 degrees of freedom using the same correlation function, and we fit the Gaussian scale mixture (2) based on our model (14). Both the Student- t model and our new model are Gaussian scale mixtures but they are not nested in each other. However, the joint tail behavior of a Student- t process with Df degrees of freedom can be considered to be close to that of the Gaussian scale mixture model (14) with $\beta = 0$ and $\gamma = \text{Df}$ because the two models have the same limiting max-stable dependence structure. Our model should therefore provide a reasonable approximation although asymptotic dependence is a boundary case. Boxplots of estimated parameters (see Supplementary Material) suggest that when Df = 1, 2, the performance of the MLE $\hat{\psi}$ is similar to that obtained in a well-specified setting. When Df = 5, 10, estimates of λ and γ are more “biased”, but the extremal dependence structure is still very well captured, as illustrated by estimated $\chi(u)$ and $\bar{\chi}(u)$ coefficients in Figure 5. In summary, our Gaussian

Figure 5: Dependence structure estimation under misspecified setting. Estimated coefficients $\chi(u)$ (left) and $\bar{\chi}(u)$ (right), $u \in [0.9, 1]$, for data simulated from a Student- t process with $Df = 10$ degrees of freedom and correlation function $\rho(\mathbf{s}_1, \mathbf{s}_2) = \exp\{-(\|\mathbf{s}_2 - \mathbf{s}_1\|/\lambda)^\nu\}$ with $\lambda = 0.5$ and $\nu = 1$ for two points at distance $\|\mathbf{s}_2 - \mathbf{s}_1\| = 0.5$. Estimation is either nonparametric (yellow) or parametric based on the (misspecified) Gaussian scale mixture model (2) with random scale (14), using $D = 5$ (red), 10 (green) or 15 (blue) uniform locations in $[0, 1]^2$. The number of replicates is $n = 1000$. Solid lines show means of 500 simulations, while shaded areas are 95% overall confidence envelopes. True curves are in solid black, and the threshold $v = 0.95$ used in (16) is the vertical dashed line.

scale mixture model provides a very flexible tail-dependent structure described by a relatively small number of parameters; it can be consistently estimated from high threshold exceedances using a full pseudo-likelihood approach with partial censoring, although it remains fairly computer intensive for large dimensions.

5 Application

To illustrate the benefits of our new modeling approach, we analyze hourly wind speed extremes recorded during 2012–2014 in the Pacific Northwest, US, a region with large wind energy resources. The dataset was compiled from data archived by the Bonneville Power Administration. Data are available year-round at 20 meteorological towers along the border between Oregon and Washington. To avoid modeling complex spatio-temporal non-stationary patterns, we restrict our attention to winter months (DJF) for the 12 stations located on the East side of the Cascade mountain range; see Figure 6. Selected data comprise up to 6504 hourly observations at each site, with about 8% of values missing. Owing to important East-

Figure 6: Left: Topographic map with meteorological towers selected in our study and state boundaries (black). The ellipses overlaid, centered at Goodnoe Hills (GDH), correspond to the isocontours of the fitted coefficients $\chi(u) = 0.5, 0.4, 0.3, 0.2$ (from the center outward) using the thresholds $u = 0.97$ (red) and $u = 0.99$ (blue), for the best anisotropic model. Right: Wind rose of winter (DJF) wind speeds for the 12 stations, preliminarily transformed to the uniform scale. The color scale corresponds to different marginal quantile ranges.

West pressure gradients in this region and the special nature of the orography, wind patterns are mainly characterized by easterly and westerly winds (Kazor and Hering, 2015). A wind rose for the data at the 12 stations reveals that extreme winds blow mostly from the West or South-West, suggesting that simple anisotropic models might perform well; see Figure 6. More details about data and monitoring stations are in the Supplementary Material.

Hourly wind speeds show strong temporal dependence. Since our focus is on spatial dependence, we ignore temporal dependence for parameter estimation but account for it in uncertainty assessment using a block bootstrap. We use the anisotropic correlation function $\rho(\mathbf{s}_1, \mathbf{s}_2) = \exp\{-(h_{12}/\lambda)^\nu\}$, where h_{12} denotes the Mahalanobis distance, that is,

$$h_{12}^2 = (\mathbf{s}_1 - \mathbf{s}_2)^T \Omega^{-1} (\mathbf{s}_1 - \mathbf{s}_2), \quad \Omega = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \lambda_{12}^{-2} \end{pmatrix} \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}^T.$$

The Mahalanobis distance defines elliptical isocontours; λ and λ_{12} are respectively the length of one principal axis and the length ratio of the two principal axes, while θ is the angle with respect to the West-East direction. When $\lambda_{12} = 1$, the model is isotropic. We consider three main (isotropic or anisotropic) model classes:

1. RW: Our new Gaussian scale mixture copula model defined through (2) and (14) (Model 2), where we distinguish the unrestricted ($\beta \geq 0$) and restricted ($\beta = 0$) cases.

The reason why we consider Model 2 instead of Model 1 proposed in §3.2 is that extreme

Table 2: Estimated parameters $\hat{\psi} = (\hat{\lambda}, \hat{\lambda}_{12}, \hat{\theta}, \hat{\nu}, \hat{\beta}, \hat{\gamma})^T$ for the unrestricted anisotropic Gaussian scale mixture copula model with corresponding 95%-confidence intervals (CIs).

Parameters	λ [km]	λ_{12}	θ [rad]	ν	β	γ
Estimates	287	3.71	0.23	0.46	1.96	0.05
95%-CIs	[95, 1127]	[2.64, 5.08]	[0.18, 0.34]	[0.32, 0.76]	[0.96, 8.47]	[0, 0.32]

wind speeds (and wind gusts in particular) are known to be very localized, suggesting that a flexible asymptotically independent model might perform well (however, Model 2 does not preclude the possibility of asymptotic dependence arising when $\beta = 0$).

2. Student- t : The Student- t copula with $Df = 1, \dots, 30$ degrees of freedom and the same anisotropic correlation function $\rho(\mathbf{s}_1, \mathbf{s}_2)$.
3. Gaussian: The Gaussian copula (same as 2., but with $Df = \infty$ fixed).

All models, summarized in the Supplementary Material, were fitted to the wind data using the censored likelihood estimator with threshold $\mathbf{v} = (v, \dots, v)$, $v = 0.95$; see §4.1. This yields about $6504 \times 0.05 \approx 325$ correlated exceedances at each site. The log profile censored likelihood of the (an)isotropic Student- t copula is shown in the left panel of Figure 7, and it is compared to the maximized log likelihoods of the (an)isotropic Gaussian and RW copula models. Deviances are not straightforward to interpret owing to temporal dependence, but anisotropic models seem to outperform isotropic models by a large margin. This is not surprising given the wind patterns in the study region. The best Student- t model has $Df = 13$ degrees of freedom, a result that is consistent for isotropic and anisotropic models. This suggests that the asymptotic dependence strength is quite weak although the asymptotically independent Gaussian copula model is outperformed by the extra flexibility of the Student- t copula. Although the likelihood differences in Figure 7 need to be interpreted with care, the best model overall seem to be the anisotropic unrestricted RW copula (with $\beta \geq 0$); the difference in log likelihoods is about 7 and 9, respectively, with respect to the best Student- t model and the restricted RW model with $\beta = 0$.

Estimated parameters for our best (anisotropic unrestricted RW) model, with 95%-confidence intervals computed using a weekly block bootstrap with 100 replicates respecting the missing value patterns are reported in Table 2. Figure 8 shows a conditional simulation

Figure 7: Left: Negative log profile censored likelihood for the Student- t copula with Df degrees of freedom (black), and negative maximized log censored likelihood for the RW copula with $\beta \geq 0$ (blue) and $\beta = 0$ (red). The Gaussian copula corresponds to Df = ∞ . Isotropic and anisotropic models correspond to thin and thick lines, respectively. The best negative log likelihood value has been subtracted from all curves. The vertical grey line at Df = 13 represents the best Student- t model. Right: Estimated probability $\Pr(U_1 > u | U_2 > u)$, as a function of the threshold u , with U_1, U_2 representing the hourly wind speed data at Goodnoe Hills (GDH) and Augspurger (AUG), preliminarily transformed to the uniform scale. Curves display the empirical estimate (yellow), and the model-based estimates for the anisotropic unrestricted RW with $\beta \geq 0$ (red), restricted RW with $\beta = 0$ (green), Student- t with Df = 13 (purple) and Gaussian (blue) models. 95% overall confidence envelopes are shown for the empirical, unrestricted RW and Gaussian fits (using similar colors). Similar envelopes were obtained for the other models but are omitted here for readability.

Figure 8: Left: Conditional simulation for our best fitted RW copula model with parameters given in Table 2 with conditioning values observed at the twelve stations (dots) on February 22, 2012, a day of very strong wind. Middle and right: corresponding 25% and 75%-conditional quantiles based on 500 simulations. The color scale indicates quantile levels.

Figure 9: Quantile-quantile plots for the sum $\sum_{j \in A} X_j$ of the observations at five stations, preliminarily transformed to follow the marginal distribution G_j of each Gaussian scale mixture model considered here, namely the anisotropic Gaussian (left panel), Student- t with $Df = 13$ degrees of freedom (second panel), unrestricted RW with $\beta \geq 0$ (third panel) and restricted RW (right panel) models. The set of stations A was chosen to correspond to locations with no missing values, i.e., Hood River (HOO), Horse Heaven (HOR), Roosevelt (ROO), Shaniko (SHA) and Sunnyside (SUN). QQ-plots zoom into the upper tail region, i.e., where the sum exceeds its 95%-quantile. Dashed lines are 95% overall confidence bands.

and corresponding 25% and 75%-conditional quantiles (calculated based on 500 simulations using the algorithm described in §2.4) of the wind speed field over the region of study for February 22, 2012, which corresponds to the day of strongest average wind speeds observed at the twelve stations. The estimate $\hat{\lambda}_{12} = 3.71$ implies that the anisotropy is strong, and $\hat{\theta} = 0.23$ indicates that the direction of strongest correlation is toward the North-East, as illustrated by the isocontours of the fitted coefficient $\chi(u)$ overlaid on the left panel of Figure 6, and by the wind patterns simulated in Figure 8. Interestingly, this agrees with the wind rose, although the wind directions were not used in the fitting procedure. The value $\hat{\nu} = 0.46$ indicates that the hourly wind field shows small scale variability, as expected. The estimates $\hat{\beta} = 1.96$ and $\hat{\gamma} = 0.05$ for the random scale parameters strongly support the assumption of asymptotic independence, given that the confidence interval for β excludes zero by far. This also formally justifies preferring our unrestricted model with $\beta > 0$ over the restricted ($\beta = 0$) model or the Student- t model, both of which have one less parameter and similar log-likelihood values. Furthermore, for all bootstrap replicates, the likelihood value was always lower when fixing $\beta = 0$. This also indicates that standard asymptotics (with $\beta > 0$) prevail in this case. The values of $\hat{\beta}$ and $\hat{\gamma}$ also suggest that the extremal dependence structure is quite far from being Gaussian.

For model validation beyond relative comparison of models, the elliptical structure $\mathbf{X} = (X_1, \dots, X_D)^T \stackrel{D}{=} R\mathbf{W}$, $\mathbf{W} \sim \mathcal{N}_D(\mathbf{0}, \Sigma)$, of multivariate Gaussian scale mixtures is useful for constructing tools such as visual model diagnostics or formal Monte-Carlo tests. For example, Figure 9 shows quantile-quantile (QQ) plots based on the sum of a subset of components $\sum_{j \in A} X_j$, $A \subset \{1, \dots, D\}$, exploiting the fact that the latter follows a univariate Gaussian scale mixture RW^{Sum} with the same radial variable R and underlying Gaussian variable W^{Sum} with zero mean and variance equal to $\mathbf{1}^T \Sigma_{A:A} \mathbf{1}$, where $\mathbf{1} = (1, \dots, 1)^T$. As these QQ-plots reveal a relatively good agreement between empirical and model-based upper-tail quantiles for all fitted anisotropic models (Gaussian, Student- t model with Df = 13 degrees of freedom, restricted and unrestricted RW model), this suggests that they all capture the extremal dependence structure reasonably well, although the fit of the Gaussian copula is maybe slightly worse than the others. More generally, one could also exploit the structure of Gaussian scale mixtures to construct QQ-plots based on the conditional distribution obtained for a specific station given the other stations (also leading to an elliptical distribution that can be deduced from Theorem 1); see Bortot et al. (2000) for alternative model diagnostics.

To further validate our fitted models, the right panel of Figure 7 shows the estimated probability $\Pr(U_1 > u \mid U_2 > u)$, which is more easily interpretable than $\chi(u)$ yet asymptotically equivalent as $u \rightarrow 1$, plotted as a function of the threshold $u \in [0.9, 1]$, where U_1 and U_2 represent the hourly wind speed data at Goodnoe Hills (GDH) and Augspurger (AUG), preliminarily transformed to the uniform scale. This plot suggests that although all models seem to perform similarly and reasonably well at moderately high levels, they nevertheless predict very different behaviors for very large extremes with $u \rightarrow 1$, highlighting the need for flexible tail models covering asymptotic independence and dependence scenarios, such as those proposed in §3.2. This figure also clearly shows that our model-based approach allows for a considerable reduction in the uncertainty compared to a fully nonparametric approach.

We finally assess the ability of each fitted anisotropic model to predict extreme wind speeds at unobserved locations using cross-validation. More precisely, we withhold each station at a time from the dataset, and we refit all models based on the data observed at the eleven remaining stations. We then use the fitted models to predict wind speed extremes at the hold-out station. More precisely, we rank the wind speed data observed at each hold-

Table 3: CRPS values for each fitted model, obtained by predicting extreme wind speeds at each station. Each value is the average CRPS over the 30 most extreme days at each station. For each specific day and station, the CRPS values are calculated from 1000 conditional simulations based on the fitted models, withholding the data at the prediction site. For each station, the best model (i.e., lowest CRPS value) appears in bold.

Station	Acronym	Gaussian	Student, Df = 13	RW, $\beta \geq 0$	RW, $\beta = 0$
Augspurger	AUG	2.82	2.67	2.65	2.71
Butler Grade	BUT	2.48	2.45	2.55	2.46
Chinook	CNK	2.39	2.31	2.31	2.33
Goodnoe Hills	GDH	3.33	3.11	3.00	3.15
Hood River	HOO	1.62	1.54	1.48	1.58
Horse Heaven	HOR	3.28	3.18	3.16	3.12
Kennewick	KEN	3.97	3.88	4.04	3.87
Roosevelt	ROO	3.79	3.74	3.72	3.75
Seven Mile	SML	4.41	4.30	4.23	4.31
Shaniko	SHA	3.90	3.89	4.12	3.63
Sunnyside	SUN	4.86	4.82	4.95	4.79
Wasco	WAS	4.38	4.14	4.07	4.23

out station and then extract the 30 most extreme days containing no missing values at the twelve stations and such that at least seven of the twelve stations (including the hold-out station itself) exceed the 95% marginal quantile. For each selected day, we then use the algorithm developed in §2.4 to perform 1000 simulations at the hold-out station conditional on the values observed at all other eleven sites. This yields, for each station, selected day and model, 1000 predicted wind speeds. We then assess the predictive quality of each model by computing the continuous ranked probability score (CRPS; [Gneiting and Raftery, 2007](#)), and averaging these scores over the 30 prediction days. Table 3 reports the results. Although CRPS values appear to be fairly similar across models, our new Gaussian scale mixture copula model is often selected as the best model. Overall, the unrestricted ($\beta \geq 0$) and restricted ($\beta = 0$) RW models provide the most accurate predictions for 7 and 4 of the 12 stations, respectively, whereas the Student- t model is the best for 1 station and the Gaussian model is often the worst. This demonstrates that although our new RW model has more parameters than the Gaussian and Student- t models, it does not overfit, and thanks to its high tail flexibility, it provides some improvement in terms of spatial prediction of extreme events in most of the cases considered in our cross-validation study.

6 Discussion

Starting with a detailed study of the tail behavior of general Gaussian scale mixtures, we have proposed new parsimonious and flexible subasymptotic copula models to achieve a smooth transition between asymptotic independence and asymptotic dependence. Unlike approaches often used in multivariate analysis comparing the fit of the asymptotically dependent Student- t model to that of the asymptotically independent Gaussian model, we give strong attention to appropriately capturing the tail decay in asymptotically independent scenarios, while keeping a highly flexible asymptotically dependent submodel, and making sure that the transition between the two asymptotic regimes is continuous. The gain in tail flexibility as compared to Gaussian or classical asymptotic extreme-value models may also allow fixing lower thresholds in exceedance-based modeling, which might improve the estimation of the extremal dependence structure by gaining statistical efficiency without introducing too much bias. Furthermore, our new elliptic copula models may also be useful for the modeling of the full data range, arising in a much wider spectrum of applications.

In addition to providing more flexibility than max-stable models, inference for our model is also facilitated and may be performed using a censored pseudo-likelihood, although the latter is expressed in terms of integrals whose numerical approximation is quite intensive to compute. Overall, computations are of the same order as for the censored Poisson and Pareto likelihoods advocated by [Wadsworth and Tawn \(2014\)](#) and [Thibaud and Opitz \(2015\)](#), which are only valid for asymptotically dependent data. The asymptotically dependent restrictions of the models that we propose are closely related to the elliptic Pareto process of [Thibaud and Opitz \(2015\)](#) while their asymptotically independent counterparts provide alternatives and more flexible parametric extensions to the Laplace model of [Opitz \(2016\)](#). Estimating the value of the random scale's shape parameter allows the data to provide evidence about the asymptotic dependence class without fixing it a priori.

In our application, we found evidence of asymptotic independence in hourly wind speed data in the Pacific Northwest, US, and we showed that the extremal dependence structure was well captured by our proposed copula model. Over large regions, a limitation of our model is that independence at large distances cannot be captured unless the random scale's

distribution is degenerate. To circumvent this issue, our model could be extended through a random partitioning approach (Morris et al., 2017), but inference would become awkward. It would be interesting to investigate non-stationary (Huser and Genton, 2016) and spatio-temporal (Huser and Davison, 2014) extensions of our new Gaussian scale mixture models.

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Appendix: Proofs

The following lemmas from the literature unify notations and adapt results to our context.

Lemma 3 (Gumbel domain of attraction; Hashorva, 2006, 2010). *For a random variable $X \sim F_X$ with infinite upper endpoint, suppose that a weight function $w(\cdot) > 0$ satisfies*

$$\frac{1 - F_X\{u + x/w(u)\}}{1 - F_X(u)} \rightarrow \exp(-x), \quad u \rightarrow \infty. \quad (17)$$

Then F_X is in the maximum domain of attraction of the unit Gumbel distribution, that is $F_X^n(a_n x + b_n) \rightarrow \exp\{-\exp(-x)\}$, $x \in \mathbb{R}$, where we can choose $b_n = F^{-1}(1 - 1/n)$ and $a_n = 1/w(b_n)$ for $n > 1$. If X has Weibull-type tail decay $\Pr(X \geq x) \sim \alpha x^\gamma \exp(-\delta x^\beta)$ for $x \rightarrow \infty$ and constants $\alpha > 0$, $\beta > 0$, $\gamma \in \mathbb{R}$ and $\delta > 0$, then (17) holds and we may take

$$w(u) \sim \delta \beta u^{\beta-1}, \quad u \rightarrow \infty. \quad (18)$$

Lemma 4 (Joint tail decay for Weibull-type random scale; Hashorva, 2010). *Assume that the bivariate elliptic random vector $\mathbf{X} = (X_1, X_2)^T \stackrel{D}{=} R \Sigma^{1/2} (U_1, U_2)^T$ with correlation matrix Σ , $\rho = \Sigma_{1,2} \in (-1, 1)$, has radial distribution $R \sim F$ in the Gumbel domain of attraction with weight function $w(\cdot)$ as in (18) and with copula C . Then*

$$\frac{\overline{C}(1 - x_1/u, 1 - x_2/u)}{\overline{C}(1 - 1/u, 1 - 1/u)} \rightarrow (x_1 x_2)^{1/(2\eta)}, \quad \eta = \{(1 + \rho)/2\}^{\beta/2}, \quad u \rightarrow \infty, \quad (19)$$

where η is the coefficient of tail dependence ([Ledford and Tawn, 1996](#)) of $\mathbf{X}$. More specifically, if R has Weibull-type tail with parameters as in (12), then

$$\begin{aligned}\bar{C}(1 - 1/x, 1 - 1/x) &\sim K_1 (\log x)^{K_2} x^{-1/\eta}, \quad x \rightarrow \infty, \\ K_1 &= \alpha^{-1}(1 - \rho)^{-2} \{2/(1 + \rho)\}^{\gamma/2 - \beta/2 + 1} (1 - \rho^2)^{3/2} \delta^{(1/\eta - 1)\gamma/\beta} \{\alpha^2/(2\pi\beta)\}^{1 - 1/(2\eta)}, \\ K_2 &= (1 - 1/\eta)\gamma/\beta + 1/(2\eta) - 1.\end{aligned}\tag{20}$$

Lemma 5 (Tail decay of products of Weibull-type variables; [Arendarczyk and Debicki, 2011](#)). *If two independent random variables R_1 and R_2 have Weibull-type tails such that*

$$\Pr(R_i \geq r) \sim \alpha_i r^{\gamma_i} \exp(-\delta_i r^{\beta_i}), \quad r \rightarrow \infty, \quad i = 1, 2,$$

where $\alpha_i > 0$, $\beta_i > 0$, $\gamma_i \in \mathbb{R}$, $\delta_i > 0$ for $i = 1, 2$, then, as $r \rightarrow \infty$,

$$\Pr(R_1 R_2 \geq r) \sim \sqrt{2\pi} \sqrt{\frac{\beta_2 \delta_2}{\beta_1 + \beta_2}} \alpha_1 \alpha_2 A^{0.5\beta_2 + \gamma_2 - \gamma_1} r^{\frac{2\beta_2 \gamma_1 + 2\beta_1 \gamma_2 + \beta_1 \beta_2}{2(\beta_1 + \beta_2)}} \exp\left(-B r^{\frac{\beta_1 \beta_2}{\beta_1 + \beta_2}}\right), \tag{21}$$

where $A = \{(\beta_1 \delta_1)/(\beta_2 \delta_2)\}^{1/(\beta_1 + \beta_2)}$, $B = \delta_1^{1-a} \delta_2^a \{(\beta_2/\beta_1)^a + (\beta_1/\beta_2)^{1-a}\}$ and $a = \beta_1/(\beta_1 + \beta_2)$.

Proof. Without loss of generality, consider $i = 1$. We have $X_1 \stackrel{D}{=} \sqrt{\sigma_{11}} R U_1$ with $U_1^2 \sim \text{Beta}\{1/2, (D - 1)/2\}$, see [Cambanis et al. \(1981, Lemma 2\)](#). The tail representation (12) is stable under power transformations of variables like $R \mapsto R^p$, $p \in (0, \infty)$, where some of the parameters in (12) are different for the transformed variable R^p . We can fix $p = 2$ and apply Theorem 4.1 of [Hashorva and Pakes \(2010\)](#) to X_1^2 and R^2 , which provides results concerning the random scaling with a Beta-distributed variable. Indeed, X_1^2 is Weibull-tailed if R^2 is so, and X_1^2 and R^2 have the same weight function $w(\cdot)$ in (18) and therefore the same parameter β . Switching back to $|X_1|$ and R using $p = 1/2$ and taking into account the symmetry of X_1 with respect to 0 then proves the statement of the lemma. $\square$

Proof of Theorem 1. The formulas for conditional elliptic distributions follow from the related standard theory ([Cambanis et al., 1981, Corollary 5 and Section 4](#)). The density f_{RRW} of the radial variable of a Gaussian scale mixture distribution is obtained as follows:

$$\begin{aligned}f_{RRW}(r) &= \frac{\partial}{\partial r} \int_0^\infty F(r/s) f_{RW}(s) ds = \int_0^\infty F(1/s) \frac{\partial}{\partial r} f_{RW}(rs) r ds \\ &= \int_0^\infty F(1/s) \{f_{RW}(rs) + f'_{RW}(rs) rs\} ds.\end{aligned}$$

The density of R conditional to $\mathbf{X} = \mathbf{x}$ is based on the change of variables from $(r, \mathbf{w})$ to $(r, \mathbf{x}) = (r, r\mathbf{w})$ in $f_{R, \mathbf{W}} = f \times \varphi_D(\cdot; \Sigma)$, yielding $f_{R, \mathbf{X}}(r, \mathbf{x}) = f(r) \times \varphi_D(\mathbf{x}/r; \Sigma) \times r^{-D}$, and $f_{R|\mathbf{X}=\mathbf{x}}(r) = f_{R, \mathbf{X}}(r, \mathbf{x})/f_{\mathbf{X}}(\mathbf{x})$ yields the final expression. $\square$

Proof of Theorem 2. The bivariate standard Gaussian vector $\mathbf{W} = (W_1, W_2)^T$ has elliptic representation $R_W(U_1, \rho U_1 + \sqrt{1 - \rho^2} U_2)^T$ with $R_W \perp\!\!\!\perp (U_1, U_2)^T$ and $(U_1, U_2)^T$ distributed uniformly on the unit circle. We let parameter and variable subscripts such as in R_W and α_W refer to the vector $\mathbf{W}$. Since $R_W^2 \sim \chi_2^2 = \Gamma(a = 1, b = 2)$ is gamma distributed, we transform the gamma tail characterized in Lemma 1 of the Supplementary Materials to obtain

$$\Pr(R_W > r) = \Pr(R_W^2 > r^2) \sim (r/2)^{1-1} \{\Gamma(1)\}^{-1} \exp(-r^2/2) = \exp(-r^2/2), \quad r \rightarrow \infty.$$

Therefore, R_W has Weibull-type tail as in (12) with parameters $\alpha_W = 1$, $\beta_W = 2$, $\gamma_W = 0$, $\delta_W = 1/2$ in obvious notation. We write $\mathbf{X} = R\mathbf{W} = R^*(U_1, \rho U_1 + \sqrt{1 - \rho^2} U_2)^T$ with $R^* = RR_W$ a product of two independent Weibull-type variables. The tail expansion for such products, given in Lemma 5, yields a Weibull-type tail as in (12) for R^* . Setting $R_1 = R$ and $R_2 = R_W$ in Lemma 5, the constant in (21) is $A = \{(\beta\delta)/(2 \times 1/2)\}^{1/(\beta+2)} = (\beta\delta)^{1/(2+\beta)}$ and $\Pr(R^* > r) \sim \alpha^* r^{\gamma^*} \exp(-\delta^* r^{\beta^*})$ for $r \rightarrow \infty$, with

$$\begin{aligned} \alpha^* &= \left(\frac{2\pi\beta_W\delta_W}{\beta_W + \beta} \right)^{1/2} A^{\beta_W/2 + \gamma_W - \gamma} \alpha \alpha_W = \left(\frac{2\pi}{2 + \beta} \right)^{1/2} A^{1-\gamma} \alpha, \\ \beta^* &= 2\beta/(2 + \beta), \quad \gamma^* = \frac{2\gamma + \beta}{2 + \beta}, \\ \delta^* &= \delta^{1-\beta/(\beta_W + \beta)} \delta_W^{\beta/(\beta_W + \beta)} \{(\beta_W/\beta)^{\beta/(\beta_W + \beta)} + (\beta/\beta_W)^{1-\beta/(\beta_W + \beta)}\} \\ &= \delta^{2/(2+\beta)} 2^{-\beta/(2+\beta)} \{(2/\beta)^{\beta/(2+\beta)} + (\beta/2)^{2/(2+\beta)}\}. \end{aligned}$$

We derive $\bar{\chi}$ using Lemma 4. The weight function of R^* in (18) is $w^*(u) = \delta^* \beta^* u^{\beta^*-1}$. Lemma 4 yields the coefficient of tail dependence (Ledford and Tawn, 1996) of $\mathbf{X}$ given as $\eta = \{(1 + \rho)/2\}^{\beta/(2+\beta)}$. We obtain $\bar{\chi} = 2(\eta - 1) = 2\{(1 + \rho)/2\}^{\beta/(2+\beta)} - 1$. Lemma 4 helps derive the joint tail representation of the survival copula $\bar{C}$. By combining (19) and (20) (with $x_1 = x_2 = 1$) and substituting R in Lemma 4 by R^* (with its tail parameters $\alpha^*, \beta^*, \gamma^*, \delta^*$), we obtain that $\bar{C}(1 - 1/x, 1 - 1/x) \sim \bar{C}(1 - 1/x, 1 - 1/x) = K (\log x)^{(1-1/\eta)\gamma^*/\beta^* + 1/(2\eta) - 1} x^{-1/\eta}$ as $x \rightarrow \infty$, with constant term

$$K = (\alpha^*)^{-1} (1 - \rho)^{-2} \{2/(1 + \rho)\}^{\gamma^*/2 - \beta^*/2 + 1} (1 - \rho^2)^{3/2} (\delta^*)^{(1/\eta - 1)\gamma^*/\beta^*} \{(\alpha^*)^2/(2\pi\beta^*)\}^{1-1/(2\eta)}.$$

Finally, $\chi = 0$ follows from $\bar{\chi} < 1$. □

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