The Effects of Initial Rise and Axial Loads on MEMS Arches

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Abstract

Arch microbeams have been utilized and proposed for many uses over the past few years due to their large tunability and bistability. However, recent experimental data have shown different mechanical behavior of arches when subjected to axial loads, i.e., their stiffness may increase or decrease with applied axial loads. This paper aims to investigate in depth the influence of the competing effects of initial rise and axial loads on the mechanical behavior of micromachined arches; mainly their static deflection and resonant frequencies. Based on analytical solutions, the static response and eigenvalue problems are analyzed for various values of initial rises and axial loads. Universal curves showing the variation of the first three resonance frequencies of the arch are generated for various values of initial rise under both tensile and compressive axial loads. This study shows that increasing the tensile or compressive axial loads for different values of initial rise may lead to either increase in the stiffness of the beam or initial decrease in the stiffness, which later increases as the axial load is increased depending on the dominant effect of the initial rise of the arch and the axial load. The obtained universal curves represent useful design tools to predict the tunability of arches under axial loads for various values of initial rises. The use of the universal curves is demonstrated with an experimental case study. Analytical formulation is developed to predict the point of minimum where the trend of the resonance frequency versus axial loads changes qualitatively due to the competing effects of axial loads and initial curvature.

I. Introduction

The static and dynamic behaviors of micro-electromechanical systems (MEMS) resonators are affected by several design parameters that require thorough investigation during modeling. Examples of these parameters are bias voltage, axial load, anchor conditions, and excitation frequency, which influence the response of MEMS resonators and determine their usability and range of operation [1-9]. The bistability of an arch beam [10-13], for instance, is one of those phenomena that require tuning of design parameters to predict its occurrence. One of the advantages of MEMS bistable structures, such as buckled beams and arches, is their large strokes compared to monostable and straight structures [14]. However, arches have more advantages over buckled beams because their responses are more controllable and predictable since their initial curvature is predetermined from fabrication. The existence of two different stable states at the

same force, commonly referred to as bistability, is an intrinsic property of curved beams that provides the basis of operation for several micro/nano devices; primarily in electrostatic switches and recently in logic and memory devices [15-17]. Loading a bistable structure beyond a critical value leads to snap-through motion of the structure (the structure jumps from one stable state to another stable state). It is essential to be able to predict the minimum critical value required for the existence of snap-through.

During fabrication process, microbeams of immovable edges are subjected to residual stress, which can induce deformation and introduces axial force in the system in the case of clamped-clamped beams [18-20]. Besides the axial force induced during fabrication, an additional transversal force can also be applied to the arch beam through other means, such as electrostatic actuation [21]. The axial force can alter the stiffness of the microbeams and accordingly affect the natural frequencies. The effects of axial loads on the static and dynamic behavior of microbeams have been investigated extensively in the literature [22-26]. Elata et al. [24] proposed a novel method for measuring the residual stress based on the bifurcation response of a clamped-clamped straight beam under electrostatic forces. They showed the critical voltage required for the bifurcation response as a monotonic function of the residual stress. Alkharabsheh et al. [26] investigated the statics and dynamics of MEMS arches under axial forces and electrostatic forces. They indicated that the effect of axial force on MEMS arches can either increase or decrease their stiffness, and thereby increase or decrease the natural frequency depending on the applied electrostatic and axial forces. Nayfeh et al [27] investigated analytically and experimentally the natural frequencies and modeshapes of buckled beams. They obtained the exact solutions for the linear modes and associated frequencies of initially buckled beam for different boundary conditions. Several studies have also investigated the nonlinear vibrations of arch beams [21, 28-38]. Most of these studies are based on the Galerkin discretization [13, 14, 20, 39, 40]. Krakover et al [41] demonstrated a displacement sensing technique by monitoring the resonant frequency of a curved beam. They showed experimentally a pre-buckling decrease and post-buckling increase of the natural frequency of an arch beam subjected to an electrostatic actuation and demonstrated an increase in sensitivity by dynamically operating the beam near the beginning of snap-through. Curved beams under electrostatic actuation and compressive axial load were also investigated in [42, 43], where coexistence of many competing in-well and cross-well attractors were reported.

It is well known that the stiffness of arches increases with their initial rise and curvature. On the other hand, applying tensile axial loads on arches reduces their curvature, and hence reduces their stiffness for some regimes before the tension effect dominates. Despite the several works on MEMS arches, the literature lacks a thorough investigation on the competing effects of initial rise of the arch and the axial loads on its stiffness and natural frequencies. Practical design guidelines and curves that can guide the designers to regimes where the natural frequencies can increase monotonically, which is an essential requirement for tunable resonators, are not available. Hence, one aim of this work is to generate universal curves for the first three symmetric resonance modes of an arch beam under both tensile and compressive axial loads for different values of initial curvature. Also, we aim to derive analytical equations that can predict the minimum point where the behavior of the natural frequency of the arch changes qualitatively with the axial load.

The organization of the rest of the paper is as follows: Section II introduces the governing equation of an arch beam. Static deflection analysis is discussed in Section III. Section IV discusses the exact

solutions to the eigenvalue problem. In Section V, the proposed analytical expression to obtain the point of minimum where the trend of the resonance frequency against the axial loads changes qualitatively due to the competing effects of axial loads and initial curvature are presented. Demonstration with experimental case study is discussed in Section VI and Section VII shows conclusions of this paper.

II. Problem Formulation

The system under consideration, Fig. 1, consists of a clamped-clamped arch beam under compressive axial load, $\hat{P}$. Since this study is focused on the effects of initial curvature and axial loads on the statics and dynamics of arch beams, the exact solution of an arch beam is adopted following the procedures in [27, 39]. Design parameters for the problem can be defined as follow; length L , width b , thickness h , cross-sectional area A , moment of inertia I , young modulus E , density ρ , coefficient of the damping force c and the initial midpoint elevation of the beam a . The governing equation of motion, using an Euler-Bernoulli beam model, describing the transverse deflection of the arch beam under consideration can be expressed as [40]

$$EI \frac{\partial^4 \hat{w}(\hat{x}, \hat{t})}{\partial \hat{x}^4} + \rho A \frac{\partial^2 \hat{w}(\hat{x}, \hat{t})}{\partial \hat{t}^2} + \hat{c} \frac{\partial \hat{w}(\hat{x}, \hat{t})}{\partial \hat{t}} - \hat{P} \left(\frac{\partial^2 \hat{w}(\hat{x}, \hat{t})}{\partial \hat{x}^2} + \frac{d^2 \hat{w}_o(\hat{x})}{d\hat{x}^2} \right) = \frac{EA}{2L} \left(\frac{\partial^2 \hat{w}(\hat{x}, \hat{t})}{\partial \hat{x}^2} + \frac{d^2 \hat{w}_o(\hat{x})}{d\hat{x}^2} \right) \int_0^L \left(\left(\frac{\partial \hat{w}(\hat{x}, \hat{t})}{\partial \hat{x}} \right)^2 + 2 \frac{\partial \hat{w}(\hat{x}, \hat{t})}{\partial \hat{x}} \frac{d\hat{w}_o(\hat{x})}{d\hat{x}} \right) d\hat{x} \quad (1)$$

where $\hat{x}$ is the position along the length of the beam, $\hat{t}$ is time and $\hat{w}_o(\hat{x})$ is the initial deflection of the arch beam. Here we assume it to be in the downward configuration as

$$\hat{w}_o(\hat{x}) = -\frac{\hat{a}}{2} \left(1 - \cos \left(2\pi \frac{\hat{x}}{L} \right) \right) \quad (2)$$

Note that a positive deflection in $\hat{w}(\hat{x}, \hat{t})$ means upward deflection with respect to this configuration along the $\hat{z}$ axis.

The arch beam is subjected to the following boundary conditions

$$\hat{w}(0, \hat{t}) = \hat{w}(L, \hat{t}) = 0, \quad \frac{\partial \hat{w}(0, \hat{t})}{\partial \hat{x}} = \frac{\partial \hat{w}(L, \hat{t})}{\partial \hat{x}} = 0 \quad (3)$$

For convenience, we introduce the following nondimensional variables

$$w(x,t) = \frac{\hat{w}(\hat{x},\hat{t})}{r}, \quad w_o(x) = \frac{\hat{w}_o(\hat{x})}{r}, \quad x = \frac{\hat{x}}{L}, \quad t = \frac{\hat{t}}{T} \quad \text{where} \quad r = \sqrt{\frac{I}{A}}, \quad T = \sqrt{\frac{\rho AL^4}{EI}} \quad (4)$$

where r represents the radius of gyration and T is the time constant. By simplifying and substituting Eq. (4) in Eq. (1) and Eq. (3), we obtain the nondimensional equation of motion and the associated boundary conditions

$$\begin{aligned} & \frac{\partial^4 w(x,t)}{\partial x^4} + \frac{\partial^2 w(x,t)}{\partial t^2} + c \frac{\partial w(x,t)}{\partial t} - P \left(\frac{\partial^2 w(x,t)}{\partial x^2} + \frac{d^2 w_o(x)}{dx^2} \right) \\ &= \frac{Ar^2}{2I} \left(\frac{\partial^2 w(x,t)}{\partial x^2} + \frac{d^2 w_o(x)}{dx^2} \right) \int_0^L \left(\left(\frac{\partial w(x,t)}{\partial x} \right)^2 + 2 \frac{\partial w(x,t)}{\partial x} \frac{dw_o(x)}{dx} \right) dx \end{aligned} \quad (5a)$$

$$w_o(x) = -\frac{a}{2} (1 - \cos(2\pi x)) \quad (5b)$$

$$w(0,t) = w(L,t) = 0, \quad \frac{\partial w(0,t)}{\partial x} = \frac{\partial w(L,t)}{\partial x} = 0 \quad (6)$$

where the nondimensional parameters in Eq. (5a) and Eq. (5b) are defined as;

$$c = \frac{\hat{c}L^4}{EIT}, \quad a = \frac{\hat{a}}{r} \quad \text{and} \quad P = \frac{\hat{P}L^2}{EI}$$

III. Static Deflection

a. Exact Solution

Following [39] we outline next the static solution of the arch equation. The equation governing the static deflection of the arch beam is obtained by dropping the time dependent terms in Eq. (5) and thus, Eq. (5) and Eq. (6) become

$$\begin{aligned} & \frac{d^4 w_e}{dx^4} - P \left(\frac{d^2 w_e}{dx^2} + \frac{d^2 w_o}{dx^2} \right) \\ &= \frac{Ar^2}{2I} \left(\frac{d^2 w_e}{dx^2} + \frac{d^2 w_o}{dx^2} \right) \int_0^1 \left(\left(\frac{dw_e}{dx} \right)^2 + 2 \frac{dw_e}{dx} \frac{dw_o}{dx} \right) dx \end{aligned} \quad (7a)$$

$$w_e(0) = w_e(1) = 0, \quad \frac{dw_e(0)}{dx} = \frac{dw_e(1)}{dx} = 0 \quad (7b)$$

Next, following the procedures in [27, 39], the exact solution of Eq.(7a) and Eq. (7b) is derived, which yields an expression for the total static deflection in the form of

$$\psi = \frac{b}{2} (1 - \cos(2\pi x)) \quad (8a)$$

$$\text{where } \psi(x) = w_e(x) + w_o(x) \quad (8b)$$

and b is the amplitude of the total static deflection given by

$$b^3 + \left(\frac{4}{\pi^2} P - a^2 + 16 \right) b + 16a = 0 \quad (9)$$

It is worth to note that as an alternative approach to solving Eq. (7), one can also apply the Galerkin method to discretize the equation.

Eq. (9) can be used to have an expression for the value of an axial load at which two equilibrium state of the solutions can exist for different values of initial curvature. The critical axial load, P_c at which two equilibrium solutions of Eq. (9) coexist can be obtained by solving for P in Eq. (9) when the discriminant is zero. The critical axial load P_c can be expressed as

$$P_c = -4\pi^2 + \pi^2 \left(-3a^{\frac{2}{3}} + \frac{a^2}{4} \right) \quad (10)$$

b. Results of the Static Analysis

In Fig. 2, the relationship between P_c and the initial rise a is shown. When $a = 0$ (a straight beam), the compressive axial load requires to buckle the beam is $4\pi^2$ as classically well-known. As the initial curvature is increased, more compressive axial load is required until $a = 2\sqrt{2}$ where the tensile axial

load reaches its maximum value. This plot can be also used to predict the minimum axial load required for having only one stable state in the system when the axial load is increased beyond the minimum value. Detailed explanation of this plot will be made with subsequent plots.

Fig. 3 and Fig. 4 show the variation of the static deflection around equilibrium points versus the axial load for different values of initial midpoint elevation. Perturbed pitchfork bifurcation is observed for $a \neq 0$ as classically established. It can also be observed that the critical buckling loads in Fig. 2 correspond to the saddle node points in Fig. 3 and Fig. 4. For instance, the critical buckling load for $a = 8$ in Fig. 2 is $P = 0$ and the corresponding axial load for $a = 8$ at the saddle node point in Fig. 4 is also zero. In addition, after the saddle node points, only one stable state of the system remains which means that the critical buckling loads can be used to predict the existence of only one stable state of the system response.

IV. Eigenvalue Problem

a. Eigenvalue Problem

Following [27, 39], the eigenvalue problem can be derived by linearizing the dynamic deflection of the beam around its static equilibrium deflection. In view of this fact, the deflection of the arch beam under axial load $w(x, t)$ is assumed to be the sum of the static equilibrium deflection $w_e(x)$ and the dynamic deflection $v(x, t)$

$$w(x, t) = w_e(x) + v(x, t) \quad (11)$$

Next, Eq. (7a), Eq. (8b) and Eq. (11) are substituted into Eq. (5a), which yields

$$\begin{aligned} & \frac{\partial^4 v(x, t)}{\partial x^4} + \frac{\partial^2 v(x, t)}{\partial t^2} + c \frac{\partial v(x, t)}{\partial t} + \frac{\partial^2 v(x, t)}{\partial x^2} \left(-P - \frac{1}{2} \int_0^1 \left(\left(\frac{d\psi}{dx} \right)^2 - \left(\frac{dw_o}{dx} \right)^2 \right) dx \right) \\ & - \frac{1}{2} \frac{\partial^2 v(x, t)}{\partial x^2} \left(\int_0^1 \left(\left(\frac{\partial v(x, t)}{\partial x} \right)^2 + 2 \frac{\partial v(x, t)}{\partial x} \frac{d\psi}{dx} \right) dx \right) - \frac{d^2 \psi}{dx^2} \int_0^1 \left(\frac{\partial v(x, t)}{\partial x} \frac{d\psi}{dx} \right) dx \\ & - \frac{1}{2} \frac{d^2 \psi}{dx^2} \int_0^1 \left(\frac{\partial v(x, t)}{\partial x} \right)^2 dx = 0 \end{aligned} \quad (12)$$

By dropping the nonlinear, nonhomogeneous, and damping terms in Eq. (12), we obtain the linearized equation

$$\begin{aligned} \frac{\partial^4 v(x,t)}{\partial x^4} + \frac{\partial^2 v(x,t)}{\partial t^2} + \frac{\partial^2 v(x,t)}{\partial x^2} \left(-P - \frac{1}{2} \int_0^1 \left(\left(\frac{d\psi}{dx} \right)^2 - \left(\frac{dw_o}{dx} \right)^2 \right) dx \right) \\ - \frac{d^2 \psi}{dx^2} \int_0^1 \left(\frac{\partial v(x,t)}{\partial x} \frac{d\psi}{dx} \right) dx = 0 \end{aligned} \quad (13)$$

To determine the natural frequencies ω and their corresponding modeshapes ϕ , the dynamic deflection is assumed to be of the form

$$v = \phi(x) e^{i\omega t} \quad (14)$$

Eq. (14) is then substituted into Eq. (13) to obtain the expression for the mode equation and the associated boundary conditions

$$\frac{d^4 \phi}{dx^4} + \frac{d^2 \phi}{dx^2} \left(-P - \frac{1}{2} \int_0^1 \left(\left(\frac{d\psi}{dx} \right)^2 - \left(\frac{dw_o}{dx} \right)^2 \right) dx \right) - \omega^2 \phi - \frac{d^2 \psi}{dx^2} \int_0^1 \left(\frac{d\phi}{dx} \frac{d\psi}{dx} \right) dx = 0 \quad (15a)$$

$$\phi(0) = \phi(1) = 0 \quad \text{and} \quad \frac{d\phi(0)}{dx} = \frac{d\phi(1)}{dx} = 0 \quad (15b)$$

The integral terms in Eq. (15a) can be considered as constant terms by following the procedures in [27]. Thus, we let

$$\varsigma = \left(-P - \frac{1}{2} \int_0^1 \left(\left(\frac{d\psi}{dx} \right)^2 - \left(\frac{dw_o}{dx} \right)^2 \right) dx \right) \quad \text{and} \quad A_5 = \int_0^1 \left(\frac{d\phi}{dx} \frac{d\psi}{dx} \right) dx \quad (16)$$

Substituting Eq. (16) into Eq. (15a), we obtain

$$\frac{d^4 \phi}{dx^4} + \varsigma \frac{d^2 \phi}{dx^2} - \omega^2 \phi - A_5 \frac{d^2 \psi}{dx^2} = 0 \quad (17)$$

Using the boundary conditions in Eq. (15b), the homogeneous and the particular solutions of the mode equation are expressed in the form

$$\begin{aligned} \phi = A_1 \cos(\beta_1 x) + A_2 \cosh(\beta_2 x) + A_3 \sin(\beta_1 x) + A_4 \sinh(\beta_2 x) \\ + A_5 \frac{2b\pi^2}{4\pi^2 \varsigma + \omega^2 - 16\pi^4} \cos(2\pi x) \end{aligned} \quad (18)$$

$$\text{where, } \beta_1 = \frac{\sqrt{\zeta + \sqrt{\zeta^2 + 4\omega^2}}}{\sqrt{2}}, \quad \beta_2 = \frac{\sqrt{-\zeta + \sqrt{\zeta^2 + 4\omega^2}}}{\sqrt{2}} \quad (19)$$

The relationship between the axial load and the natural frequencies can be obtained by solving for the amplitude of the total static deflection in Eq. (9) in terms of the axial load and the initial rise and substituting it in Eq. (16) to have ζ , which can then be replaced in the mode equation.

b. Results of the Eigenvalue Analysis

In the following, we solve for the eigenvalue problem of the arch around the lower equilibrium position branch in Fig. 3 and Fig.4, since this is the more practical case where the natural frequency of the arch is desired to be tuned continuously for an extended range.

The variation of the first natural frequency for different values of initial rise is shown in Fig. 5 against both tensile and compressive forces. We recall here the sign convention adopted in this study that the axial force is assumed to be positive in tension and negative in compression. It can be observed from Fig. 5 that the minimum points of the frequency response for smaller values of initial rise (i.e. $a < 1$) are on the compressive axial load region. In these cases, the first resonant frequency increases monotonically with the increase in the tensile axial load. Under compressive axial loads, the frequency experiences an initial decrease for lower values of compressive loads and then increases as the compressive load increases until it saturates at frequency values, which approximately triples the first natural frequency. This shows a wide tunability of an arch beam under compressive axial load for smaller initial rises.

However, the minimum points of the frequency response for higher values of initial rise ($a > 3$) of the arch beam occurred in the tensile axial load regime. For these cases, the frequency decreases for lower values of the tensile axial load until it reaches a critical load, at which there is a minimum point, where the previous behavior due to the initial curvature of the beam is overcome by the increase in stiffness of the beam due to increase in the axial load and thereby increases the frequency monotonically as the tensile load increases afterwards. On the other hand, under compressive axial loads, the frequency behavior increases until it also saturates at larger values of the compressive load.

One can note that there are critical values of loads for the minimum points, where the qualitative change of behavior of the resonance frequency versus axial load occurs. These are reflection of the competition between the increase or decrease in stiffness due to the axial load change and how this affects the arch curvature. For example, for $a=3$, the arch is already curved up that for some range of axial load, applying tension leads to decrease in its curvature, and hence, reduces for some range its natural frequency. Then, at some loads, the curvature becomes too small that it is no longer influential. After the minimum point, increasing the tensile load increases the stiffness of the arch, which eventually becomes a flat beam. Hence, it continues to experience an increase in its stiffness with the increase of the tensile load.

A 3D plot that shows the variation of the frequency as the axial load is varied for different initial rises is also shown in Fig. 6. Similar analyses are made for the third and fifth resonant frequencies as shown in Fig. 7 and Fig. 8.

V. Critical Loads at the Minimum Points

a. Critical Load Analysis

It is essential to understand the effects of varying the axial load on the behavior of the arch beam. As mentioned in Section IV that there is a critical point for each initial rise at which the frequency behavior with the axial load changes qualitatively. For practical applications where continuous trend in the resonance frequency with the axial load is desirable, it is important to predict the location of these minimum points. Here, we derive the criteria for such.

In deriving an expression for the critical value, a Galerkin discretization method is applied to have approximate solutions by assuming

$$v(x, t) = \sum u_i(t) \phi_i(x) \quad (20)$$

where $u_i(t)$ is the modal coordinate and $\phi_i(x)$ is the exact modeshapes of an arch beam obtained in Eq. (18). By substituting Eq. (20) into the dynamic deflection equation and applying orthogonality conditions, integrating over the beam domain from 0 to 1, the following linearly coupled ordinary differential equations are obtained in terms of the modal coordinates:

$$\begin{aligned} \ddot{u}_j + u_j \omega_{non,j}^2 = & - \sum u_i \int_0^1 \phi_j w_o'' \int_0^1 (\phi_i' w_o') dx - \sum u_i \int_0^1 \phi_j \phi_i'' dx \left[-P - \frac{1}{2} \int_0^1 (w_e'^2 + 2w_e' w_o') dx \right] \\ & + \sum u_i \int_0^1 \phi_j (w_e'' + w_o'') \int_0^1 \phi_i' (w_e' + w_o') dx \end{aligned} \quad (21)$$

Using a single mode approximation for first mode, we obtain

$$\begin{aligned} \ddot{u}_1 + u_1 \omega_{non,1}^2 = & - u_1 \int_0^1 \phi_1 w_o'' \int_0^1 (\phi_1' w_o') dx - u_1 \int_0^1 \phi_1 \phi_1'' dx \left[-P - \frac{1}{2} \int_0^1 (w_e'^2 + 2w_e' w_o') dx \right] \\ & + u_1 \int_0^1 \phi_1 (w_e'' + w_o'') \int_0^1 \phi_1' (w_e' + w_o') dx \end{aligned} \quad (22)$$

Assuming harmonic motion of the form $u_1 = C e^{j\omega t}$, where C is a constant, Eq. (22) becomes

$$\begin{aligned} -\omega_1^2 + \omega_{non,1}^2 = & - \int_0^1 \phi_1 w_o'' \int_0^1 (\phi_1' w_o') dx - \int_0^1 \phi_1 \phi_1'' dx \left[-P - \frac{1}{2} \int_0^1 (w_e'^2 + 2w_e' w_o') dx \right] \\ & + \int_0^1 \phi_1 (w_e'' + w_o'') \int_0^1 \phi_1' (w_e' + w_o') dx \end{aligned} \quad (23)$$

By noting that the point of minimum is the point at which the derivative of the differential function is equal to zero in its domain, we differentiate Eq. (23) with respect to P and equate it to zero, recall that w_e is a function of b in Eq. (8b) and b is a function of P in Eq. (9), which gives

$$\frac{\partial \omega_1}{\partial P} = \frac{1}{4\omega_1} \left[\begin{aligned} & -2 \int \phi_1 \phi_1'' dx - \left(\int \phi_1 \phi_1'' dx \right) \int \left(2 \frac{\partial w_e'}{\partial b} \frac{\partial b}{\partial p} w_s' + 2 \frac{\partial w_e'}{\partial b} \frac{\partial b}{\partial p} w_o' \right) \\ & -2 \left(\int \phi_1 (w_e'' + w_o'') dx \right) \left(\int \phi_1' \frac{\partial w_e'}{\partial b} \frac{\partial b}{\partial p} dx \right) \\ & -2 \left(\int \phi_1 \frac{\partial w_e''}{\partial b} \frac{\partial b}{\partial p} dx \right) \left(\int \phi_1' w_e' dx \right) \\ & -2 \left(\int \phi_1 \frac{\partial w_e''}{\partial b} \frac{\partial b}{\partial p} dx \right) \left(\int \phi_1' w_o' dx \right) \end{aligned} \right] = 0 \quad (24)$$

Since b is a function of P in Eq. (9), Eq. (9) can be differentiated with respect to P to have

$$\frac{\partial b}{\partial P} = \frac{4b}{3\pi^2 b^2 - 4P - a^2 \pi^2 + 16\pi^2} \quad (25)$$

Substituting the known values of ϕ_1 , w_o and w_e into Eq. (24) and solving for $\frac{\partial b}{\partial P}$, a simple algebraic expression can be obtained for the first mode

$$\frac{\partial b}{\partial P} = \frac{11.4875}{56.1716a - 112.86b - 56.1716(a+b)} \quad (26)$$

Solving Eq. (25) and Eq. (26) simultaneously and simplifying the expression, a closed form expression can be obtained to generate the loci for the minimum points of the first natural frequency in Fig. 5.

$$P = \frac{1}{4 \int \phi_n \phi_n'' dx} \left[\int \phi_n \phi_n'' dx (3\pi^2 b^2 - a^2 \pi^2 + 16\pi^2) \right. \\ \left. + 4b \left(\left(\int \phi_n \phi_n'' dx \right) \int \left(\frac{\partial w_e'}{\partial b} w_e' + \frac{\partial w_o'}{\partial b} w_o' \right) \right) \right. \\ \left. + \left(\int \phi_n (w_e'' + w_o'') dx \right) \left(\int \phi_n' \frac{\partial w_e'}{\partial b} dx \right) + \left(\int \phi_n \frac{\partial w_e''}{\partial b} dx \right) \left(\int \phi_n' w_e' dx \right) \right. \\ \left. + \left(\int \phi_n \frac{\partial w_o''}{\partial b} dx \right) \left(\int \phi_n' w_o' dx \right) \right] \quad (27)$$

Similar expressions can be developed for other modes by replacing the modeshapes in Eq. (27).

b. Results

The variation of the critical axial load at the minimum points against the initial curvature is shown in Fig. 9 and the results of the proposed analytical expression for the critical axial load are in good agreement with the loci for each critical axial load in the first resonance frequency (i.e., Fig. 5). The proposed analytical expression can be used to evaluate the critical axial load for any resonance frequency.

VI. Demonstration with experimental case study

The universal curves are demonstrated with experimental data obtained from an electrothermally actuated arch resonator [43]. The dimensions for the arch resonator are shown in Table. 1. In the case study, the electrothermal voltage is converted to compressive axial load using the following expressions:

$$S_{Th} = \alpha EA \int_0^1 (T[x] - T_a) dx \quad (28)$$

$$T[x] = \frac{\sigma_e V_{Th}^2}{2k} \left(\frac{x}{L} - \frac{x^2}{L^2} \right) + T_a \quad (29)$$

where S_{Th} is the compressive load, α is the coefficient of thermal expansion, $T[x]$ is the temperature distribution, T_a is the ambient temperature, σ_e is the electrical conductivity of the material, and k is the thermal conductivity of the microbeam material. For the electrothermally actuated arch resonator, good agreement is achieved for nondimensional initial rise of 2.728 for the first resonance frequency as shown in Fig. 10.

VII. Conclusions

This paper investigated the effects of axial load and initial rise of an arch beam on the static deflection and resonance frequencies. It was shown that the influences of axial load and initial rise are significant on the behavior of arch beams. It is demonstrated in this paper that the effects of axial force on MEMS arches can either increase or decrease the stiffness of the arch beam and thereby increasing or decreasing the natural frequency respectively, depending on the initial rise of the arch beam and the applied tensile or compressive axial force. Analytical expressions were developed that can be used to predict the critical value of an axial load requires for the existence of snap-through and for the value of the axial load required to qualitatively change the frequency behavior of an arch beam. These closed form expressions can be used for any modeshape of an arch beam. Universal curves were also generated for the first three symmetric modes of an arch beam under both tensile and compressive axial loads for different values of initial curvature. The universal curves showed possibility of achieving wide range of tunability and predicting change in mechanical behavior of arches subjected to additional axial loads for different initial rises.

VIII. References

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Acknowledgement

This research has been support by KAUST.

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Fig. 1: Schematic diagram for axially loaded arch beam.

Fig. 2: Variation of the critical buckling load with the initial rise.

(a)

Fig. 3: (a) Variation of the static deflection around the equilibrium point against the axial load for initial rises from 0.1 to 1. (b) Enlarged view of (a) around the origin.

(a)

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(a)

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Fig 10: Comparison of the first natural frequency obtained from the universal curves and experimental data obtained from the electrothermally actuated arch resonator [43].

Parameters	Case study One
Length L (μm)	500
Thickness h (μm)	2.8
Width b (μm)	25
Initial Rise a (μm)	2.2

Table 1: Geometric dimensions of the arch resonator [43].