Wave-equation Q tomography

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ABSTRACT

Strong subsurface attenuation leads to distortion of amplitudes and phases of seismic waves propagating inside the earth. The amplitude and the dispersion losses from attenuation are often compensated for during prestack depth migration. However, most attenuation compensation or Q -compensation migration algorithms require an estimate of the background Q model. We have developed a wave-equation gradient optimization method that inverts for the subsurface Q distribution by minimizing a skeletonized misfit function ϵ , where ϵ is the sum of the squared differences between the observed and the predicted peak/centroid-frequency shifts of the early arrivals. The gradient is computed by migrating the observed traces weighted by the frequency shift residuals. The background Q model is perturbed until the predicted and the observed traces have the same peak frequencies or the same centroid frequencies. Numerical tests determined that an improved accuracy of the Q model by wave-equation Q tomography leads to a noticeable improvement in migration image quality.

INTRODUCTION

The earth is anelastic and distorts the amplitude and the phase of propagating seismic waves (Aki and Richards, 1980). Attenuation of P-waves can be quantified by a quality factor Q that accounts for the phase shift and the amplitude loss as a function of the frequency content of the propagating waves and the distance traveled. Lower values of Q imply greater energy loss of the wave per cycle or higher attenuation.

The amplitude loss caused by attenuation is often compensated for during prestack depth migration (PSDM). For example, Xin et al. (2008) and Xie et al. (2009) compensate for the attenuation loss by

ray-tracing methods. For 3D Kirchhoff migration, Traynin et al. (2008) replace the conventional acoustic traveltime expression with a frequency-dependent complex traveltime expression containing three terms. The first term carries the primary kinematic information of the arrivals, whereas the second and the third terms are amplitude- and phase-compensating terms, respectively. Dai and West (1994), Yu et al. (2002), Wang (2008), and Valenciano et al. (2011) use one-way wave-equation migration in the frequency domain for attenuation compensation. For reverse time migration, Zhang et al. (2010), Suh et al. (2012), Fletcher et al. (2012), Bai et al. (2013), Zhu et al. (2014), and Zhu and Harris (2015) propose different viscoacoustic wave equations with separate controls over phase and amplitude to compensate for the attenuation loss. Dutta and Schuster (2014), Sun et al. (2015a, 2016), and Dai et al. (2015) use viscoacoustic least-squares reverse time migration (RTM) schemes to compensate for the amplitude loss and the phase distortion caused by attenuation during migration.

Besides velocity, an additional input requirement for these Q -based migration algorithms is a reliable estimate of the background Q model. A Q model can be estimated using either data-domain- or image-domain-based tomographic techniques. In the data domain, Brzostowski and McMechan (1992) use the log of the recorded amplitudes and the source amplitudes as the input data for Q tomography. In contrast, Quan and Harris (1997) use the centroid-frequency shifts between the predicted and the observed traces and smear the shifts along raypaths to update the Q model. Hu et al. (2011) use a fitting function for the source amplitude spectrum to account for asymmetric source spectra and a multi-index box-constrained optimization technique to eliminate unrealistic Q values during Q inversion. In their approach, they use the centroid-frequency shift method that made their technique more useful for field-data applications. These frequency-shift methods rely on the high-frequency assumption made in classical ray-based tomography. Alternative data-domain approaches include using a full-waveform inversion (FWI)-like algorithm, in which an objective function is setup to invert for a Q model that minimizes the L_2 -norm of the residual between the observed and the predicted data (Liao and

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McMechan, 1995; Malinowski et al., 2007; Kamei and Pratt, 2008; Bai and Yingst, 2013; Wang and Zhang, 2014).

For most image-domain techniques, a residual image is minimized, which is the difference between the image computed by the background Q model and a target image, which is attenuation-free (Shen et al., 2014, 2015; Shen and Zhu, 2015). The image perturbation is then related to the perturbation in Q using ray-based or wave-equation-based migration velocity analysis operators. A Q -tomography technique in the interpreted-image domain was developed by Zhou et al. (2011), who generate amplitude ratio maps by comparing the reflection amplitudes associated with a lossy layer with that of a reference horizon that remains unaffected. Through ray tracing, they accumulate the attenuation effects along raypaths and a Q volume is then estimated from tomographic inversion. A similar approach was also used by He et al. (2013), who use picked events in depth-migrated common image gathers and smear along raypaths the centroid-frequency shifts associated with the picked events.

In this paper, we present a novel skeletonized wave-equation Q inversion method that finds the Q model that minimizes the difference between the peak frequencies of the observed and the predicted transmission arrivals and early arrivals. The peak frequencies are obtained from the amplitude spectra of the traces, which are a skeletonized representation of the data. The Fréchet derivative is derived using the implicit function theorem, and the gradient is numerically obtained by a zero-lag crosscorrelation between the forward-propagated source wavefield and the back-propagated residual traces. A residual trace is obtained by weighting the observed trace by the frequency shift between the trace and its corresponding predicted trace. Unlike conventional ray-based Q tomography, the wave-equation Q tomography (WQ) residuals are smeared along transmission wavepaths (Woodward, 1992) computed from finite-difference solutions to the time-domain viscoacoustic wave equation characterized by the standard linear solid (SLS) mechanism (Christensen, 1982; Carcione et al., 1988; Blanch et al., 1995). The proposed approach has no high-frequency assumptions about the data, unlike ray-based tomography methods. It is also less susceptible to cycle-skipping problems associated with any FWI-like algorithm in which the amplitude and the phase differences between the predicted and the observed traces are minimized to obtain the Q model.

This paper is organized into five sections. After the introduction, the second section describes the theory of wave-equation Q inversion. Numerical results on synthetic and field data are then presented in the third section. The limitations of the proposed method are discussed in the fourth section and the conclusions are in the last section.

THEORY OF WAVE-EQUATION Q INVERSION

The key steps in WQ are similar to the wave-equation traveltime inversion algorithm proposed in Luo and Schuster (1991a, 1991b). These steps can be generalized for any type of skeletonized data and are the following: (1) define a connective function that connects the frequency-shift residual with the pressure seismogram, (2) define a frequency-shift misfit function, and (3) derive the perturbation of the misfit function with respect to Q using the viscoacoustic wave equation and the connective function defined in the first step.

In our analysis, we assume that the wave propagation honors the 2D time-domain viscoacoustic wave equation based on the SLS mechanism (Blanch et al., 1995). For a given velocity and Q model, the pressure seismograms can be computed by

$$\begin{aligned}\frac{\partial P}{\partial t} + K \frac{\tau_\epsilon}{\tau_\sigma} (\nabla \cdot \mathbf{v}) + r_p &= S(\mathbf{x}_s, t), \\ \frac{\partial \mathbf{v}}{\partial t} + \frac{1}{\rho} \nabla P &= 0, \\ \frac{\partial r_p}{\partial t} + \frac{1}{\tau_\sigma} \left(r_p + K \left(\frac{\tau_\epsilon}{\tau_\sigma} - 1 \right) (\nabla \cdot \mathbf{v}) \right) &= 0.\end{aligned}\quad (1)$$

Here, $\mathbf{v} = \{v_x, v_y, v_z\}$ represents the particle velocity vector, P represents the pressure, r_p represents the memory variable, K represents the bulk modulus of the medium, and $S(\mathbf{x}_s, t)$ represents a band-limited point source function at $\mathbf{x} = \mathbf{x}_s$. The stress and strain relaxation parameters, τ_σ and τ_ϵ , are related to the quality factor Q , and the reference angular frequency ω (usually chosen to be the central frequency of the source wavelet) as (Robertsson et al., 1994)

$$\begin{aligned}\tau_\sigma &= \frac{\sqrt{1 + \frac{1}{Q^2}} - \frac{1}{Q}}{\omega}, \\ \tau_\epsilon &= \frac{1}{\omega^2 \tau_\sigma} = \frac{\sqrt{1 + \frac{1}{Q^2}} + \frac{1}{Q}}{\omega}.\end{aligned}\quad (2)$$

To simplify the algebraic formula, the following substitution is made:

$$\tau = \frac{\tau_\epsilon}{\tau_\sigma} - 1 \quad (3)$$

so that equation 1 becomes

$$\begin{aligned}\frac{\partial P}{\partial t} + K(\tau + 1)(\nabla \cdot \mathbf{v}) + r_p &= S(\mathbf{x}_s, t), \\ \frac{\partial \mathbf{v}}{\partial t} + \frac{1}{\rho} \nabla P &= 0, \\ \frac{\partial r_p}{\partial t} + \frac{1}{\tau_\sigma} (r_p + \tau K(\nabla \cdot \mathbf{v})) &= 0.\end{aligned}\quad (4)$$

From equations 2 and 3, the relation between τ and Q can be obtained as

$$\tau = \frac{2}{Q} \left(\frac{1}{Q} + \sqrt{1 + \frac{1}{Q^2}} \right). \quad (5)$$

Figure 1a–1c shows the variation of the relaxation parameters τ_ϵ , τ_σ , and τ with Q , respectively. It can be seen from these figures that high values of τ imply strong attenuation whereas low values indicate weak attenuation. In addition, for realistic geologic models, Q varies from 20 to 200. In this range, τ has a wider variation in its values than τ_σ and τ_ϵ . Thus, for the parameterization in WQ, τ is used because it is quite sensitive to small changes in Q .

In the next subsection, we define a connective function that connects the change in peak frequency of an arrival with the observed and the predicted pressure seismograms.

Connective function

Let $\tilde{P}_f(\mathbf{x}_r, t; \mathbf{x}_s)$ denote a predicted event for a given background Q model recorded at the receiver location $\mathbf{x}_r$ due to a source excited at time $t = 0$ and location $\mathbf{x}_s$. The subscript f is the peak frequency of this event that can be obtained from its amplitude spectrum (shown by the red curve in Figure 2). Similarly, let $P_{f-f_1}(\mathbf{x}_r, t; \mathbf{x}_s)$ denote the same event in the observed data with a peak frequency of $f - f_1$ (the spectrum of this event is shown by the blue curve in Figure 2). Here, f_1 is the shift between the peak frequencies of the predicted and the observed traces because of Q .

For a sufficiently accurate background velocity model, the similarity between the observed and predicted traces in Figure 3 can be written as

$$F_{f_1}(\mathbf{x}_r, t; \mathbf{x}_s) = \int dt P_{f-f_1}(\mathbf{x}_r, t; \mathbf{x}_s) \tilde{P}_f(\mathbf{x}_r, t; \mathbf{x}_s). \quad (6)$$

Figure 1. Variation of the parameters (a) τ_ϵ , (b) τ_σ , and (c) τ for different values of Q . The central frequency of the source wavelet is taken to be 20 Hz.

In WQ, we seek to minimize the peak-frequency shift between an observed and a predicted trace. For an accurate background Q model, the predicted and the observed arrivals will have the same peak frequency. In other words, we seek a shift value $f_1 = \Delta f$ that maximizes the crosscorrelation function in equation 6. If $\Delta f = 0$, it indicates that the correct background Q model has been found and the transmission arrivals in the predicted and observed traces have the same peak frequencies.

Because $f_1 = \Delta f$ maximizes the crosscorrelation function in equation 6, the derivative of $F_{f_1}(\mathbf{x}_r, t; \mathbf{x}_s)$ with respect to f_1 should be zero at $f_1 = \Delta f$. Thus,

$$\begin{aligned} \dot{F}_{\Delta f}(\Delta f, \tau(\mathbf{x})) &= \left[\frac{\partial F_{f_1}(\mathbf{x}_r, t; \mathbf{x}_s)}{\partial f_1} \right]_{f_1=\Delta f}, \\ &= \int dt \dot{P}_{f-\Delta f}(\mathbf{x}_r, t; \mathbf{x}_s) \tilde{P}_f(\mathbf{x}_r, t; \mathbf{x}_s) = 0, \end{aligned} \quad (7)$$

where $\dot{P}_{f-\Delta f}(\mathbf{x}_r, t; \mathbf{x}_s) = [\partial P_{f-f_1}(\mathbf{x}_r, t; \mathbf{x}_s) / \partial f_1]_{f_1=\Delta f}$. Equation 7 is the connective function that acts as an intermediate equation to connect the fundamental data (the pressure seismograms) with the skeletonized data, which are the frequency-shift residuals

Figure 2. Comparison between the amplitude spectra of a single transmission arrival in a predicted and an observed trace.

Figure 3. Comparison between a predicted and an observed trace for a single transmission arrival. A Ricker wavelet with a peak frequency of 12 Hz is used as the source wavelet, and the reference frequency here is smaller than the seismic data frequency band.

(Luo and Schuster, 1991a, 1991b). Such a connective function is required because there is no wave equation that relates the skeletonized data to a single type of model parameter. The inversion is controlled by a skeletonized misfit function formed from a reduced version of the fundamental data.

The misfit function for WQ is defined in the next section, whereas the connective function defined in this section will be later used to derive the Fréchet derivative of τ .

Misfit function

The WQ method attempts to invert for a Q model or an equivalent τ model, which predicts pressure seismograms $\tilde{P}_f(\mathbf{x}_r, t; \mathbf{x}_s)$ that minimize the misfit function

$$\epsilon = \frac{1}{2} \sum_s \sum_r \Delta f(\mathbf{x}_r, \mathbf{x}_s)^2, \quad (8)$$

where Δf is defined in the previous section and the summation in equation 8 is over all sources and receivers. The gradient $\gamma(\mathbf{x})$ is given by

$$\gamma(\mathbf{x}) = -\frac{\partial \epsilon}{\partial \tau(\mathbf{x})} = -\sum_s \sum_r \frac{\partial \Delta f}{\partial \tau(\mathbf{x})} \Delta f(\mathbf{x}_r, \mathbf{x}_s). \quad (9)$$

Figure 4 shows the frequency-shift misfit function for different values of Q . From this figure, we can see that the misfit function monotonically decreases as the hypothetical Q value approaches the correct Q value of 40. Thus, a wave-equation-based gradient optimization method using such a skeletonized misfit function can sometimes quickly converge to the global minimum.

From equation 7, we can get the following three equations:

$$\begin{aligned} \dot{F}_{\Delta f}(\Delta f, \tau(\mathbf{x})) &= 0, \\ \Rightarrow \frac{\partial \dot{F}_{\Delta f}}{\partial \Delta f} \frac{\partial \Delta f}{\partial \tau(\mathbf{x})} + \frac{\partial \dot{F}_{\Delta f}}{\partial \tau(\mathbf{x})} &= 0, \\ \Rightarrow \frac{\partial \Delta f}{\partial \tau(\mathbf{x})} &= -\frac{\frac{\partial \dot{F}_{\Delta f}}{\partial \tau(\mathbf{x})}}{\frac{\partial \dot{F}_{\Delta f}}{\partial \Delta f}}, \end{aligned} \quad (10)$$

Figure 4. Plot of the frequency-shift misfit function versus hypothetical Q values for the background model. The observed data are generated with $Q = 40$.

$$\frac{\partial \dot{F}_{\Delta f}}{\partial \Delta f} = \int dt \dot{P}_{f-\Delta f}(\mathbf{x}_r, t; \mathbf{x}_s) \tilde{P}_f(\mathbf{x}_r, t; \mathbf{x}_s), \quad (11)$$

$$\frac{\partial \dot{F}_{\Delta f}}{\partial \tau(\mathbf{x})} = \int dt \dot{P}_{f-\Delta f}(\mathbf{x}_r, t; \mathbf{x}_s) \frac{\partial \tilde{P}_f(\mathbf{x}_r, t; \mathbf{x}_s)}{\partial \tau(\mathbf{x})}. \quad (12)$$

Using equation 10, the gradient in equation 9 can be written as

$$\gamma(\mathbf{x}) = \sum_s \sum_r \frac{\frac{\partial \dot{F}_{\Delta f}}{\partial \tau(\mathbf{x})}}{\frac{\partial \dot{F}_{\Delta f}}{\partial \Delta f}} \Delta f(\mathbf{x}_r, \mathbf{x}_s). \quad (13)$$

The Fréchet derivative $[\partial \tilde{P}_f(\mathbf{x}_r, t; \mathbf{x}_s) / \partial \tau(\mathbf{x})]$ is now derived in the next subsection.

Fréchet derivative

To obtain the Fréchet derivative of the pressure field with respect to the perturbation in $\tau(\mathbf{x})$, we linearize the viscoacoustic wave equation in equation 4. A perturbation of $\tau \rightarrow \tau + \delta\tau$ will produce perturbed wavefields $\delta \tilde{P}_f$, $\delta \mathbf{v}$, and δr_p , which satisfy the linearized viscoacoustic wave equation given by (Dutta and Schuster, 2014)

$$\begin{aligned} \frac{\partial \delta \tilde{P}_f}{\partial t} + K(\tau + 1)(\nabla \cdot \delta \mathbf{v}) + \delta r_p &= -K \delta \tau (\nabla \cdot \mathbf{v}), \\ \frac{\partial \delta \mathbf{v}}{\partial t} + \frac{1}{\rho} \nabla \delta \tilde{P}_f &= 0, \\ \frac{\partial \delta r_p}{\partial t} + \frac{1}{\tau_\sigma} (\delta r_p + \tau K (\nabla \cdot \delta \mathbf{v})) &= -\frac{K}{\tau_\sigma} \delta \tau (\nabla \cdot \mathbf{v}). \end{aligned} \quad (14)$$

Using the Green's functions $g_p(\mathbf{x}_r, t; \mathbf{x}, 0)$ and $g_{r_p}(\mathbf{x}_r, t; \mathbf{x}, 0)$, equation 14 can also be expressed as

$$\begin{aligned} \delta \tilde{P}_f(\mathbf{x}_r, t; \mathbf{x}_s) &= -\left(K(\mathbf{x}) (g_p(\mathbf{x}_r, t; \mathbf{x}, 0) * \nabla \cdot \mathbf{v}(\mathbf{x}, t; \mathbf{x}_s)) \right. \\ &\quad \left. + \frac{K(\mathbf{x})}{\tau_\sigma(\mathbf{x})} (g_{r_p}(\mathbf{x}_r, t; \mathbf{x}, 0) * \nabla \cdot \mathbf{v}(\mathbf{x}, t; \mathbf{x}_s)) \right) \delta \tau(\mathbf{x}), \end{aligned} \quad (15)$$

where $*$ denotes convolution in time. Dividing by $\delta \tau(\mathbf{x})$ on both sides, we get

$$\begin{aligned} \frac{\partial \tilde{P}_f(\mathbf{x}_r, t; \mathbf{x}_s)}{\partial \tau(\mathbf{x})} &= -\left(K(\mathbf{x}) (g_p(\mathbf{x}_r, t; \mathbf{x}, 0) * \nabla \cdot \mathbf{v}(\mathbf{x}, t; \mathbf{x}_s)) \right. \\ &\quad \left. + \frac{K(\mathbf{x})}{\tau_\sigma(\mathbf{x})} (g_{r_p}(\mathbf{x}_r, t; \mathbf{x}, 0) * \nabla \cdot \mathbf{v}(\mathbf{x}, t; \mathbf{x}_s)) \right). \end{aligned} \quad (16)$$

Here, $g_p(\mathbf{x}, t; \mathbf{x}_s)$ and $g_{r_p}(\mathbf{x}, t; \mathbf{x}_s)$ are the pressure and the memory variable Green's functions, respectively, and satisfy the system of equations $\mathbf{S}\mathbf{w} = \mathbf{F}$ and $\mathbf{S}\mathbf{w}_1 = \mathbf{F}_1$ in equation 4, where $\mathbf{F} = [\delta(\mathbf{x} - \mathbf{x}')\delta(t - t') \quad 0 \quad 0]^T$ and $\mathbf{F}_1 = [0 \quad 0 \quad \delta(\mathbf{x} - \mathbf{x}')\delta(t - t')]^T$. Equation 12 can now be written as

$$\begin{aligned} \frac{\partial \dot{F}_{\Delta f}}{\partial \tau(\mathbf{x})} &= \int dt \dot{P}_{f-\Delta f}(\mathbf{x}_r, t; \mathbf{x}_s) \frac{\partial \tilde{P}_f(\mathbf{x}_r, t; \mathbf{x}_s)}{\partial \tau(\mathbf{x})}, \\ &= - \int dt K(\mathbf{x}) \left(g_p(\mathbf{x}_r, t; \mathbf{x}, 0) * \nabla \cdot \mathbf{v}(\mathbf{x}, t; \mathbf{x}_s) \right. \\ &\quad \left. + \frac{1}{\tau_\sigma(\mathbf{x})} g_{r_p}(\mathbf{x}_r, t; \mathbf{x}, 0) * \nabla \cdot \mathbf{v}(\mathbf{x}, t; \mathbf{x}_s) \right) \times \dot{P}_{f-\Delta f}(\mathbf{x}_r, t; \mathbf{x}_s). \end{aligned} \quad (17)$$

Substituting equations 11 and 17 into equation 13, the gradient $\gamma(\mathbf{x})$ can be expressed as

$$\begin{aligned} \gamma(\mathbf{x}) &= \sum_s \sum_r \frac{\partial \dot{F}_{\Delta f}}{\partial \tau(\mathbf{x})} \Delta f(\mathbf{x}_r, \mathbf{x}_s), \\ &= - \sum_s \sum_r \frac{\int dt K(\mathbf{x}) \left(g_p(\mathbf{x}_r, t; \mathbf{x}, 0) * \nabla \cdot \mathbf{v}(\mathbf{x}, t; \mathbf{x}_s) + \frac{1}{\tau_\sigma(\mathbf{x})} g_{r_p}(\mathbf{x}_r, t; \mathbf{x}, 0) * \nabla \cdot \mathbf{v}(\mathbf{x}, t; \mathbf{x}_s) \right)}{\int dt \dot{P}_{f-\Delta f}(\mathbf{x}_r, t; \mathbf{x}_s) \tilde{P}_f(\mathbf{x}_r, t; \mathbf{x}_s)} \\ &\quad \times \dot{P}_{f-\Delta f}(\mathbf{x}_r, t; \mathbf{x}_s) \Delta f(\mathbf{x}_r, \mathbf{x}_s), \\ &= -K(\mathbf{x}) \sum_s \sum_r \frac{1}{E} \int dt \left(g_p(\mathbf{x}_r, t; \mathbf{x}, 0) * \nabla \cdot \mathbf{v}(\mathbf{x}, t; \mathbf{x}_s) + \frac{1}{\tau_\sigma(\mathbf{x})} g_{r_p}(\mathbf{x}_r, t; \mathbf{x}, 0) * \nabla \cdot \mathbf{v}(\mathbf{x}, t; \mathbf{x}_s) \right) \\ &\quad \times \dot{P}_{f-\Delta f}(\mathbf{x}_r, t; \mathbf{x}_s) \Delta f(\mathbf{x}_r, \mathbf{x}_s), \end{aligned} \quad (18)$$

where

$$E = \int dt \dot{P}_{f-\Delta f}(\mathbf{x}_r, t; \mathbf{x}_s) \tilde{P}_f(\mathbf{x}_r, t; \mathbf{x}_s). \quad (19)$$

Let $\Delta P_f(\mathbf{x}_r, t; \mathbf{x}_s) = (1/E) \dot{P}_{f-\Delta f}(\mathbf{x}_r, t; \mathbf{x}_s) \Delta f(\mathbf{x}_r, \mathbf{x}_s)$ denote the data residual obtained by weighting the frequency derivative of the observed trace with the residual-frequency shift and the scaling factor E . Equation 18 now becomes

$$\begin{aligned} \gamma(\mathbf{x}) &= -K(\mathbf{x}) \sum_s \sum_r \int dt \left(g_p(\mathbf{x}_r, t; \mathbf{x}, 0) * \nabla \cdot \mathbf{v}(\mathbf{x}, t; \mathbf{x}_s) \right. \\ &\quad \left. + \frac{1}{\tau_\sigma(\mathbf{x})} g_{r_p}(\mathbf{x}_r, t; \mathbf{x}, 0) * \nabla \cdot \mathbf{v}(\mathbf{x}, t; \mathbf{x}_s) \right) \times \Delta P_f(\mathbf{x}_r, t; \mathbf{x}_s). \end{aligned} \quad (20)$$

Using the identity

$$\int dt [f(t) * g(t)] h(t) = \int dt g(t) [f(-t) * h(t)], \quad (21)$$

equation 20 can be rewritten as

$$\begin{aligned} \gamma(\mathbf{x}) &= -K(\mathbf{x}) \sum_s \sum_r \int dt \underbrace{\text{source}}_{(\nabla \cdot \mathbf{v}(\mathbf{x}, t; \mathbf{x}_s))} \underbrace{\text{backpropagated residual}}_{(g_p(\mathbf{x}_r, -t; \mathbf{x}, 0) * \Delta P_f(\mathbf{x}_r, t; \mathbf{x}_s))} \\ &\quad + \frac{1}{\tau_\sigma(\mathbf{x})} \underbrace{\text{source}}_{(\nabla \cdot \mathbf{v}(\mathbf{x}, t; \mathbf{x}_s))} \underbrace{\text{backpropagated residual}}_{(g_{r_p}(\mathbf{x}_r, -t; \mathbf{x}, 0) * \Delta P_f(\mathbf{x}_r, t; \mathbf{x}_s))}, \\ &= -K(\mathbf{x}) \sum_s \int dt \underbrace{\text{source}}_{(\nabla \cdot \mathbf{v}(\mathbf{x}, t; \mathbf{x}_s))} \sum_r \underbrace{\text{backpropagated residual}}_{(g_p(\mathbf{x}_r, -t; \mathbf{x}, 0) * \Delta P_f(\mathbf{x}_r, t; \mathbf{x}_s))} \\ &\quad + \frac{1}{\tau_\sigma(\mathbf{x})} \underbrace{\text{source}}_{(\nabla \cdot \mathbf{v}(\mathbf{x}, t; \mathbf{x}_s))} \sum_r \underbrace{\text{backpropagated residual}}_{(g_{r_p}(\mathbf{x}_r, -t; \mathbf{x}, 0) * \Delta P_f(\mathbf{x}_r, t; \mathbf{x}_s))}, \\ &= -K(\mathbf{x}) \sum_s \int dt \left(\nabla \cdot \mathbf{v}(\mathbf{x}, t; \mathbf{x}_s) q(\mathbf{x}, t; \mathbf{x}_s) + \frac{1}{\tau_\sigma(\mathbf{x})} \nabla \cdot \mathbf{v}(\mathbf{x}, t; \mathbf{x}_s) s(\mathbf{x}, t; \mathbf{x}_s) \right), \\ &= -K(\mathbf{x}) \sum_s \int dt \nabla \cdot \mathbf{v}(\mathbf{x}, t; \mathbf{x}_s) \left(q(\mathbf{x}, t; \mathbf{x}_s) + \frac{s(\mathbf{x}, t; \mathbf{x}_s)}{\tau_\sigma(\mathbf{x})} \right). \end{aligned} \quad (22)$$

Here, q and s are the adjoint-state variables of P and r_p , respectively. The gradient in equation 22 can be numerically computed by a zero-lag crosscorrelation of a forward-propagated source-wavefield term $\nabla \cdot \mathbf{v}(\mathbf{x}, t; \mathbf{x}_s)$ and back-propagated residual-wavefield terms $q(\mathbf{x}, t; \mathbf{x}_s)$ and $s(\mathbf{x}, t; \mathbf{x}_s)$ that are computed by backward propagating the data residual $\Delta P_f(\mathbf{x}_r, t; \mathbf{x}_s)$ from the receiver side. An alternative derivation of the gradient for WQ using the adjoint-state method is shown in Appendix A.

For diving waves with overlapping multiple arrivals, the frequency spectra for an observed trace and a predicted trace are shown in Figure 5. It is evident that in such cases, the peak frequency for the arrivals cannot be accurately estimated because each arrival has its own peak frequency. In such cases, the objective function in equation 8 can be modified to minimize the centroid-frequency shifts between the predicted and the observed traces as

$$\begin{aligned} \epsilon &= \frac{1}{2} \sum_s \sum_r \Delta f_{\text{centroid}}(\mathbf{x}_r, \mathbf{x}_s)^2, \\ &= \frac{1}{2} \sum_s \sum_r \left(\frac{\int_0^f f_p A(f_p)}{\int_0^f A(f_p)} - \frac{\int_0^f f_o A(f_o)}{\int_0^f A(f_o)} \right)^2. \end{aligned} \quad (23)$$

The subscripts o and p stand for observed and predicted, respectively, and $A(f)$ is the amplitude for a frequency f . The connective function, the Fréchet derivative and the gradient can be similarly derived as shown in the previous subsections.

WQ algorithm

The following steps are carried out for numerically implementing WQ using a preconditioned gradient-based method, in which the preconditioner is a source-side illumination factor (Plessix and Mulder, 2004):

- 1) Form the misfit function ϵ as

$$\epsilon = \frac{1}{2} \sum_s \sum_r \Delta f(\mathbf{x}_r, \mathbf{x}_s)^2. \quad (24)$$

- 2) Estimate the residual trace $\Delta P_f(\mathbf{x}_r, t; \mathbf{x}_s)$ as

$$\Delta P_f(\mathbf{x}_r, t; \mathbf{x}_s) = \frac{1}{E} \dot{P}_{f-\Delta f}(\mathbf{x}_r, t; \mathbf{x}_s) \Delta f(\mathbf{x}_r, \mathbf{x}_s), \quad (25)$$

Figure 5. Comparison between the amplitude spectra of predicted and observed traces for diving waves.

where E is defined in the previous subsection. In this implementation, we estimate the residual trace by weighting the observed trace by the frequency-shift residual and ignore the frequency derivative over $P_{f-\Delta f}$ and the factor E . This is done because there is no obvious analytic expression for finding the frequency-derivative terms $\dot{P}_{f-\Delta f}(\mathbf{x}_r, t; \mathbf{x}_s)$ and $\dot{P}_{f-\Delta f}(\mathbf{x}_r, t; \mathbf{x}_s)$. However, as will be shown in the next section, the Q tomograms obtained using this approximation are still reasonably accurate.

- Compute the gradient by

$$\gamma(\mathbf{x}) = -K(\mathbf{x}) \sum_s \int dt \underbrace{\text{source}}_{(\nabla \cdot \mathbf{v}(\mathbf{x}, t; \mathbf{x}_s))} \sum_r \underbrace{\text{backpropagated residual}}_{(g_p(\mathbf{x}_r, -t; \mathbf{x}, 0) * \Delta P_f(\mathbf{x}_r, t; \mathbf{x}_s))} + \frac{1}{\tau_\sigma(\mathbf{x})} \underbrace{\text{source}}_{\nabla \cdot \mathbf{v}(\mathbf{x}, t; \mathbf{x}_s)} \sum_r \underbrace{\text{backpropagated residual}}_{(g_{rp}(\mathbf{x}_r, -t; \mathbf{x}, 0) * \Delta P_f(\mathbf{x}_r, t; \mathbf{x}_s))}. \quad (26)$$

- Estimate the step length α by any backtracking line-search method (Nocedal and Wright, 1999).
- Update the tau model $\tau(\mathbf{x})$ using the iterative steepest descent formula:

Figure 6. (a) True Q model and (b) inverted Q model using WQ.

Figure 7. Cross sections of the $1/Q$ anomaly at $X = 2$ km in the true model and in the inverted Q model.

$$\tau(\mathbf{x})^{(k+1)} = \tau(\mathbf{x})^{(k)} - \alpha P(\mathbf{x}) \frac{\partial \epsilon}{\partial \tau(\mathbf{x})}, \quad (27)$$

where k represents the iteration index and $P(\mathbf{x})$ is the preconditioning factor. At every iteration, the background $\tau(\mathbf{x})$ is updated and the update in τ is then mapped to Q using equation 5.

NUMERICAL RESULTS

The effectiveness of WQ is now demonstrated with synthetic and field data records from a crosswell experiment in Friendswood, Texas. The synthetic examples are for two models with strong attenuation: (1) a crosswell example with a Gaussian Q anomaly and (2) a surface-seismic example in which there are shallow Q anomalies that hinder the imaging of the reflectors below the anomaly.

In the synthetic examples, the observed data are generated by an $O(2,8)$ time-space-domain staggered-grid solution of the viscoacoustic wave equation in equation 4. We use a Ricker wavelet with a peak frequency of 15 Hz as the source wavelet. For all iterations, the minimum and maximum values of the inverted τ model are fixed at 0.0002 and 0.22, respectively. In addition, in the marine synthetic experiments, the parameter updates are frozen from the sea surface to the sea bottom.

Crosswell Gaussian Q model

Figure 6a shows a homogeneous model with a velocity of 2 km/s and a Gaussian Q anomaly embedded at the center of the model. The Q value at the center of the anomaly is approximately 35, and the source and the receiver wells are offset by 4 km. There are 60 evenly spaced sources in the source well and 200 evenly spaced receivers in the receiver well. For WQ, the starting Q model is taken to be homogeneous with $Q = 1000$. Figure 6b shows the final Q tomogram obtained from WQ, and Figure 7 compares the cross sections of the $1/Q$ anomaly between the true and the inverted Q models. In the inverted tomogram, the Q value recovered at the center of the anomaly is approximately 32. It is evident that the Gaussian Q anomaly is successfully reconstructed by WQ.

Surface-seismic model

The WQ method is now tested on a more complex 2D section of the 3D SEG/EAGE overthrust model. Figure 8a and 8b shows the true velocity and Q models, respectively, used for generating the observed data. A smooth version of the true velocity model, shown in Figure 8c, is used as the background velocity model for WQ, whereas the starting Q model is taken to be homogeneous with $Q = 1000$, as shown in Figure 8d. The observed data are generated by 200 shots evenly distributed on the surface, and there are 400 surface receivers recording the data for 6 s. The receiver interval is 30 m.

For WQ, the diving waves in the predicted and the observed traces are separated using the water velocity. The objective function in equation 23 is used to update the background Q model. The inverted Q tomogram after 30 iterations is shown in Figure 8e, and a comparison

between cross sections of the $1/Q$ anomaly in the true and inverted models is shown in Figure 9. The convergence curve of the WQ iterations is also shown in Figure 10. It is evident that WQ can successfully recover the low-intermediate wavenumber details of the background Q model. Comparison between the peak frequencies for different source-receiver pairs in the observed and the predicted

traces after WQ, shown in Figure 11, also validate the accuracy of the inverted Q tomogram.

Viscoacoustic least-squares reverse time migration or Q least-squares reverse time migration (Q -LSRTM) (Dutta and Schuster, 2014) is now used to check the fidelity of the inverted Q tomogram. The acoustic RTM and LSRTM images are shown in Figure 12a and

Figure 8. (a) True velocity and (b) Q models used for generating the observed data. (c) Velocity model used for WQ, (d) starting Q model, and (e) inverted Q model from WQ.

Figure 9. Cross sections of the $1/Q$ anomaly at $X = 8.25$ km in the true model and in the inverted Q tomogram.

Figure 10. Convergence curve for the WQ method on the surface-seismic model. The Y -axis represents the normalized frequency-shift data residual.

12b, respectively. The black boxes in these figures indicate the areas that have been affected by the shallow Q anomalies. The amplitudes as well as the phases of the events in these areas have been distorted because of attenuation. The Q -RTM and Q -LSRTM images are shown in Figure 12c and 12d, respectively, where the amplitude and the phase distortions are corrected in the Q -LSRTM image. This example demonstrates that the inverted Q tomogram from

WQ can be used as the background Q model for any Q -PSDM algorithm to obtain images with better resolution and better balanced amplitudes than standard migration techniques.

Friendswood crosswell field data

In our final example, we apply WQ to the Friendswood crosswell data (Chen et al., 1990). Two 305 m deep cased wells separated by

Figure 11. The peak frequencies for different source-receiver pairs in (a) the observed data, (b) the predicted data from WQ, and (c) their difference.

Figure 12. Acoustic (a) RTM and (b) LSRTM images obtained from the viscoacoustic data. (c) Q -RTM and (d) Q -LSRTM images for the same data using the tomogram obtained from WQ as the background Q model. The black boxes delineate the areas where improvements in imaging can be seen with WQ and Q -LSRTM.

183 m were used as the source and receiver wells. Downhole explosive sources of 10 g charges were fired at intervals of 3 m from 305 to 9 m in the source well and the receiver well contained evenly spaced 96 receivers placed at depths ranging from 293 to 3 m. The data were recorded with a sampling interval of 0.25 ms for a total recording time of 0.375 s.

The following processing steps are first applied to the data:

- 1) The recorded data are corrected from 3D to 2D format by scaling the amplitudes by $\sqrt{t}$ to approximate geometric spreading. A phase correction is applied to the data by multiplying the spectrum of the observed seismogram with the filter $\sqrt{i/\omega}$ (Zhou et al., 1995).
- 2) A directional nine-point median filter is used to eliminate the tube waves, which are seen as linear events in the common-shot gathers (CSGs).
- 3) A band-pass filter of 80–600 Hz is applied to the data to remove any extreme noise from the data.
- 4) Because the bandwidth of 80–600 Hz is too broad for waveform tomography and WQ, the data are Wiener filtered to transform the original wavelet to a Ricker wavelet with a 150 Hz peak frequency. This bandwidth is chosen based on the frequency content of the data in which most of the signal is concentrated between 80 and 250 Hz. Figure 13a and 13b compares the raw and the processed CSGs, respectively.

Figure 13. (a) A raw CSG from the Friendswood crosswell data and (b) the same CSG after applying all the data processing steps.

Figure 14. (a) The first-arrival traveltime picks for the Friendswood data and (b) the peak frequencies for different source-receiver pairs.

Figure 15. (a) Velocity model used for WQ and (b) the Q tomogram obtained from WQ.

- 5) The first-arrival traveltimes are then picked from all the CSGs and inverted to get a starting velocity model for early arrival waveform inversion. The picked traveltimes are shown in Figure 14a. The early arrival waveform tomogram, shown in Figure 15a, is used as the background velocity model for WQ.

Figure 14a and 14b shows the traveltime picks and the peak-frequency picks for the raw field data set. We can see that the low peak frequencies along the diagonal elements in this figure correlate with the high-traveltime/low-velocity regions in the model. This suggests that the low-velocity formations in this model have high attenuation or low Q values.

For WQ, the starting Q model is taken to be homogeneous with $Q = 1000$. The Q tomogram obtained after 30 iterations is shown in Figure 15b, and the convergence curve is shown in Figure 16. There is a good agreement geologically between the velocity and the Q tomograms. The high-attenuation regions in the Q tomogram correspond to the low-velocity regions in the FWI tomogram. This is also consistent with the traveltime and the peak-frequency picks shown in Figure 14a and 14b, respectively. There is also a reasonable agreement between the peak frequencies of the observed and the predicted traces as shown in Figures 14b and 17b.

The Q -LSRTM image, shown in Figure 18d, is obtained by using the velocity and the WQ tomograms in Figure 15. Similar to the synthetic example, the Q -LSRTM image using the WQ tomogram

Figure 16. Convergence curve for the WQ method on the Friendswood crosswell data. The Y-axis represents the normalized frequency-shift data residual.

Figure 17. The peak frequencies for different source-receiver pairs for the predicted data (a) before and (b) after WQ.

has events with better balanced amplitudes than the acoustic RTM and LSRTM images in Figure 18a and 18b, respectively.

DISCUSSION

Synthetic and field-data examples demonstrate that WQ can be used to invert for Q tomograms that can be used as the background Q model for any Q -PSDM algorithm. The background velocity model is needed as an input to compute the viscoacoustic Green's functions for forward propagating the source and for back projecting the weighted data residuals. For the synthetic example in Figure 8, we obtained the velocity model for WQ by smoothing the true slowness model, whereas for the field-data example, FWI was used to obtain the velocity model. Thus, in all of our examples, we ensured that a sufficiently accurate background velocity model was used during WQ. If the background velocity has significant errors, it is likely to lead to significant inaccuracies in the inverted Q tomogram.

To illustrate this point, the synthetic example in Figure 8 is repeated again. However, errors are now introduced in the background velocity by applying a triangular smoothing filter with increasing window lengths to the true model in Figure 8a. The WQ tomograms for the different velocity models are shown in the right panel in Figure 19. It is evident from these tomograms that as the errors in the background velocity model increase, the Q anomalies are not located at the right locations. Thus, for wrong background velocity models, ray-based Q tomography approaches should be preferred over wave-equation-based approaches because in any wave-equation tomography approach, the transmission or reflection wavepaths along which the residuals are smeared are strongly influenced by the background velocity. It is also recommended to first invert for the background velocity model followed by an inversion for Q . As noted by Liao and McMechan (1995), Malinowski et al. (2007), and Kamei and Pratt (2008), the reliability of any Q tomography method is dependent on the accuracy of the starting velocity model. Their observation is also consistent with what we have observed with our WQ method. Virieux and Operto (2009) state that accurate velocity models are required before inverting for Q , so that the Q inversion can discriminate the intrinsic attenuation from the extrinsic attenuation.

For field-data applications, if the recorded data are contaminated by noise, the spectra can be quite rough, as shown in Figure 14b. Accurate picking of the peak frequencies is not trivial in such cases. To mitigate this problem, the amplitude spectra can be smoothed before the peak frequencies for different source-receiver pairs are

Figure 18. Images from (a) acoustic RTM, (b) acoustic LSRTM, (c) Q -RTM, and (d) Q -LSRTM using the tomogram obtained from WQ as the background Q model. The black boxes delineate the areas where improvements in imaging can be seen with WQ and Q -LSRTM.

Figure 19. The Q tomograms obtained from using different background velocity models. The panel on the left shows the velocity models used, whereas the one on the right shows the corresponding Q tomograms obtained from WQ.

estimated. [Gamar et al. \(2015\)](#) propose a robust way of estimating the peak frequencies in the observed and the predicted signals by taking the autocorrelation of the signal in short-time windows and finding the peak frequency only at the part of the signal around the maximum peak of the autocorrelation function. We expect the inverted Q tomogram obtained from such a robust estimation of the peak frequencies and from their shifts to be more accurate.

In all our examples, we used the SLS-based time-domain viscoacoustic wave equation characterized by the SLS mechanism with one relaxation function for computing the viscoacoustic Green's functions. For typical exploration problems, where the bandwidth of the data used for migration and tomography is approximately 3–30 Hz, the use of a single-relaxation mechanism has been shown to be sufficient to accurately model the effect of Q ([Blanch et al., 1995](#); [Zhu et al., 2013](#)). However, for a wider bandwidth of the data, a single-relaxation mechanism may not be sufficient to accurately model the effect of constant Q . To obtain a constant Q response over a broad frequency range, the number of relaxation mechanisms should be increased to three. Equations 14–22 and the mapping from τ to Q should then be modified according to the number of relaxation mechanisms used.

The viscoacoustic wave equation without memory variables proposed by [Bai et al. \(2013\)](#) or the decoupled Q equations involving fractional Laplacians and its low-rank formulation ([Zhu and Harris, 2014](#); [Sun et al., 2015b](#)) can also be used to compute the Green's functions. The WQ method proposed in this paper is generic and can be used to obtain reliable Q tomograms, as long as the modeling equations satisfy a constant Q behavior within the same frequency band ([Kjartansson, 1979](#)).

In this paper, we have formulated the WQ method only for transmission arrivals. Unlike diving waves, reflection arrivals provide important information about the deeper parts of the subsurface. To invert for deeper Q anomalies in the subsurface which are beyond the reach of diving waves, we are presently investigating the extension of this method to reflection-based WQ.

CONCLUSION

A novel WQ method is presented where a skeletonized representation of the data, i.e., the difference between the peak frequencies or the centroid frequencies of the observed and the predicted transmission arrivals are inverted to estimate the background Q model. The gradient for WQ is derived using the implicit function theorem, and is numerically obtained by a zero-lag crosscorrelation between the forward-propagated

source wavefield and the back-projected observed pressure seismograms that are weighted by the frequency-shift residuals. Numerical results on synthetic and crosswell field data sets demonstrate that if the recorded data suffer from strong attenuation, the WQ method can be used to accurately estimate the background Q model. The inverted Q model can be used with any Q -PSDM algorithm to obtain images with balanced amplitudes and high resolution in areas, where there is strong attenuation. An input requirement for WQ is an accurate estimate of the background velocity model, which can be obtained by conventional velocity model building algorithms.

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APPENDIX A

ADJOINT-STATE METHOD FOR WQ

For a 2D medium, equation 4 can be written in matrix-vector notation as

$$\begin{bmatrix} \frac{\partial}{\partial t} & K(1+\tau)\frac{\partial}{\partial x} & K(1+\tau)\frac{\partial}{\partial z} & 1 \\ \frac{1}{\rho}\frac{\partial}{\partial x} & \frac{\partial}{\partial t} & 0 & 0 \\ \frac{1}{\rho}\frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial t} & 0 \\ 0 & \frac{\tau}{\tau_\sigma}K\frac{\partial}{\partial x} & \frac{\tau}{\tau_\sigma}K\frac{\partial}{\partial z} & \frac{\partial}{\partial t} + \frac{1}{\tau_\sigma} \end{bmatrix} \begin{bmatrix} P \\ v_x \\ v_z \\ r_p \end{bmatrix} = \begin{bmatrix} S \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad (\text{A-1})$$

or in a more compact form

$$\mathbf{S}(\mathbf{m})\mathbf{w}(\mathbf{m}) = \mathbf{F}, \quad (\text{A-2})$$

where

$$\mathbf{S} = \begin{bmatrix} \frac{\partial}{\partial t} & K(1+\tau)\frac{\partial}{\partial x} & K(1+\tau)\frac{\partial}{\partial z} & 1 \\ \frac{1}{\rho}\frac{\partial}{\partial x} & \frac{\partial}{\partial t} & 0 & 0 \\ \frac{1}{\rho}\frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial t} & 0 \\ 0 & \frac{\tau}{\tau_\sigma}K\frac{\partial}{\partial x} & \frac{\tau}{\tau_\sigma}K\frac{\partial}{\partial z} & \frac{\partial}{\partial t} + \frac{1}{\tau_\sigma} \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} P \\ v_x \\ v_z \\ r_p \end{bmatrix},$$

$$\mathbf{F} = \begin{bmatrix} S \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{m} = [K \quad \rho \quad \tau \quad \tau_\sigma]^T. \quad (\text{A-3})$$

The perturbation of the state variable $\mathbf{w}$ with regard to the model variable $\mathbf{m}$ is given by $\partial\mathbf{w}(\mathbf{m})/\partial\mathbf{m}$ and can be obtained from

$$\begin{aligned} \mathbf{S}(\mathbf{m})\mathbf{w}(\mathbf{m}) &= \mathbf{F} \\ \Rightarrow \frac{\partial\mathbf{S}(\mathbf{m})}{\partial\mathbf{m}}\mathbf{w}(\mathbf{m}) + \mathbf{S}(\mathbf{m})\frac{\partial\mathbf{w}(\mathbf{m})}{\partial\mathbf{m}} &= 0 \\ \Rightarrow \frac{\partial\mathbf{w}(\mathbf{m})}{\partial\mathbf{m}} &= -\mathbf{S}^{-1}(\mathbf{m})\frac{\partial\mathbf{S}(\mathbf{m})}{\partial\mathbf{m}}\mathbf{w}(\mathbf{m}). \end{aligned} \quad (\text{A-4})$$

The connective function for WQ can also be written as

$$\dot{\mathbb{F}}_{\Delta f} = \langle \mathbf{S}_r \tilde{\mathbf{w}}_f(\mathbf{x}, t; \mathbf{x}_s), \dot{\mathbf{w}}_{f-\Delta f}(\mathbf{x}_r, t; \mathbf{x}_s) \rangle, \quad (\text{A-5})$$

where $\langle \mathbf{x}, \mathbf{y} \rangle$ denotes the scalar product between the vectors $\mathbf{x}$ and $\mathbf{y}$ and $\mathbf{S}_r$ is a restriction operator onto the receiver position and it depends on the spatial coordinates. The restriction operator samples the wavefield at the receiver locations, whereas the adjoint of it $\mathbf{S}_r^*$ sprays the recorded data from the receiver coordinates to the model coordinates. Here, $\tilde{\mathbf{w}}_f$ denotes the predicted event for a given background Q model recorded at the receiver location $\mathbf{x}_r$ due to a source excited at time $t = 0$ and at location $\mathbf{x}_s$ and $\mathbf{w}_{f-\Delta f}$ denotes the same event in the observed data.

From equation 8, the misfit function for WQ is given by

$$J = \frac{1}{2} \sum_s \sum_r \Delta f(\mathbf{x}_r, \mathbf{x}_s)^2. \quad (\text{A-6})$$

The gradient of J is given by

$$\begin{aligned} \frac{\partial J}{\partial \tau(\mathbf{x})} &= -\sum_s \sum_r \frac{\partial \Delta f(\mathbf{x}_r, \mathbf{x}_s)}{\partial \tau(\mathbf{x})} \Delta f(\mathbf{x}_r, \mathbf{x}_s), \\ &= \sum_s \sum_r \frac{1}{\mathbf{E}} \frac{\partial \dot{\mathbb{F}}_{\Delta f}}{\partial \tau(\mathbf{x})} \Delta f(\mathbf{x}_r, \mathbf{x}_s), \quad (\text{Using equations 10 and 19}) \\ &= \sum_s \sum_r \frac{1}{\mathbf{E}} \frac{\partial}{\partial \tau(\mathbf{x})} (\mathbf{S}_r \tilde{\mathbf{w}}_f(\mathbf{x}, t; \mathbf{x}_s), \dot{\mathbf{w}}_{f-\Delta f}(\mathbf{x}_r, t; \mathbf{x}_s)) \Delta f(\mathbf{x}_r, \mathbf{x}_s), \quad (\text{Using equation A-5}) \\ &= \sum_s \sum_r \left\langle \mathbf{S}_r \frac{\partial \tilde{\mathbf{w}}_f(\mathbf{x}, t; \mathbf{x}_s)}{\partial \tau(\mathbf{x})}, \dot{\mathbf{w}}_{f-\Delta f}(\mathbf{x}_r, t; \mathbf{x}_s) \Delta f(\mathbf{x}_r, \mathbf{x}_s) \frac{1}{\mathbf{E}} \right\rangle, \\ &= -\sum_s \sum_r \left\langle \mathbf{S}_r \mathbf{S}^{-1} \frac{\partial \mathbf{S}}{\partial \tau(\mathbf{x})} \tilde{\mathbf{w}}_f(\mathbf{x}, t; \mathbf{x}_s), \dot{\mathbf{w}}_{f-\Delta f}(\mathbf{x}_r, t; \mathbf{x}_s) \Delta f(\mathbf{x}_r, \mathbf{x}_s) \frac{1}{\mathbf{E}} \right\rangle, \quad (\text{Using equation A-4}) \\ &= -\sum_s \left\langle \frac{\partial \mathbf{S}}{\partial \tau(\mathbf{x})} \tilde{\mathbf{w}}_f(\mathbf{x}, t; \mathbf{x}_s), (\mathbf{S}^{-1})^* \sum_r (\mathbf{S}_r^* \dot{\mathbf{w}}_{f-\Delta f}(\mathbf{x}_r, t; \mathbf{x}_s) \Delta f(\mathbf{x}_r, \mathbf{x}_s) \frac{1}{\mathbf{E}}) \right\rangle, \\ &= -\sum_s \left\langle \frac{\partial \mathbf{S}}{\partial \tau(\mathbf{x})} \tilde{\mathbf{w}}_f(\mathbf{x}, t; \mathbf{x}_s), \mathbf{w}^*(\mathbf{x}, t; \mathbf{x}_s) \right\rangle. \end{aligned} \quad (\text{A-7})$$

Here,

$$\mathbf{E} = \begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \end{bmatrix} = \begin{bmatrix} \int dt \dot{P}(\mathbf{x}_r, t; \mathbf{x}_s)_{\text{obs}} P(\mathbf{x}_r, t; \mathbf{x}_s)_{\text{calc}} \\ \int dt \ddot{v}_x(\mathbf{x}_r, t; \mathbf{x}_s)_{\text{obs}} v_x(\mathbf{x}_r, t; \mathbf{x}_s)_{\text{calc}} \\ \int dt \ddot{v}_z(\mathbf{x}_r, t; \mathbf{x}_s)_{\text{obs}} v_z(\mathbf{x}_r, t; \mathbf{x}_s)_{\text{calc}} \\ \int dt \ddot{r}_p(\mathbf{x}_r, t; \mathbf{x}_s)_{\text{obs}} r_p(\mathbf{x}_r, t; \mathbf{x}_s)_{\text{calc}} \end{bmatrix}, \quad (\text{A-8})$$

and $\mathbf{w}^* = [q \quad u_x \quad u_z \quad s]^T$ is the adjoint-state variable of $\mathbf{w} = [P \quad v_x \quad v_z \quad r_p]^T$, and it is numerically obtained by a finite-difference solution of the adjoint-state equations

$$\begin{aligned}
\frac{\partial q}{\partial t} + \nabla \cdot \left(\frac{1}{\rho} \mathbf{u} \right) &= - \sum_r \mathbf{S}_r^* \left(\dot{P}(\mathbf{x}_r, t; \mathbf{x}_s)_{\text{obs}} \Delta f(\mathbf{x}_r, \mathbf{x}_s) \frac{1}{E_1} \right), \\
\frac{\partial \mathbf{u}}{\partial t} + \nabla(K(1+\tau)q) + \nabla \left(\frac{1}{\tau_\sigma} K \tau s \right) \\
&= - \sum_r \mathbf{S}_r^* \left(\dot{\mathbf{v}}(\mathbf{x}_r, t; \mathbf{x}_s)_{\text{obs}} \Delta f(\mathbf{x}_r, \mathbf{x}_s) \frac{1}{E_2} \right), \\
\frac{\partial s}{\partial t} - \frac{s}{\tau_\sigma} - q &= - \sum_r \mathbf{S}_r^* \left(\dot{r}_p(\mathbf{x}_r, t; \mathbf{x}_s)_{\text{obs}} \Delta f(\mathbf{x}_r, \mathbf{x}_s) \frac{1}{E_3} \right), \quad (\text{A-9})
\end{aligned}$$

where $\mathbf{u} = \{u_x, u_z\}$ for a 2D medium. Assuming that we only record pressure seismograms, equation A-9 simplifies to

$$\begin{aligned}
\frac{\partial q}{\partial t} + \nabla \cdot \left(\frac{1}{\rho} \mathbf{u} \right) &= - \sum_r \mathbf{S}_r^* \Delta P(\mathbf{x}_r, t; \mathbf{x}_s), \\
\frac{\partial \mathbf{u}}{\partial t} + \nabla(K(1+\tau)q) + \nabla \left(\frac{1}{\tau_\sigma} K \tau s \right) &= 0, \\
\frac{\partial s}{\partial t} - \frac{s}{\tau_\sigma} - q &= 0, \quad (\text{A-10})
\end{aligned}$$

where $\Delta P(\mathbf{x}_r, t; \mathbf{x}_s) = (1/E_1) \dot{P}(\mathbf{x}_r, t; \mathbf{x}_s)_{\text{obs}} \Delta f(\mathbf{x}_r, \mathbf{x}_s)$ denotes a residual trace. From equation A-3, we have

$$\begin{aligned}
\mathbf{S} &= \begin{bmatrix} \frac{\partial}{\partial t} & K(1+\tau) \frac{\partial}{\partial x} & K(1+\tau) \frac{\partial}{\partial z} & 1 \\ \frac{1}{\rho} \frac{\partial}{\partial x} & \frac{\partial}{\partial t} & 0 & 0 \\ \frac{1}{\rho} \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial t} & 0 \\ 0 & \frac{\tau}{\tau_\sigma} K \frac{\partial}{\partial x} & \frac{\tau}{\tau_\sigma} K \frac{\partial}{\partial z} & \frac{\partial}{\partial t} + \frac{1}{\tau_\sigma} \end{bmatrix}, \\
\frac{\partial \mathbf{S}}{\partial \tau} &= \begin{bmatrix} 0 & K \frac{\partial}{\partial x} & K \frac{\partial}{\partial z} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{K}{\tau_\sigma} \frac{\partial}{\partial x} & \frac{K}{\tau_\sigma} \frac{\partial}{\partial z} & 0 \end{bmatrix}. \quad (\text{A-11})
\end{aligned}$$

Using equations A-3 and A-11, we can get

$$\begin{aligned}
\frac{\partial J}{\partial \tau(\mathbf{x})} &= - \sum_s \left\langle \frac{\partial \mathbf{S}}{\partial \tau(\mathbf{x})} \tilde{\mathbf{w}}_f(\mathbf{x}, t; \mathbf{x}_s), \mathbf{w}^*(\mathbf{x}, t; \mathbf{x}_s) \right\rangle, \\
&= - \sum_s \left\langle \begin{bmatrix} 0 & K \frac{\partial}{\partial x} & K \frac{\partial}{\partial z} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{K}{\tau_\sigma} \frac{\partial}{\partial x} & \frac{K}{\tau_\sigma} \frac{\partial}{\partial z} & 0 \end{bmatrix} \begin{bmatrix} P \\ v_x \\ v_z \\ r_p \end{bmatrix}, \begin{bmatrix} q \\ u_x \\ u_z \\ s \end{bmatrix} \right\rangle, \\
&= - \int_s dt K(\mathbf{x}) \nabla \cdot \mathbf{v}(\mathbf{x}, t; \mathbf{x}_s) \left(q(\mathbf{x}, t; \mathbf{x}_s) + \frac{s(\mathbf{x}, t; \mathbf{x}_s)}{\tau_\sigma(\mathbf{x})} \right). \quad (\text{A-12})
\end{aligned}$$

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