Application of perturbation theory to a P-wave eikonal equation in orthorhombic media

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ABSTRACT

The P-wave eikonal equation for orthorhombic (ORT) anisotropic media is a highly nonlinear partial differential equation requiring the solution of a sixth-order polynomial to obtain traveltimes, resulting in complex and time-consuming numerical solutions. To alleviate this complexity, we approximate the solution of this equation by applying a multiparametric perturbation approach. We also investigated the sensitivity of traveltime surfaces in ORT media with respect to three anelliptic parameters. As a result, a simple and accurate P-wave traveltime approximation valid for ORT media was derived. Two different possible anelliptic parameterizations were compared. One of the parameterizations includes anelliptic parameters defined at zero offset: η_1 , η_2 , and η_{xy} . Another parameterization includes anelliptic parameters defined for all symmetry planes: η_1 , η_2 , and η_3 . The azimuthal behavior of sensitivity coefficients with different parameterizations was used to analyze the crosstalk between anelliptic parameters.

INTRODUCTION

The seismic model with orthorhombic (ORT) anisotropy is recognized as the most encompassing, and yet somewhat practical, anisotropic representation of earth's complexity. It combines the natural vertical layering due to gravity with the horizontal one due to tectonic stresses, salt intrusion pressure, or vertical fractures in potential reservoir regions (Schoenberg and Helbig, 1997). The ORT velocity model with vertical and horizontal symmetry planes is an adequate approximation for many geologic models. However, the computation of traveltimes in ORT anisotropic media is not

trivial due to the computational complexity of the eikonal equation in such media even for the acoustic case (Alkhalifah, 2003).

To approximate the eikonal in ORT media, we propose to apply perturbation theory with respect to three anelliptic parameters that we assume to be small. A similar approach was earlier applied for a transversely isotropic medium with vertical (Alkhalifah, 2011a) and tilted (Alkhalifah, 2011b; Stovas and Alkhalifah, 2012) symmetry axes. Recently, Masmoudi and Alkhalifah (2016) propose a similar approach with parameterization based on a weak difference between the parameters defined in the vertical symmetry planes. The application of perturbation theory reduces the complexity of computing the traveltime from solving a partial differential equation (PDE) of a sixth-order polynomial in traveltime to a quadratic in traveltime corresponding to an elliptical background, in addition to nine linear first-order PDEs for the perturbation coefficients. These perturbation coefficients represent the sensitivity of the traveltime in ORT media to the perturbation parameters, and thus give insight into the dependency of the data to the different model parameters. For ORT medium, we select two different sets of anelliptic parameters for the analysis. The first one is related to anellipticity function defined at zero offset: η_1 ($[xz]$ symmetry plane), η_2 ($[yz]$ symmetry plane), and η_{xy} (the crossterm anelliptic parameter controlling the azimuthal behavior of anellipticity between vertical symmetry planes) (Stovas, 2015). The second one is related to anellipticity parameters defined for all symmetry planes: η_1 , η_2 , and η_3 (Tsvankin, 1997). Assuming a homogeneous ORT medium, the perturbation coefficients can be computed analytically, and the perturbation series results in simple and accurate moveout approximation based on the Shanks transform (Bender and Orszag, 1978). The accuracy of the proposed approximation for both parameterizations is tested against the traveltime approximation proposed by Vasconcelos and Tsvankin (2006) using two ORT models. The results show superior accuracy of proposed approximation.

In our approach, for simplicity, we focus on the vertical ORT model only. However, this method can be extended to the tilted ORT

Manuscript received by the Editor 25 February 2016; revised manuscript received 11 June 2016; published online 12 October 2016.

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medium in a similar way as shown in Hao and Stovas (2016). There are some pitfalls in specifying the orientation of the ORT symmetry planes (Fowler, 2015), but within the frame of our method, it is not important.

THE ORT P-WAVE EIKONAL EQUATION

An eikonal equation for acoustic P-waves in ORT media with vertical symmetry axis, which governs traveltime in the high-frequency asymptotic approximation, can easily be derived from the slowness surface equation (Alkhalifah, 2003; Stovas, 2015; Masmoudi and Alkhalifah, 2016):

$$\begin{aligned} & v_0^2 \left(\frac{\partial \tau}{\partial z} \right)^2 + v_1^2 (1 + 2\eta_1) \left(\frac{\partial \tau}{\partial x} \right)^2 + v_2^2 (1 + 2\eta_2) \left(\frac{\partial \tau}{\partial y} \right)^2 \\ & - 2\eta_1 v_0^2 v_1^2 \left(\frac{\partial \tau}{\partial z} \right)^2 \left(\frac{\partial \tau}{\partial x} \right)^2 \\ & - 2\eta_2 v_0^2 v_2^2 \left(\frac{\partial \tau}{\partial z} \right)^2 \left(\frac{\partial \tau}{\partial y} \right)^2 - ((1 + 2\eta_1)(1 + 2\eta_2) \\ & - (1 + \eta_{xy})^2) v_1^2 v_2^2 \left(\frac{\partial \tau}{\partial x} \right)^2 \left(\frac{\partial \tau}{\partial y} \right)^2 \\ & + v_0^2 v_1^2 v_2^2 (4\eta_1 \eta_2 - \eta_{xy}^2) \left(\frac{\partial \tau}{\partial z} \right)^2 \left(\frac{\partial \tau}{\partial x} \right)^2 \left(\frac{\partial \tau}{\partial y} \right)^2 = 1, \end{aligned} \quad (1)$$

where $\tau(x, y, z)$ is the traveltime measured from the source at the origin to a point in space (x, y, z) ; v_0, v_1, v_2 are the vertical P-wave velocity and P-wave NMO velocities defined in vertical symmetry planes $[x, z]$ and $[y, z]$, respectively; and $\eta_1, \eta_2, \eta_{xy}$ are the anelliptic parameters, where the first two are defined in vertical symmetry planes $[x, z]$ and $[y, z]$, respectively, and the third one is given by Stovas (2015)

$$\eta_{xy} = \sqrt{\frac{(1 + 2\eta_1)(1 + 2\eta_2)}{(1 + 2\eta_3)}} - 1, \quad (2)$$

where η_3 is an anelliptic parameter defined in horizontal symmetry plane $[x, y]$ (Vasconcelos and Tsvankin, 2006). Note that indices (1,2) for NMO velocities and anellipticity parameters are different from those defined by Tsvankin (1997). The notations for anisotropy parameters are given in Appendix A.

Because no angles are involved in equation 1, we assume the vertical ORT model, and orientation of the symmetry planes in ORT medium, are not changing with spatial coordinates.

APPROXIMATE SOLUTION

Assuming that the anellipticity parameters η_1, η_2 , and η_{xy} are small, we approximate the traveltime by the following trial solution of the eikonal equation:

$$\begin{aligned} \tau(x, y, z) & \approx \tau_0(x, y, z) + \sum_{j=1,2,3} a_j(x, y, z) \eta_j \\ & + \sum_{\substack{j,k=1,2,3 \\ j \leq k}} b_{jk}(x, y, z) \eta_j \eta_k, \end{aligned} \quad (3)$$

where index $3 \equiv xy$. A similar perturbation approach is used in Hao and Stovas (2016) to develop an approximation for a slowness surface in tilted ORT media.

Masmoudi and Alkhalifah (2016) use similar perturbation series but using another parameter $\Delta\chi$ instead of η_3 . This parameter can be expressed as $\Delta\chi = (\sqrt{(1 + 2\eta_2)/((1 + 2\eta_1)(1 + 2\eta_3))} - 1)(v_2/v_1)$. Being compared with η_{xy} from equation 2, one can see that parameter $\Delta\chi$ depends on the specification of vertical symmetry planes and normalized with the NMO velocities. One more difference is that the second-order coefficients related to parameter $\Delta\chi$ are neglected, $b_{13} = b_{23} = b_{33} = 0$.

In equation 3, we introduce 10 coefficients: the zeroth-order (or ellipsoidal) one, three first-order coefficients a_j , and six second-order coefficients b_{jk} . The zeroth-order coefficient $\tau_0(x, y, z)$ is the solution of the elliptic eikonal equation

$$v_0^2 \left(\frac{\partial \tau}{\partial z} \right)^2 + v_1^2 \left(\frac{\partial \tau}{\partial x} \right)^2 + v_2^2 \left(\frac{\partial \tau}{\partial y} \right)^2 = 1, \quad (4)$$

that is, for a homogeneous medium, given by a simple hyperbolic equation,

$$\tau_0(x, y, z) = \sqrt{\frac{x^2}{v_1^2} + \frac{y^2}{v_2^2} + \frac{z^2}{v_0^2}}. \quad (5)$$

The functions $a_j(x, y, z)$ and $b_{jk}(x, y, z)$ can be obtained using the same scheme used in Alkhalifah (2011a) and Stovas and Alkhalifah (2012) as follows. By substituting trial solution 3 into equation 1 and equating the terms for the same order of anelliptic parameters, we can obtain the PDEs for coefficients a_j and b_{jk} . The first-order coefficients a_j are computed from the linear first-order PDEs:

$$\begin{aligned} & v_0^2 \left(\frac{\partial \tau_0}{\partial z} \right) \left(\frac{\partial a_j}{\partial z} \right) + v_1^2 \left(\frac{\partial \tau_0}{\partial x} \right) \left(\frac{\partial a_j}{\partial x} \right) \\ & + v_2^2 \left(\frac{\partial \tau_0}{\partial y} \right) \left(\frac{\partial a_j}{\partial y} \right) = f_j \left(\frac{\partial \tau_0}{\partial x}, \frac{\partial \tau_0}{\partial y}, \frac{\partial \tau_0}{\partial z} \right), \end{aligned} \quad (6)$$

where the right-side functions f_j are given by the zeroth-order coefficient τ_0 computed from equation 4:

$$\begin{aligned} f_1 \left(\frac{\partial \tau_0}{\partial x}, \frac{\partial \tau_0}{\partial y}, \frac{\partial \tau_0}{\partial z} \right) & = v_1^2 \left(\frac{\partial \tau_0}{\partial x} \right)^2 \left(-1 + v_0^2 \left(\frac{\partial \tau_0}{\partial z} \right)^2 + v_2^2 \left(\frac{\partial \tau_0}{\partial y} \right)^2 \right), \\ f_2 \left(\frac{\partial \tau_0}{\partial x}, \frac{\partial \tau_0}{\partial y}, \frac{\partial \tau_0}{\partial z} \right) & = v_2^2 \left(\frac{\partial \tau_0}{\partial y} \right)^2 \left(-1 + v_0^2 \left(\frac{\partial \tau_0}{\partial z} \right)^2 + v_1^2 \left(\frac{\partial \tau_0}{\partial x} \right)^2 \right), \\ f_3 \left(\frac{\partial \tau_0}{\partial x}, \frac{\partial \tau_0}{\partial y}, \frac{\partial \tau_0}{\partial z} \right) & = -v_1^2 v_2^2 \left(\frac{\partial \tau_0}{\partial x} \right)^2 \left(\frac{\partial \tau_0}{\partial y} \right)^2, \end{aligned} \quad (7)$$

The second-order coefficients are computed from the linear first-order PDEs given by

$$\begin{aligned} & v_0^2 \left(\frac{\partial \tau_0}{\partial z} \right) \left(\frac{\partial b_{jk}}{\partial z} \right) + v_1^2 \left(\frac{\partial \tau_0}{\partial x} \right) \left(\frac{\partial b_{jk}}{\partial x} \right) + v_2^2 \left(\frac{\partial \tau_0}{\partial y} \right) \left(\frac{\partial b_{jk}}{\partial y} \right) \\ & = g_{jk} \left(\frac{\partial \tau_0}{\partial x}, \frac{\partial \tau_0}{\partial y}, \frac{\partial \tau_0}{\partial z}; \frac{\partial a_{j,k}}{\partial x}, \frac{\partial a_{j,k}}{\partial y}, \frac{\partial a_{j,k}}{\partial z} \right), \end{aligned} \quad (8)$$

where the right-side functions g_{jk} are given by the zeroth-order coefficient τ_0 computed from equation 4 and the first-order coefficients

a_j and a_k are obtained from equation 6. The functions g_{jk} are algebraically complicated analytical functions (Appendix B).

Note that equations 6 and 8 are linear first-order PDEs with respect to coefficients a_j and b_{jk} . Therefore, the proposed approximation suggests solving one second-order PDE (equation 4) and nine first-order PDE (equations 6 and 8) instead of solving one sixth-order PDE (equation 1).

A HOMOGENEOUS MODEL TEST

In a homogeneous medium, using equation 5, all the coefficients in equation 3 can be computed analytically. The first-order coefficients are given by

$$a_1 = -\frac{t_x^4}{\tau_0^3}, \quad a_2 = -\frac{t_y^4}{\tau_0^3}, \quad a_3 = -\frac{t_x^2 t_y^2}{\tau_0^3}, \quad (9)$$

and the second-order coefficients result in the following solutions:

$$\begin{aligned} b_{11} &= \frac{3t_x^6(\tau_0^2 + 3(t_x^2 + t_y^2))}{2\tau_0^7}, \quad b_{12} = -\frac{9t_x^4 t_y^4}{\tau_0^7}, \quad b_{13} = \frac{3t_x^4 t_y^2(2\tau_0^2 - 3t_x^2)}{\tau_0^7}, \\ b_{22} &= \frac{3t_y^6(\tau_0^2 + 3(t_x^2 + t_y^2))}{2\tau_0^7}, \quad b_{23} = \frac{3t_x^2 t_y^4(2\tau_0^2 - 3t_y^2)}{\tau_0^7}, \\ b_{33} &= \frac{3t_x^2 t_y^2(\tau_0^2(\tau_0^2 - t_0^2) - 3t_x^2 t_y^2)}{2\tau_0^7}, \end{aligned} \quad (10)$$

where $t_x^2 = x^2/v_1^2$, $t_y^2 = y^2/v_2^2$, and $t_0^2 = z^2/v_0^2$ with velocities being defined after equation 1.

These coefficients can be considered as sensitivity coefficients for traveltimes in ORT media. They can be used for a simple traveltimes approximation for ORT media based on the Shanks transform (Bender and Orszag, 1978):

$$t(x, y) = \frac{A_0 A_2 - A_1^2}{A_0 - 2A_1 + A_2}, \quad (11)$$

with

$$\begin{aligned} A_0 &= \tau_0(x, y, z = v_0 t_0), \\ A_1 &= A_0 + \sum_{j=1,2,3} a_j(x, y, z = v_0 t_0) \eta_j, \\ A_2 &= A_1 + \sum_{\substack{j,k=1,2,3 \\ j \leq k}} b_{jk}(x, y, z = v_0 t_0) \eta_j \eta_k. \end{aligned} \quad (12)$$

Note that along one of the symmetry planes, the coefficients 9 and 10, and approximations 11 and 12 reduce to the ones derived in Alkhalifah (2011a).

ALTERNATIVE PARAMETERIZATION

The problem described above can also be formulated in terms of another set of anellipticity parameters, specifically in terms of η_1 , η_2 , and η_3 . In this case, the trial solution takes the following form:

$$\begin{aligned} \tau(x, y, z) &\approx \tau_0(x, y, z) + \sum_{j=1,2,3} a'_j(x, y, z) \eta_j \\ &+ \sum_{\substack{j,k=1,2,3 \\ j \leq k}} b'_{jk}(x, y, z) \eta_j \eta_k, \end{aligned} \quad (13)$$

where index 3 corresponds to parameter η_3 . To obtain the coefficients in equation 13, we expand equation 2 in terms of η_1 , η_2 , and η_3 up to second order:

$$\eta_{xy} = \eta_1 + \eta_2 - \eta_3 - \frac{1}{2}\eta_1^2 - \frac{1}{2}\eta_2^2 + \frac{3}{2}\eta_3^2 + \eta_1\eta_2 - \eta_1\eta_3 - \eta_2\eta_3 + \dots \quad (14)$$

By using equation 14, we can rearrange the functions f_j and g_{jk} in equations 6 and 8. Considering a homogeneous model, equation 13 results in new first-order coefficients a'_j , which can be written as a linear combination of the coefficients defined in equation 9:

$$\begin{aligned} a'_1 &= a_1 + a_3 = -\frac{t_x^2(t_x^2 + t_y^2)}{\tau_0^3}, \\ a'_2 &= a_2 + a_3 = -\frac{t_y^2(t_x^2 + t_y^2)}{\tau_0^3}, \\ a'_3 &= -a_3 = \frac{t_x^2 t_y^2}{\tau_0^3}. \end{aligned} \quad (15)$$

The second-order coefficients b'_{jk} are given by

$$\begin{aligned} b'_{11} &= b_{11} + b_{13} + b_{33} - \frac{1}{2}a_3 \\ &= \frac{t_x^2(t_0^4 t_y^2 + t_0^2(t_x^2 + t_y^2)(12t_x^2 + 5t_y^2) + (t_x^2 + t_y^2)^2(3t_x^2 + 4t_y^2))}{2\tau_0^7}, \\ b'_{22} &= b_{22} + b_{23} + b_{33} - \frac{1}{2}a_3 \\ &= \frac{t_y^2(t_0^4 t_x^2 + t_0^2(t_x^2 + t_y^2)(5t_x^2 + 12t_y^2) + (t_x^2 + t_y^2)^2(4t_x^2 + 3t_y^2))}{2\tau_0^7}, \\ b'_{33} &= b_{33} + \frac{3}{2}a_3 = -\frac{t_x^2 t_y^2(t_0^4 + t_0^2(t_x^2 + t_y^2) + 3t_x^2 t_y^2)}{2\tau_0^7}, \\ b'_{12} &= b_{12} + b_{13} + b_{23} + 2b_{33} + a_3 \\ &= -\frac{t_x^2 t_y^2(t_0^4 - 7t_0^2(t_x^2 + t_y^2) + (t_x^2 + t_y^2)^2)}{\tau_0^7}, \\ b'_{13} &= -b_{13} - 2b_{33} - a_3 \\ &= \frac{t_x^2 t_y^2(t_0^4 - t_0^2(7t_x^2 + t_y^2) + (t_x^2 + t_y^2)(t_x^2 - 2t_y^2))}{\tau_0^7}, \\ b'_{23} &= -b_{23} - 2b_{33} - a_3 \\ &= \frac{t_x^2 t_y^2(t_0^4 - t_0^2(t_x^2 + 7t_y^2) + (t_x^2 + t_y^2)(-2t_x^2 + t_y^2))}{\tau_0^7}. \end{aligned} \quad (16)$$

Note that equations for coefficients a'_j and b'_{jk} are more complicated compared with coefficients a_j and b_{jk} . The Shanks transforms 11 and 12 can be defined in a similar manner using the new coefficients a'_j (equation 15) and b'_{jk} (equation 16) substituted into equation 13.

NUMERICAL EXAMPLES

To illustrate the sensitivity of the traveltimes to all the anelliptic parameters (η_1 , η_2 , and η_{xy}), we plot the coefficients a_j (Figure 1) and b_{jk} (Figure 2) versus (offsets to depth ratios) x/z and y/z for ORT model with velocities $v_0 = 2$ km/s, $v_1 = 2.2$ km/s, and $v_2 = 2.5$ km/s. This set of velocities results in rather large values for anisotropic parameters δ_1 and δ_2 ; however, we select these numbers to illustrate nonsymmetric behavior of sensitivity parameters. We can see that the magnitude of the first-order coefficients a_1 and a_2 are larger than the magnitude of the coefficient a_3 . A similar behavior happens for the second-order coefficients b_{11} , b_{22} , and b_{33} (Figure 2a and 2c). The crossterm second-order coefficients b_{12} , b_{13} , and b_{23} (Figure 2b and 2d) are smaller compared with coefficients b_{11} and b_{22} . The first-order coefficients a'_1 , a'_2 , and a'_3 computed for parameterization (η_1 , η_2 , and η_3) are shown in Figure 3. One can see that they are slightly different from their

counterparts shown in Figure 1. The second-order coefficients b'_{11} , b'_{22} , and b'_{33} (Figure 4a and 4c) and b'_{12} , b'_{13} , and b'_{23} (Figure 4b and 4d) are very different from their counterparts shown in Figure 2. In particular, we notice anomalies in coefficient b'_{12} , whereas coefficient b_{12} possesses a very smooth behavior. These anomalies are the result of crosstalk between the crossterm coefficient b_{12} and coefficients a_3 and b_{33} (equation 16).

To test the accuracy of the proposed traveltimes approximations 11 and 12 with the different parameterizations in anelliptic parameters (η_1 , η_2 , and η_{xy}) and (η_1 , η_2 , and η_3), we choose two ORT models with vertical and NMO velocities mentioned above, anelliptic parameters $\eta_1 = 0.15$, $\eta_2 = 0.1$, and two different parameters $\eta_3 = 0.2$ ($\eta_{xy} = 0.056$) and $\eta_3 = -0.1$ ($\eta_{xy} = 0.396$). The choice of these models were made to highlight the accuracy of the two parameterizations with respect to the choice of the third anellipticity parameter, mainly controlled by magnitude of crossterm anelliptic parameter η_{xy} . To illustrate the anisotropy behavior of the two mod-

Figure 1. (a-c) The first-order coefficients a_1 , a_2 , and a_3 plotted versus offset/depth projections. The model is defined by three velocities $v_0 = 2$ km/s, $v_1 = 2.2$ km/s, and $v_2 = 2.5$ km/s.

Figure 2. (a-c) The second-order coefficients b_{11} , b_{22} , and b_{33} and (d-f) b_{12} , b_{13} , and b_{23} plotted versus offset/depth projections. The model is defined by three velocities $v_0 = 2$ km/s, $v_1 = 2.2$ km/s, and $v_2 = 2.5$ km/s.

els, we plot the NMO ellipse and two anellipticities (Stovas, 2015) in polar plots. Models 1 and 2 have the same NMO ellipse (Figure 5a), but different anellipticities (Figure 5b). One can see that model 2 is more anelliptical than model 1. The accuracy plots from approximations 11 and 12 with the two parameterizations of the anelliptic parameters applied for models 1 and 2 are shown in Figure 6. The exact traveltimes are computed from parametric equations for P-waves in acoustic ORT media (Stovas, 2015).

For model 1 with weak crossterm anelliptic parameter η_{xy} , the accuracy of the traveltimes approximations 11 and 12 is very similar for these parameterizations in anelliptic parameters (Figure 6a and 6c). However, for model 2 with strong crossterm anelliptic parameter η_{xy} , the accuracy of the approximation based on the parameterization with η_{xy} is better (Figure 6b and 6d). For both models, the relative error in traveltimes is less than 0.1% regardless to the choice of parameterization. To compare the accuracy, we also perform the computation with the traveltimes approximation proposed by Vasconcelos and Tsvankin (2006). The results are shown in Figure 7.

One can see that for both models, the accuracy of our approximation is much better.

DISCUSSION

The sensitivity coefficients a_1 and a_2 equal zero in mutually orthogonal vertical symmetry planes, $a_1 = 0$ when $x = 0$, and $a_2 = 0$ when $y = 0$. The sensitivity coefficient $a_3 = 0$ when $x = 0$ or $y = 0$. However, between the symmetry planes, $x, y \neq 0$, these coefficients do not equal zero. Ideally, the azimuthal dependence of coefficients a_1 and a_2 should be mostly focused with the corresponding symmetry plane, whereas the azimuthal dependence of coefficient a_3 should be mostly focused at azimuth of approximately $\theta = \pi/4$, with as little overlapping as possible.

This is important for inversion of anelliptic parameters from traveltimes data. The overlapping of polar graphs for sensitivity coefficients indicates the azimuth range with similar influence of

Figure 3. (a-c) The first-order coefficients a'_1 , a'_2 , and a'_3 plotted versus offset/depth projections. The model is defined by three velocities $v_0 = 2$ km/s, $v_1 = 2.2$ km/s, and $v_2 = 2.5$ km/s.

Figure 4. (a-c) The second-order coefficients b'_{11} , b'_{22} , and b'_{33} and (d-f) b'_{12} , b'_{13} , and b'_{23} plotted versus offset/depth projections. The model is defined by three velocities $v_0 = 2$ km/s, $v_1 = 2.2$ km/s, and $v_2 = 2.5$ km/s.

anelleptic parameters on traveltime, which can be considered as the azimuthal crosstalk between these parameters.

The crosstalk between anelleptic parameters η_1 and η_2 with different parameterizations discussed above can be illustrated by plotting coefficients a_j and a'_j (b_{jk} and b'_{jk}) in polar plots showing their azi-

muthal dependence. To do that, we substitute lateral coordinates with $x = r \cos \theta$ and $y = r \sin \theta$, where r is the radial offset and θ is the azimuth. For given $t_0 = 1$ s, $r = 5$ km, and velocities $v_1 = 2.2$ km/s and $v_2 = 2.5$ km/s, we plot the coefficients a_j and a'_j (b_{jk} and b'_{jk}) in Figure 8. The first-order coefficients from both

Figure 5. The NMO ellipse (a) and anellepticity (b) for ORT models 1 and 2.

Figure 6. The relative error (in %) in traveltime from equations 11 and 12 plotted versus offset/depth projections. Two parameterizations (η_1 , η_2 and η_{xy}) and (η_1 , η_2 and η_3) are used in computation for two models with three velocities $v_0 = 2$ km/s, $v_1 = 2.2$ km/s, and $v_2 = 2.5$ km/s, anellepticity parameters $\eta_1 = 0.15$ and $\eta_2 = 0.1$ with $\eta_3 = 0.2$ ($\eta_{xy} = 0.056$) for model 1 and $\eta_3 = -0.1$ ($\eta_{xy} = 0.396$) for model 2. The results with first parameterization applied for models 1 and 2 are shown in (a) and (c), respectively. The results with second parameterization applied for models 1 and 2 are shown in (b) and (d), respectively.

Figure 7. The relative error (in %) in traveltime approximation from Vasconcelos and Tsvankin (2006) plotted versus offset/depth projections. The model is defined by three velocities $v_0 = 2$ km/s, $v_1 = 2.2$ km/s, $v_2 = 2.5$ km/s and anellepticity parameters (a) $\eta_1 = 0.15$, $\eta_2 = 0.1$, and $\eta_3 = 0.2$ ($\eta_{xy} = 0.056$) and (b) $\eta_3 = -0.1$ ($\eta_{xy} = 0.396$).

Figure 8. (a) The absolute values of the first-order coefficients a_1 (red), a_2 (blue), and a_3 (black) are shown by solid lines, and a'_1 (red), a'_2 (blue), and a'_3 (black) are shown by dashed lines computed from different parameterizations and plotted in a polar plot. (b) The absolute values of the second-order coefficients b_{11} (red), b_{22} (blue), and b_{12} (black) are shown by solid lines, and b'_{11} (red), b'_{22} (blue), and b'_{12} (black) are shown by dashed lines computed from different parameterizations and plotted in a polar plot. (c) The absolute values of the second-order coefficients b_{13} (red), b_{23} (blue), and b_{33} (black) are shown by solid lines, and b'_{13} (red), b'_{23} (blue), and b'_{33} (black) are shown by dashed lines computed from different parameterizations and plotted in a polar plot. The model is defined by three velocities $v_0 = 2$ km/s, $v_1 = 2.2$ km/s, $v_2 = 2.5$ km/s, and radial offset is 5 km.

parameterizations are shown in Figure 8a. The crosstalk between anelliptic parameters η_1 and η_2 can be seen as the overlapping area from a_1 and a_2 (a'_1 and a'_2). One can see that parameterization with η_{xy} results in a smaller overlapping area compared with the one obtained from parameterization with η_3 . A similar result can be observed from comparison of second-order coefficients b_{11} and b_{22} (b'_{11} and b'_{22}), as is shown in Figure 8b. Both crossterm the second-order coefficients b_{12} and b'_{12} are oriented for an azimuth of approximately $\theta = \pi/4$. The behavior of second-order coefficients b_{13} (b'_{13}) and b_{23} (b'_{23}) is more complicated (Figure 8c). They have two petals, but b_{13} and b_{23} are less detached to corresponding axes compared with b'_{13} and b'_{23} indicating less influence of η_1 and η_2 when estimating η_{xy} .

CONCLUSION

We have developed the traveltime perturbation method for the acoustic ORT eikonal equation. The perturbation coefficients are used to analyze the sensitivity of the traveltime to anelliptic parameters. This analysis reveals that traveltime is more sensitive to η_1 and η_2 within $[xz]$ and $[yz]$ symmetry planes, and sensitive to η_{xy} at a lesser degree mainly along the 45° azimuth direction. For a homogeneous ORT medium, this method results in simple and accurate moveout approximation. We also developed the perturbation solution for another parameterization in anelliptic parameters in terms of η_1 , η_2 , and η_3 . The comparison between two parameterizations shows that for models with weak crossterm anelliptic parameter η_{xy} , both approximations result in similar accuracy, whereas for models with strong η_{xy} , the parameterization with η_{xy} gives better results than the one using η_3 . The comparison made with alternative traveltime approximation derived under assumption of azimuthally dependent VTI medium shows that our approximation is much more accurate for both tested ORT models. The analysis of crosstalk between η_1 and η_2 shows that parameterization with η_{xy} gives better discrimination between anelliptic parameters.

ACKNOWLEDGMENTS

A. Stovas would like to acknowledge the ROSE Project for financial support. N. Masmoudi and T. Alkhalifah would like to thank KAUST for financial support.

APPENDIX A

ORT ANISOTROPY PARAMETER NOTATIONS

In this paper, the following notations for anisotropy parameters are used. Note that they are slightly different from commonly used notations from Tsvankin (1997). The NMO velocities defined in vertical $[xz]$ and $[yz]$ symmetry planes are, respectively, given by

$$\begin{aligned} v_1^2 &= v_0^2(1 + 2\delta_1), \\ v_2^2 &= v_0^2(1 + 2\delta_2). \end{aligned} \quad (\text{A-1})$$

The anellipticity parameters defined in vertical $[xz]$ and $[yz]$ symmetry planes are, respectively, given by

$$\begin{aligned} \eta_1 &= \frac{\varepsilon_1 - \delta_1}{1 + 2\delta_1}, \\ \eta_2 &= \frac{\varepsilon_2 - \delta_2}{1 + 2\delta_2}, \end{aligned} \quad (\text{A-2})$$

and the anellipticity parameter defined in horizontal symmetry plane $[xy]$ is

$$\eta_3 = \frac{\varepsilon_2 - \varepsilon_1 - \delta_3(1 + 2\varepsilon_1)}{(1 + 2\delta_3)(1 + 2\varepsilon_1)} \quad (\text{A-3})$$

The anisotropy parameters ε_1 and ε_2 used in equations above are given by

$$\begin{aligned}\varepsilon_1 &= \frac{c_{11} - c_{33}}{2c_{33}}, \\ \varepsilon_2 &= \frac{c_{22} - c_{33}}{2c_{33}}.\end{aligned}\quad (\text{A-4})$$

The anisotropy parameters δ_1 , δ_2 , and δ_3 used in equations above are given by

$$\begin{aligned}\delta_1 &= \frac{(c_{13} + c_{55})^2 - (c_{33} - c_{55})^2}{2c_{33}(c_{33} - c_{55})}, \\ \delta_2 &= \frac{(c_{23} + c_{44})^2 - (c_{33} - c_{44})^2}{2c_{33}(c_{33} - c_{44})}, \\ \delta_3 &= \frac{(c_{12} + c_{66})^2 - (c_{11} - c_{66})^2}{2c_{11}(c_{11} - c_{66})}.\end{aligned}\quad (\text{A-5})$$

The stiffness coefficients in equations A-4 and A-5 are annotated as c_{ij} .

APPENDIX B

RIGHT-SIDE COEFFICIENTS g_{jk} FROM EQUATION 8

In the case of a heterogeneous medium, the right-side coefficients g_{jk} in equation 8 can explicitly be written in terms of the zeroth-order coefficient and first-order coefficients computed in equations 4 and 6, respectively. On-diagonal second-order coefficients g_{jj} are given by

$$\begin{aligned}2g_{11}\left(\frac{\partial\tau_0}{\partial x}, \frac{\partial\tau_0}{\partial y}, \frac{\partial\tau_0}{\partial z}; \frac{\partial a_1}{\partial x}, \frac{\partial a_1}{\partial y}, \frac{\partial a_1}{\partial z}\right) &= -v_0^2\left(\frac{\partial a_1}{\partial z}\right)^2 \\ &- v_1^2\left(\frac{\partial a_1}{\partial x}\right)\left(4\frac{\partial\tau_0}{\partial x} + \frac{\partial a_1}{\partial x}\right) - v_2^2\left(\frac{\partial a_1}{\partial y}\right)^2 \\ &+ 4v_0^2v_1^2\frac{\partial\tau_0}{\partial x}\frac{\partial\tau_0}{\partial z}\left(\frac{\partial\tau_0}{\partial z}\frac{\partial a_1}{\partial x} + \frac{\partial\tau_0}{\partial x}\frac{\partial a_1}{\partial z}\right) \\ &+ 4v_1^2v_2^2\frac{\partial\tau_0}{\partial x}\frac{\partial\tau_0}{\partial y}\left(\frac{\partial\tau_0}{\partial y}\frac{\partial a_1}{\partial x} + \frac{\partial\tau_0}{\partial x}\frac{\partial a_1}{\partial y}\right),\end{aligned}\quad (\text{B-1})$$

$$\begin{aligned}2g_{22}\left(\frac{\partial\tau_0}{\partial x}, \frac{\partial\tau_0}{\partial y}, \frac{\partial\tau_0}{\partial z}; \frac{\partial a_2}{\partial x}, \frac{\partial a_2}{\partial y}, \frac{\partial a_2}{\partial z}\right) &= -v_0^2\left(\frac{\partial a_2}{\partial z}\right)^2 \\ &- v_1^2\left(\frac{\partial a_2}{\partial x}\right)^2 - v_2^2\left(\frac{\partial a_2}{\partial y}\right)\left(4\frac{\partial\tau_0}{\partial y} + \frac{\partial a_2}{\partial y}\right) \\ &+ 4v_0^2v_2^2\frac{\partial\tau_0}{\partial y}\frac{\partial\tau_0}{\partial z}\left(\frac{\partial\tau_0}{\partial z}\frac{\partial a_2}{\partial y} + \frac{\partial\tau_0}{\partial y}\frac{\partial a_2}{\partial z}\right) \\ &+ 4v_1^2v_2^2\frac{\partial\tau_0}{\partial x}\frac{\partial\tau_0}{\partial y}\left(\frac{\partial\tau_0}{\partial y}\frac{\partial a_2}{\partial x} + \frac{\partial\tau_0}{\partial x}\frac{\partial a_2}{\partial y}\right),\end{aligned}\quad (\text{B-2})$$

$$\begin{aligned}2g_{33}\left(\frac{\partial\tau_0}{\partial x}, \frac{\partial\tau_0}{\partial y}, \frac{\partial\tau_0}{\partial z}; \frac{\partial a_3}{\partial x}, \frac{\partial a_3}{\partial y}, \frac{\partial a_3}{\partial z}\right) &= -v_0^2\left(\frac{\partial a_3}{\partial z}\right)^2 \\ &- v_1^2\left(\frac{\partial a_3}{\partial x}\right)^2 - v_2^2\left(\frac{\partial a_3}{\partial y}\right)^2 \\ &+ v_1^2v_2^2\frac{\partial\tau_0}{\partial x}\frac{\partial\tau_0}{\partial y}\left(\frac{\partial\tau_0}{\partial x}\frac{\partial\tau_0}{\partial y} + 4\left(\frac{\partial\tau_0}{\partial x}\frac{\partial a_3}{\partial y} + \frac{\partial\tau_0}{\partial y}\frac{\partial a_3}{\partial x}\right)\right) \\ &- v_0^2v_1^2v_2^2\left(\frac{\partial\tau_0}{\partial x}\right)^2\left(\frac{\partial\tau_0}{\partial y}\right)^2\left(\frac{\partial\tau_0}{\partial z}\right)^2.\end{aligned}\quad (\text{B-3})$$

Off-diagonal second-order coefficients g_{jk} , $k > j$ are slightly more complicated

$$\begin{aligned}g_{12}\left(\frac{\partial\tau_0}{\partial x}, \frac{\partial\tau_0}{\partial y}, \frac{\partial\tau_0}{\partial z}; \frac{\partial a_{1,2}}{\partial x}, \frac{\partial a_{1,2}}{\partial y}, \frac{\partial a_{1,2}}{\partial z}\right) &= -v_0^2\left(\frac{\partial a_1}{\partial z}\right)\left(\frac{\partial a_2}{\partial z}\right) \\ &- v_1^2\left(\frac{\partial a_2}{\partial x}\right)\left(2\frac{\partial\tau_0}{\partial x} + \frac{\partial a_1}{\partial x}\right) - v_2^2\left(\frac{\partial a_1}{\partial y}\right)\left(2\frac{\partial\tau_0}{\partial y} + \frac{\partial a_2}{\partial y}\right) \\ &+ 2v_0^2v_1^2\frac{\partial\tau_0}{\partial x}\frac{\partial\tau_0}{\partial z}\left(\frac{\partial\tau_0}{\partial z}\frac{\partial a_2}{\partial x} + \frac{\partial\tau_0}{\partial x}\frac{\partial a_2}{\partial z}\right) + 2v_0^2v_2^2\frac{\partial\tau_0}{\partial z}\frac{\partial\tau_0}{\partial y} \\ &\times\left(\frac{\partial\tau_0}{\partial z}\frac{\partial a_1}{\partial y} + \frac{\partial\tau_0}{\partial y}\frac{\partial a_1}{\partial z}\right) + 2v_1^2v_2^2\frac{\partial\tau_0}{\partial x}\frac{\partial\tau_0}{\partial y} \\ &\times\left(\frac{\partial\tau_0}{\partial x}\frac{\partial\tau_0}{\partial y} + \frac{\partial\tau_0}{\partial x}\left(\frac{\partial a_1}{\partial y} + \frac{\partial a_2}{\partial y}\right) + \frac{\partial\tau_0}{\partial y}\left(\frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial x}\right)\right) \\ &- 2v_0^2v_1^2v_2^2\left(\frac{\partial\tau_0}{\partial x}\right)^2\left(\frac{\partial\tau_0}{\partial y}\right)^2\left(\frac{\partial\tau_0}{\partial z}\right)^2,\end{aligned}\quad (\text{B-4})$$

$$\begin{aligned}g_{13}\left(\frac{\partial\tau_0}{\partial x}, \frac{\partial\tau_0}{\partial y}, \frac{\partial\tau_0}{\partial z}; \frac{\partial a_{1,3}}{\partial x}, \frac{\partial a_{1,3}}{\partial y}, \frac{\partial a_{1,3}}{\partial z}\right) &= -v_0^2\left(\frac{\partial a_1}{\partial z}\right)\left(\frac{\partial a_3}{\partial z}\right) - v_1^2\left(\frac{\partial a_3}{\partial x}\right)\left(2\frac{\partial\tau_0}{\partial x} + \frac{\partial a_1}{\partial x}\right) \\ &- v_2^2\left(\frac{\partial a_1}{\partial y}\right)\left(\frac{\partial a_3}{\partial y}\right) + 2v_0^2v_1^2\frac{\partial\tau_0}{\partial x}\frac{\partial\tau_0}{\partial z}\left(\frac{\partial\tau_0}{\partial z}\frac{\partial a_3}{\partial x} + \frac{\partial\tau_0}{\partial x}\frac{\partial a_3}{\partial z}\right) \\ &- 2v_1^2v_2^2\frac{\partial\tau_0}{\partial x}\frac{\partial\tau_0}{\partial y}\left(\frac{\partial\tau_0}{\partial x}\left(\frac{\partial a_1}{\partial y} - \frac{\partial a_3}{\partial y}\right) + \frac{\partial\tau_0}{\partial y}\left(\frac{\partial a_1}{\partial x} - \frac{\partial a_3}{\partial x}\right)\right),\end{aligned}\quad (\text{B-5})$$

$$\begin{aligned}g_{23}\left(\frac{\partial\tau_0}{\partial x}, \frac{\partial\tau_0}{\partial y}, \frac{\partial\tau_0}{\partial z}; \frac{\partial a_{2,3}}{\partial x}, \frac{\partial a_{2,3}}{\partial y}, \frac{\partial a_{2,3}}{\partial z}\right) &= -v_0^2\left(\frac{\partial a_2}{\partial z}\right)\left(\frac{\partial a_3}{\partial z}\right) - v_1^2\left(\frac{\partial a_2}{\partial x}\right)\left(\frac{\partial a_3}{\partial x}\right) \\ &- v_2^2\left(\frac{\partial a_3}{\partial y}\right)\left(2\frac{\partial\tau_0}{\partial y} + \frac{\partial a_2}{\partial y}\right) + 2v_0^2v_2^2\frac{\partial\tau_0}{\partial y}\frac{\partial\tau_0}{\partial z}\left(\frac{\partial\tau_0}{\partial y}\frac{\partial a_3}{\partial z} + \frac{\partial\tau_0}{\partial z}\frac{\partial a_3}{\partial y}\right) \\ &- 2v_1^2v_2^2\frac{\partial\tau_0}{\partial x}\frac{\partial\tau_0}{\partial y}\left(\frac{\partial\tau_0}{\partial y}\left(\frac{\partial a_2}{\partial x} - \frac{\partial a_3}{\partial x}\right) + \frac{\partial\tau_0}{\partial x}\left(\frac{\partial a_2}{\partial y} - \frac{\partial a_3}{\partial y}\right)\right).\end{aligned}\quad (\text{B-6})$$

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