Author's Accepted Manuscript

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PII: S0020-7462(16)30182-2
DOI: <http://dx.doi.org/10.1016/j.ijnonlinmec.2016.09.007>
Reference: NLM2707

To appear in: *International Journal of Non-Linear Mechanics*

Received date: 12 June 2016
Revised date: 18 September 2016
Accepted date: 21 September 2016

Cite this article as: Feras K. Alfosail, Ali H. Nayfeh and Mohammad I. Younis, Natural Frequencies and Mode Shapes of Statically Deformed Inclined Risers, *International Journal of Non-Linear Mechanics*, <http://dx.doi.org/10.1016/j.ijnonlinmec.2016.09.007>

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Natural Frequencies and Mode Shapes of Statically Deformed Inclined Risers

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Abstract

We investigate numerically the linear vibrations of inclined risers using the Galerkin approach. The riser is modeled as an Euler-Bernoulli beam accounting for the nonlinear mid-plane stretching and self-weight. After solving for the initial deflection of the riser due to self-weight, we use a Galerkin expansion employing 15 axially loaded beam mode shapes to solve the eigenvalue problem of the riser around the static equilibrium configuration. This yields the riser natural frequencies and corresponding exact mode shapes for various values of inclination angles and tension. The obtained results are validated against a boundary-layer analytical solution and are found to be in good agreement. This constitutes a basis to study the nonlinear forced vibrations of inclined risers.

Keywords: Inclined riser; Natural frequency; Mode shapes; Galerkin method

1. INTRODUCTION

The study of vibrations of marine risers is important for gaining understanding of their behavior and avoiding their catastrophic failure by fatigue [1]. Marine risers are exposed to environmental static and dynamic loads including vortex-induced forces [2]. Therefore, studying the vibrations of riser structures is essential for examining their structural integrity under these circumstances. Determining accurately the natural frequencies and mode shapes of risers represents an essential step towards analyzing their dynamical behaviors.

Many researchers developed different approaches to study the linear vibrations of vertical marine risers since the early work of Dareing and Huang [3] who used power series expansions to find the natural frequencies and mode shapes of risers. Kirk et al. [4] employed the Rayleigh-Ritz method to compute the natural frequencies of risers. The obtained beam mode shapes are in good agreement with those of Dareing and Huang [3]. Most of the recent research that examines the linear vibration of vertical risers has focused on the mathematical aspects of the methods, such as the segmentation method [5, 6], differential transformation methods [7, 8], and the variational iteration method [9]. In all of the previous research, the axial stretch was neglected, which motivated Mazzilli [10] to use the method of multiple scales [11] to examine the nonlinear modes of vertical risers. More recently, Clementi et al. [11] extended the work of Mazzilli [10] to risers with non-uniform cross sections.

Inclined risers are different from vertical risers because part of the self-weight force induces a distributed load in the inclined riser that causes a bending moment and an initial static deflection. Therefore, we need first to determine their equilibrium configurations. Perturbation methods have been used to examine the linear vibrations of risers with static deflection. Kim and

Triantafillou [12] used the WKB approach to develop an asymptotic solution to the linear vibration problem of a string. Pesce et al. [13] also used the WKB approach to develop a closed-form solution for the mode shapes and frequencies of catenary risers. In contrast, Chatjigeorgiou [14] used the boundary-layer perturbation method to find the mode shapes and natural frequencies of marine risers with variable tension. In all of these perturbation studies, the mode shapes and natural frequencies are based on a string approximation, which can be inaccurate for higher-order modes.

Patrikalakis and Kriezis [15] used finite differences with an embedding technique to solve the linearized equations of risers around their static configurations. Ming-Guang [16] examined the linear vibration problem of post-buckled marine risers. They used the Galerkin approach employing the buckling modes of the risers. On the other hand, Patel and Vaz [17] used the mode shapes of a straight beam in the Galerkin approximation to examine the linear vibration problem of deflected marine risers under current forces. Chucheeepsakul et al. [18] studied the nonlinear vibrations of marine risers accounting for the effect of mid-plane stretching. They recommended the use of a shooting optimization in addition to the weighted residual method to examine the natural frequencies and extract the mode shapes of the structures.

Neglecting the variable tension term in the linear vibration problem, a number of researchers investigated the mode shapes and natural frequencies of risers. Clauss et al. [19] investigated the linear vibrations of risers during laying operations accounting for a constant mid-plane stretching. Nayfeh et al. [20] developed an exact solution to the natural frequencies and mode shapes of buckled beams accounting for mid-plane stretching. Sinir [21] extended the work of Nayfeh et al. [20] by including the effect of internal velocity.

The objective of this work is to develop an analytical-numerical approach to calculate the natural frequencies and mode shapes of inclined risers accounting for their exact static deflections under self-weight. The static profile is obtained using a boundary-layer perturbation method. Then, the dynamic problem is solved using a Galerkin discretization around the equilibrium profile.

2. PROBLEM FORMULATION

Figure 1. Schematic of the clamped-clamped inclined riser.

We investigate the natural frequencies and mode shapes of the inclined riser shown in figure 1. The riser is modeled as an Euler-Bernoulli beam with linearly varying axial tension accounting for its self-weight and its mid-plane stretching. With these assumptions, the equation of motion of the riser can be written as [22-24]

$$\begin{aligned}
 m \frac{\partial^2 \hat{y}}{\partial t^2} + EI \frac{\partial^4 \hat{y}}{\partial \hat{x}^4} + 2m_f U_i \frac{\partial^2 \hat{y}}{\partial \hat{x} \partial t} + \left(m_f U_i^2 - (T_e - W_e \sin(\theta)(L - \hat{x})) - \frac{EA_p}{2L} \int_0^L \left(\frac{\partial \hat{y}}{\partial \hat{x}} \right)^2 d\hat{x} \right) \left(\frac{\partial^2 \hat{y}}{\partial \hat{x}^2} \right) \\
 + c \frac{\partial \hat{y}}{\partial t} - W_e \sin(\theta) \left(\frac{\partial \hat{y}}{\partial \hat{x}} \right) = F_d - W_e \cos(\theta)
 \end{aligned} \quad (1)$$

where m is the total mass per unit length defined as $m = m_f + m_p + m_a$, where m_f is the mass of the internal fluid per unit length, m_p is the mass of riser structure, m_a is the fluid added mass, EI is the flexural rigidity, U_i is the internal fluid velocity, T_e is the applied tension, θ is the inclination angle defined as $\theta = \arctan\left(\frac{\text{WaterDepth}}{\text{RiserOffset}}\right)$, L is the length of the pipe, A_p is the cross-section area of the pipe defined as $A_p = \frac{\pi}{4}(D_o^2 - D_i^2)$, c is the damping coefficient which includes structural damping and fluid added damping, F_d is the external dynamic force, and W_e is the apparent weight of the structure per unit length and is expressed as $W_e = (\rho_i A_i + \rho_p A_p - \rho_e A_e)g$ where ρ_i is the internal fluid density, A_i internal area of the riser pipe defined as $A_i = \frac{\pi}{4} D_i^2$, ρ_e is seawater density, A_e is riser pipe external area and defined as $A_e = \frac{\pi}{4} D_o^2$, g is the gravitational constant, D_i is the internal riser pipe diameter, and D_o is the external riser pipe diameter.

Unlike the vertical riser, the self-weight in the inclined riser introduces a lateral distributed weight. This weight causes an initial deflection of the riser. The static profile $\hat{y}_s$ of the inclined riser is obtained from equation (1) by neglecting the time dependent terms; the result is

$$EI \frac{d^4 \hat{y}_s}{d\hat{x}^4} + \left(m_f U_i^2 - (T_e - W_e \sin(\theta)(L - \hat{x})) - \frac{EA_p}{2L} \int_0^L \left(\frac{d\hat{y}_s}{d\hat{x}} \right)^2 d\hat{x} \right) \left(\frac{d^2 \hat{y}_s}{d\hat{x}^2} \right) - W_e \sin(\theta) \left(\frac{d\hat{y}_s}{d\hat{x}} \right) = -W_e \cos(\theta) \quad (2)$$

We solve equation (2) using the method of matched asymptotic expansions [11] with clamped-clamped boundary conditions [25]. The static solution is exact and complete.

To derive the equation governing the dynamics of the riser structure around the equilibrium configuration, we split the solution of equation (1) into $\hat{y}_s$ and a dynamic component $\hat{y}_d$ such that

$$\hat{y}_t(\hat{x}, t) = \hat{y}_s(\hat{x}) + \hat{y}_d(\hat{x}, t) \quad (3)$$

Substituting equation (3) into equation (1) and using the solution of equation (2), we obtain

$$\begin{aligned} m \frac{\partial^2 \hat{y}_d}{\partial \hat{t}^2} + EI \frac{\partial^4 \hat{y}_d}{\partial \hat{x}^4} + 2m_f U_i \frac{\partial^2 \hat{y}_d}{\partial \hat{x} \partial \hat{t}} - W_e \sin(\theta) \left(\frac{\partial \hat{y}_d}{\partial \hat{x}} \right) + c \frac{\partial \hat{y}_d}{\partial \hat{t}} - \left(\frac{EA_p}{2L} \left(\int_0^L 2 \frac{d\hat{y}_s}{d\hat{x}} \frac{\partial \hat{y}_d}{\partial \hat{x}} d\hat{x} + \int_0^L \left(\frac{\partial \hat{y}_d}{\partial \hat{x}} \right)^2 d\hat{x} \right) \right) \left(\frac{d^2 \hat{y}_s}{d\hat{x}^2} \right) + \\ \left(m_f U_i^2 - (T_e - W_e \sin(\theta)(L - \hat{x})) - \frac{EA_p}{2L} \left(\int_0^L \left(\frac{\partial \hat{y}_s}{\partial \hat{x}} \right)^2 d\hat{x} + \int_0^L \left(\frac{\partial \hat{y}_d}{\partial \hat{x}} \right)^2 d\hat{x} + \int_0^L 2 \frac{d\hat{y}_s}{d\hat{x}} \frac{\partial \hat{y}_d}{\partial \hat{x}} d\hat{x} \right) \right) \left(\frac{\partial^2 \hat{y}_d}{\partial \hat{x}^2} \right) = F_d \end{aligned} \quad (4)$$

Next, we introduce the following dimensionless variables:

$$y = \frac{\hat{y}}{D} \quad x = \frac{\hat{x}}{L} \quad t = \sqrt{\frac{EI}{mL^4}} \hat{t} \quad (5)$$

where the variables with a “hat” are dimensional and the variables without the “hat” are dimensionless. Substituting equation (5) into equation (4) results in

$$\begin{aligned} \frac{\partial^2 y_d}{\partial t^2} + \frac{\partial^4 y_d}{\partial x^4} + 2\sqrt{\beta \bar{V}} \frac{\partial^2 y_d}{\partial x \partial t} + \left(\bar{V}^2 - (\bar{T} - \bar{\sigma}(1-x)) - \bar{\eta} \left(\int_0^1 \left(\frac{\partial y_s}{\partial x} \right)^2 dx + \int_0^1 \left(\frac{\partial y_d}{\partial x} \right)^2 dx + \int_0^1 2 \frac{dy_s}{dx} \frac{\partial y_d}{\partial x} dx \right) \right) \left(\frac{\partial^2 y_d}{\partial x^2} \right) \\ + \bar{c} \frac{\partial y_d}{\partial t} - \bar{\sigma} \left(\frac{\partial y_d}{\partial x} \right) - \left(\bar{\eta} \left(\int_0^1 2 \frac{dy_s}{dx} \frac{\partial y_d}{\partial x} dx + \int_0^1 \left(\frac{\partial y_d}{\partial x} \right)^2 dx \right) \right) \left(\frac{d^2 y_s}{dx^2} \right) = \bar{F}_d \end{aligned} \quad (6)$$

where

$$\begin{aligned} \beta = \frac{m_f}{m}, \quad \bar{V} = \sqrt{\frac{m_f}{EI}} L U_i, \quad \bar{\sigma} = \frac{W_e \sin(\theta) L^3}{EI}, \quad \bar{\eta} = \frac{A_p D^2}{2I}, \quad \bar{c} = \frac{c L^2}{\sqrt{mEI}}, \\ \bar{F}_d = \frac{F_d L^4}{EID}, \quad \bar{T} = \frac{T_e L^2}{EI} \end{aligned} \quad (7)$$

We note that equation (6) contains quadratic and cubic nonlinearities, which arise due to the mid-plane stretching of the riser.

To determine the natural frequencies and mode shapes of the riser, we drop the forcing, damping, and nonlinearities from equation (6) and obtain

$$\begin{aligned} & \frac{\partial^2 y_d}{\partial t^2} + \frac{\partial^4 y_d}{\partial x^4} + 2\sqrt{\beta}\bar{v} \frac{\partial^2 y_d}{\partial x \partial t} + \left(\bar{v}^2 - (\bar{T} - \bar{\sigma}(1-x)) - \bar{\eta} \int_0^1 \left(\frac{\partial y_s}{\partial x} \right)^2 dx \right) \left(\frac{\partial^2 y_d}{\partial x^2} \right) - \left(\bar{\eta} \int_0^1 2 \frac{dy_s}{dx} \frac{\partial y_d}{\partial x} dx \right) \left(\frac{d^2 y_s}{dx^2} \right) \\ & - \bar{\sigma} \left(\frac{\partial y_d}{\partial x} \right) = 0 \end{aligned} \quad (8)$$

We consider the case in which the axial applied tension is much greater than the weight (i.e., $\bar{T} \gg \bar{\sigma}$) and the internal velocity is in the subcritical range. In the analysis, we denote the static mid-plane stretching in equation (8) as

$$\Gamma_s = \bar{\eta} \int_0^1 \left(\frac{dy_s}{dx} \right)^2 dx \quad (9)$$

Next, we seek a solution of equation (8) in the form

$$y_d(x, t) = \phi(x) e^{i\omega t} \quad (10)$$

where ω is the dimensionless natural frequency and ϕ is the corresponding mode shape.

Substituting equation (10) into equation (8) yields

$$\frac{d^4 \phi}{dx^4} + \left[\bar{v}^2 - (\bar{T} - \bar{\sigma}(1-x)) - \Gamma_s \right] \left(\frac{d^2 \phi}{dx^2} \right) - \bar{\sigma} \left(\frac{d\phi}{dx} \right) - \omega^2 \phi(x) = \left(\bar{\eta} \int_0^L 2 \frac{dy_s}{dx} \frac{d\phi}{dx} dx \right) \left(\frac{d^2 y_s}{dx^2} \right) \quad (11a)$$

$$2i\omega\sqrt{\beta}\bar{v} \frac{d\phi}{dx} = 0 \quad (11b)$$

where $i = \sqrt{-1}$. The effect of the Coriolis term of the internal velocity (11b) appears only at internal fluid flow velocities above or near the critical range [21, 26]. It provides linear coupling

between the different mode shapes. In this study, we only consider the subcritical range case.

The boundary conditions associated with equation (11a) are

$$\phi(x)=0 \quad \text{and} \quad \frac{d\phi(x)}{dx}=0 \quad \text{at } x=0 \text{ and } x=1 \quad (12)$$

Due to the limitation imposed by using the boundary layer perturbation method to obtain the analytical static solution of the inclined riser, the boundary condition considered in this analysis is restricted to study clamped-clamped ends. Future research can be directed towards studying other boundary conditions.

3. METHOD OF SOLUTION

The integro-differential equation (11a) and associated boundary conditions, equation (12), constitute an eigenvalue problem for the natural frequencies and associated mode shapes of inclined risers under static deformation. In the absence of an exact solution to this problem, one turns to approximate methods. One possible method is finite differences. However, the variable axial tension governed by $\bar{\sigma}$ destabilizes the numerical scheme [27]. Another possible method is the method of matched asymptotic expansions [14]. However, the lack of bending in the perturbation solution makes the mode-shape solution insufficient and incomplete. Therefore, we use the Galerkin approximation and seek an expansion in the form

$$\phi_i(x) = \sum_{i=1}^n \alpha_i f_i(x) \quad (13)$$

where the $f_i(x)$ are quasi-comparison functions [28] and the α_i are constant coefficients. We choose the $f_i(x)$ to be the eigenfunctions of

$$\frac{d^4 f}{dx^4} + (\bar{\nu}^2 - \bar{T}) \left(\frac{d^2 f}{dx^2} \right) - \omega^2 f(x) = 0 \quad (14)$$

$$f(x)=0 \quad \text{and} \quad \frac{df(x)}{dx}=0 \quad \text{at } x=0 \text{ and } x=1 \quad (15)$$

We use 15 basis functions to ensure convergence of the natural frequencies of the lowest three modes and to capture accurately characteristics of the higher-order modes (see appendix A).

Substituting equation (13) into equation (11a) yields the following system of algebraic equations:

$$\begin{aligned} & \sum_{i=1}^n \alpha_i \frac{d^4 f_i(x)}{dx^4} + \left[\bar{v}^2 - (\bar{T} - \bar{\sigma}(1-x)) - \Gamma_s \right] \sum_{i=1}^n \alpha_i \frac{d^2 f_i(x)}{dx^2} - \bar{\sigma} \sum_{i=1}^n \alpha_i \frac{df_i(x)}{dx} - \omega^2 \sum_{i=1}^n \alpha_i f_i(x) \\ &= \left(\bar{\eta} \int_0^L \left[2 \frac{dy_s}{dx} \sum_{i=1}^n \alpha_i \frac{df_i(x)}{dx} \right] dx \right) \left(\frac{d^2 y_s}{dx^2} \right) \end{aligned} \quad (15)$$

Equation (15) is reduced to a set of algebraic equations in the α_i using the orthogonality of the basis functions. The determinant of the coefficient matrix of the resulting equations gives the natural frequencies of the inclined riser and the associated mode shapes are obtained from the eigenvectors.

4. NUMERICAL RESULTS

We present the frequencies of the riser whose properties are listed in Table 1 for various depths and internal inclination angles. We neglect the effect of the internal velocity by setting $\bar{v}=0$.

Table 1. Properties of the riser used in the analysis

Property	Value
Outside Diameter, D	0.26 m
Inside Diameter, D_i	0.2 m
Modulus of Elasticity, E	207 GPa
Density of Riser, ρ_p	7850 Kg/m ³
Density of Sea Water, ρ_e	1025 Kg/m ³
Density of Internal Fluid, ρ_i	998 Kg/m ³

4.1. Validation Results

We use the boundary-layer static solution of equation (2) [25] to validate the proposed Galerkin approximation utilizing five basis functions. The riser static profiles shown in figure 2 are obtained for a large value of the nonlinear mid-plane stretching for various inclination angles and tensions. In all of the configurations, a good agreement is obtained between the perturbation solution [25] and the five-mode Galerkin approximation.

Figure 2 Riser static profiles : (‘—’) boundary-layer solution and (‘- - -’) five-mode Galerkin approximation. (a) Depth = 150 m , $\theta = 15^\circ$, $\bar{T} = 3500$, (b) depth = 150 m , $\theta = 30^\circ$, $\bar{T} = 850$, (c) depth = 150 m , $\theta = 45^\circ$, $\bar{T} = 400$, and (d) depth = 150 m , $\theta = 60^\circ$, $\bar{T} = 350$.

4.2. Natural Frequencies of the Inclined Riser

In this part, we examine the value of $\bar{\sigma}$ for each riser configuration and define the effective tension on the riser as $T_{eff} = (\bar{v}^2 - \bar{T} + \bar{\sigma} - \Gamma_s)$ and the tension ratio as $T_r = \frac{T_{eff}}{\bar{\sigma}}$. In this way, the natural frequencies of the riser are obtained for cases where the tension $\bar{T} > \bar{\sigma}$ to ensure stable static configurations. The depths that are selected in this study are 50, 75, 100, 125, and 150 meters and the angles are 15° , 30° , 45° , and 75° . The results of the natural frequencies for the lowest five modes are tabulated for applied tension values of $\bar{T} = 2\bar{\sigma}$, $\bar{T} = 5\bar{\sigma}$, and $\bar{T} = 10\bar{\sigma}$ in Tables 2.

Table 2a: Natural frequencies of the riser at $\theta = 15^\circ$.

Depth	$\bar{\sigma}$	$\bar{T}$	T_r	ω_1	ω_2 / ω_1	ω_3 / ω_1	ω_4 / ω_1	ω_5 / ω_1
50	89.23	$2\bar{\sigma}$	31.18	290.07	1.208	1.836	2.465	3.126
		$5\bar{\sigma}$	32.28	286.06	1.245	1.891	2.539	3.928
		$10\bar{\sigma}$	34.27	279.93	1.308	1.985	2.665	4.117
75	301.17	$2\bar{\sigma}$	27.94	498.17	1.202	1.825	2.421	3.040
		$5\bar{\sigma}$	29.03	490.56	1.244	1.885	2.503	3.143
		$10\bar{\sigma}$	31.03	479.05	1.315	1.990	2.646	3.321
100	713.87	$2\bar{\sigma}$	25.66	730.13	1.205	1.829	2.416	3.027
		$5\bar{\sigma}$	26.76	718.20	1.250	1.894	2.506	3.139
		$10\bar{\sigma}$	28.77	700.34	1.328	2.008	2.662	3.334
125	1394.28	$2\bar{\sigma}$	23.97	981.76	1.208	1.834	2.419	3.028
		$5\bar{\sigma}$	25.06	964.82	1.256	1.904	2.516	3.148
		$10\bar{\sigma}$	27.10	939.73	1.340	2.025	2.683	3.357
150	2409.31	$2\bar{\sigma}$	22.64	1250.39	1.211	1.839	2.424	3.033
		$5\bar{\sigma}$	23.74	1227.81	1.262	1.913	2.526	3.160
		$10\bar{\sigma}$	25.79	1194.69	1.351	2.041	2.704	3.381

Table 2b: Natural frequencies of the riser at $\theta = 30^\circ$.

Depth	$\bar{\sigma}$	$\bar{T}$	T_r	ω_1	ω_2 / ω_1	ω_3 / ω_1	ω_4 / ω_1	ω_5 / ω_1
50	23.91	$2\bar{\sigma}$	16.88	114.69	1.296	2.044	2.900	3.894
		$5\bar{\sigma}$	18.10	112.15	1.362	2.143	3.031	4.056
		$10\bar{\sigma}$	16.44	115.74	1.271	2.008	2.851	3.834
75	80.70	$2\bar{\sigma}$	15.88	197.86	1.241	1.899	2.588	3.332
		$5\bar{\sigma}$	17.07	192.86	1.315	2.009	2.735	3.514
		$10\bar{\sigma}$	27.26	182.58	1.719	2.607	3.526	4.487
100	191.28	$2\bar{\sigma}$	14.87	290.67	1.228	1.866	2.504	3.173
		$5\bar{\sigma}$	16.04	282.62	1.308	1.985	2.664	3.373
		$10\bar{\sigma}$	18.30	272.89	1.440	2.180	2.926	3.699
125	373.60	$2\bar{\sigma}$	14.02	391.37	1.225	1.858	2.475	3.117
		$5\bar{\sigma}$	15.20	380.16	1.309	1.983	2.644	3.328
		$10\bar{\sigma}$	25.85	364.42	1.760	2.648	3.540	4.441
150	645.57	$2\bar{\sigma}$	13.32	498.74	1.225	1.858	2.466	3.096
		$5\bar{\sigma}$	14.5	483.88	1.315	1.989	2.644	3.319
		$10\bar{\sigma}$	17.76	462.19	1.516	2.285	3.046	3.819

Table 2c: Natural frequencies of the riser at $\theta = 45^\circ$.

Depth	$\bar{\sigma}$	$\bar{T}$	T_r	ω_1	ω_2 / ω_1	ω_3 / ω_1	ω_4 / ω_1	ω_5 / ω_1
50	11.96	$2\bar{\sigma}$	10.06	66.12	1.474	2.497	3.752	5.308
		$5\bar{\sigma}$	9.61	66.83	1.440	2.448	3.684	5.223
		$10\bar{\sigma}$	14.39	63.08	1.721	2.842	4.063	5.855
75	40.35	$2\bar{\sigma}$	10.28	115.11	1.315	2.071	2.932	3.928
		$5\bar{\sigma}$	11.59	111.28	1.426	2.235	3.150	4.196
		$10\bar{\sigma}$	15.06	107.40	1.643	2.554	3.563	4.694
100	95.64	$2\bar{\sigma}$	9.88	169.42	1.274	1.957	2.689	3.493
		$5\bar{\sigma}$	11.17	163.21	1.393	2.135	2.925	3.782
		$10\bar{\sigma}$	13.87	157.55	1.587	2.422	3.304	4.247
125	186.80	$2\bar{\sigma}$	9.44	228.29	1.259	1.918	2.595	3.314
		$5\bar{\sigma}$	10.72	243.43	1.250	1.899	2.567	3.270
		$10\bar{\sigma}$	13.44	211.77	1.591	2.411	3.253	4.129
150	322.79	$2\bar{\sigma}$	8.87	281.82	1.836	2.764	3.706	4.660
		$5\bar{\sigma}$	10.32	279.39	2.101	2.820	4.343	5.148
		$10\bar{\sigma}$	14.18	269.73	1.664	2.511	3.369	4.246

Table 2d: Natural frequencies of the riser at $\theta = 60^\circ$.

Depth	$\bar{\sigma}$	$\bar{T}$	T_r	ω_1	ω_2/ω_1	ω_3/ω_1	ω_4/ω_1	ω_5/ω_1
50	7.97	$2\bar{\sigma}$	5.60	43.86	1.769	3.193	5.010	7.291
		$5\bar{\sigma}$	7.40	43.29	1.886	3.343	5.203	7.520
		$10\bar{\sigma}$	11.08	44.38	2.014	3.473	5.321	7.595
75	26.90	$2\bar{\sigma}$	6.53	76.93	1.440	2.371	3.494	4.860
		$5\bar{\sigma}$	8.04	74.14	1.599	2.600	3.795	5.227
		$10\bar{\sigma}$	11.35	73.96	1.809	2.890	4.151	5.628
100	63.76	$2\bar{\sigma}$	6.52	113.62	1.348	2.120	2.998	4.011
		$5\bar{\sigma}$	7.97	108.67	1.523	2.379	3.338	4.426
		$10\bar{\sigma}$	11.21	107.92	1.762	2.725	3.779	4.947
125	125.53	$2\bar{\sigma}$	6.34	153.24	1.315	2.026	2.797	3.649
		$5\bar{\sigma}$	11.49	146.06	1.778	2.710	3.691	4.735
		$10\bar{\sigma}$	11.03	145.28	1.755	2.677	3.650	4.686
150	215.19	$2\bar{\sigma}$	6.12	195.45	1.301	1.986	2.704	3.474
		$5\bar{\sigma}$	10.92	185.81	1.764	2.673	3.611	4.589
		$10\bar{\sigma}$	10.85	185.64	1.761	2.668	3.604	4.580

Table 2e: Natural frequencies of the riser at $\theta = 75^\circ$.

Depth	$\bar{\sigma}$	$\bar{T}$	T_r	ω_1	ω_2/ω_1	ω_3/ω_1	ω_4/ω_1	ω_5/ω_1
50	6.41	$2\bar{\sigma}$	2.64	31.47	2.185	4.100	6.619	9.768
		$5\bar{\sigma}$	4.92	32.49	2.260	4.139	6.594	9.652
		$10\bar{\sigma}$	9.46	36.49	2.244	3.964	6.182	8.923
75	21.62	$2\bar{\sigma}$	3.46	52.09	1.683	2.926	4.486	6.428
		$5\bar{\sigma}$	5.40	51.69	1.891	3.191	4.808	6.784
		$10\bar{\sigma}$	9.61	57.46	2.027	3.303	4.838	6.668
100	51.25	$2\bar{\sigma}$	7.72	78.46	1.905	3.003	4.258	5.700
		$5\bar{\sigma}$	5.47	74.85	1.761	2.818	4.055	5.509
		$10\bar{\sigma}$	9.59	83.08	1.957	3.061	4.303	5.710
125	100.105	$2\bar{\sigma}$	3.66	103.95	1.431	2.257	3.202	4.296
		$5\bar{\sigma}$	5.43	100.32	1.717	2.673	3.737	4.934
		$10\bar{\sigma}$	9.55	112.07	1.936	2.971	4.085	5.295
150	172.98	$2\bar{\sigma}$	3.59	132.24	1.412	2.184	3.033	3.981
		$5\bar{\sigma}$	5.35	127.69	1.703	2.614	3.593	4.657
		$10\bar{\sigma}$	9.50	143.97	1.930	2.936	3.988	5.100

The effect of the riser configuration angle on the natural frequencies of the structure can be observed from Tables 2a, 2b, 2c, 2d, and 2e. At lower angles, the riser's weight produces a large static deflection, which in turn results in a higher frequency range. In addition, we observe the possibility of internal resonances for some deflections due to commensurability of the natural frequencies. The interaction between the different modes with commensurable or near commensurable frequencies is expected to lead to complex dynamics [30-32]. The extent of this coupling depends on the values of the mid-plane stretching Γ_s , the applied tension $\bar{T}$, the apparent weight $\bar{\sigma}$, which are reflected in the value of T_r , and the static deflection of the riser. We also note that lowering the depth below 50 m for higher inclination angles makes $\bar{\sigma}$ small enough so that the internal velocity cannot be neglected.

Figure 3. Shift in the first natural frequency of the riser with the internal fluid velocity normalized by the weight for different values of θ and applied tension of $\bar{T} = 5\bar{\sigma}$ (a) depth=150 m and (b) depth=50 m

The effect of the internal fluid velocity is studied by plotting in figure 3 the shift in the first natural frequency from the tabulated values in Tables 2 with the internal velocity normalized by the weight for an applied tension of $\bar{T} = 5\bar{\sigma}$ and various inclination angles. We note from figures 3a and 3b that for the angles $15^\circ, 30^\circ, 45^\circ$, and 60° , as the velocity is increased the natural

frequency of the riser increases for both depth cases of 150 *m* and 50 *m*. On the other hand, at an inclination angle of 75° the inclined riser starts to exhibit characteristic behavior similar to those of vertical riser where the natural frequency decreases as the internal velocity is increased due to the smaller static deflection compared to other inclination angles. This phenomenon is very apparent in the 50 *m* case more than the 150 *m* due to the less static deflection observed in the 50 *m* case. These results agree qualitatively with the previously published work in [26].

We further examine the natural frequencies by plotting in figure 4 variation of the first and second natural frequencies with the applied tension for different inclination angles θ and a depth of 150 *m*. We observe from figures 4a and 4b that an increase in the inclination angle results in a decrease in the characteristic length of the riser, which decreases the frequency ranges. In addition, we observe from figure 4a that the first natural frequency decreases slightly and then increases with increasing applied tension. This is mainly due to the interaction between the axial stretch and the applied tension and the static deflection. The natural frequency of the inclined riser is not only affected by the effective tension, but also by the static deflection of the riser. Increasing the applied tension decreases the maximum static deflection, thus, it will contribute by softening the structure and decreasing the natural frequency. Increasing the applied tension further reduces the riser's deflection and the riser becomes near straight, in which case the stiffness of the structure increases and the frequency increases. This effect becomes less dominant in higher order modes as appears clearly in figure 4b since the second natural frequency tends to increase in all of the cases. The behavior observed in the second natural frequency occurs for all of the higher-order modes.

Figure 4. Variation of the natural frequency of the riser with the applied tension for different values of θ and depth=150 m : (a) first natural frequency and (b) second natural frequency.

Next, we examine the results at an inclination angle of $\theta=72^\circ$ and a depth of 150m. We observe from figure 5a that the range of the lowest five natural frequencies of the riser is very narrow at the lower applied tension range (i.e., 200-500). We observe from figure 5b that as the applied tension increases, the effect of the static deflection becomes negligible and the inclined riser has characteristics similar to those of tension-dominated vertical risers. Then, the frequency ratio of the second mode to the first mode tends to integers, which might result in internal resonances and hence complex dynamics [30-32]. These results confirm the result depicted previously in figure 4.

Figure 5. Variation of the natural frequencies of the riser with the applied tension for $\theta=72^\circ$ and depth=150 m : (a) lowest five natural frequencies and (b) normalized lowest five natural frequencies values with respect to the first natural frequency.

One can note from the previous results that this riser behaves similarly to shallow arches. Low values of tensile forces decreases the curvature (the static deformation) of the riser; and hence decreases its stiffness. This effect overcomes the other opposite effect from tension, which tends to increase the stiffness (as in strings and vertical straight beams). For higher values of tensile forces, as the curved riser gets closer to a straight shape, the effect of reduction in the curvature and static deformation becomes increasingly less until completely disappearing (when the riser becomes straight). Hence after that, tension will lead to an increase in the stiffness, and hence the natural frequencies of the riser. Another point is worth to be mentioned is regarding the convergence of all the natural frequencies such that their ratios are integer numbers. This is expected since the riser becomes dominated by tension, and hence, the natural frequencies become similar to those of a string.

4.3. Mode Shapes of the Inclined Riser

The lowest three mode shapes of the riser are plotted in figure 6 for an inclination angle $\theta = 72^\circ$ and a depth of 150 m. The mode shapes are nearly symmetric due to the configuration of the static equilibrium, which has the maximum deflection at $x = 0.48$.

Figure 6. The lowest three mode shapes of the riser at $\theta=72^\circ$ and a depth=150 m.

5. SUMMARY AND CONCLUSION

The Galerkin approach is used to extract the mode shapes and natural frequencies of inclined marine risers. The mode shapes of a straight beam under axial tension are used as basis functions in the discretization. The obtained static results using the Galerkin approach are in excellent agreement with the exact ones obtained from a boundary-layer method. The numerical study demonstrated the effect of the initial static deflection on the dynamics of the inclined riser structure. At angles away from the vertical configuration, the riser showed softening behavior in the values of the natural frequency which decreases as the configuration becomes close to the vertical configuration. The study of the internal velocity demonstrated the softening effect from static deflection on higher depth side more than the lower depth. These results suggest why it is important to include bending moment in the static and dynamic analysis of risers. Furthermore, analysis of the natural frequencies shows potential complex dynamics involving various internal resonances at high values of applied tension. These resonances will be investigated by including the nonlinearities of the risers in the future.

Acknowledgments

This research made use of the resources of the IT Research Computing at King Abdullah University of Science & Technology (KAUST) in Thuwal, Saudi Arabia. In addition, the first author acknowledges the support of Saudi Aramco.

APPENDIX A

MODE SHAPES OF AXIALLY LOADED PIPE

In this Appendix, we compute the basis functions used in the Galerkin discretization. The basis functions are solutions of the eigenvalue problem

$$\frac{\partial^4 f}{\partial x^4} + (\bar{v}^2 - \bar{T}) \left(\frac{\partial^2 f}{\partial x^2} \right) - \omega^2 f(x) = 0 \quad (\text{A.1})$$

$$f(x) = 0 \quad \text{and} \quad \frac{df(x)}{dx} = 0 \quad \text{at } x = 0 \text{ and } x = 1 \quad (\text{A.2})$$

Equation (A.1) is a fourth-order linear differential equation which has solutions of the following form:

$$f(x) \propto e^{\lambda x} \quad (\text{A.3})$$

Substituting equation (A.3) into equation (A.1) gives the following characteristic equation for the roots:

$$\lambda_1^2 = \frac{(\bar{T} - \bar{v}^2) + \sqrt{(\bar{T} - \bar{v}^2)^2 + 4\omega^2}}{2}, \lambda_2^2 = \frac{(\bar{T} - \bar{v}^2) - \sqrt{(\bar{T} - \bar{v}^2)^2 + 4\omega^2}}{2} \quad (\text{A.4})$$

Hence, the general solution of equation (A.1) can be expressed as

$$f(x) = A \cosh(\lambda_1 x) + B \sinh(\lambda_1 x) + C \cos(\text{Re}[\lambda_2]x) + D \sin(\text{Re}[\lambda_2]x) \quad (\text{A.5})$$

Substituting equation (A.5) into the boundary conditions equation (A.2) yields a set of four linear algebraic equations. Setting the determinant of the coefficient matrix of these equations equal to zero yields a characteristic equation for the ω 's and hence the corresponding mode shapes. Convergence of the lowest three natural frequencies of the inclined riser using the mode shapes of equation (A.5) in equation (15) is shown in figure A.1.

For higher-order mode shapes, the limited computer word length becomes insufficient to capture the oscillation characteristics of the mode shapes (i.e., the trigonometric terms). Therefore, we use 50 working precision in Mathematica [29] with the advantage of utilizing parallel computing to improve the computation time.

Figure A.1 Convergence of the lowest three natural frequencies using 15 mode shapes from equation (A.5) in equation (15) for $\theta=72^\circ$, depth=150 m, and $\bar{T}=650$: (—■—) first natural frequency, (—●—) second natural frequency, and (—▲—) third natural frequency.

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Highlights

The aim of this research is to

- Investigate numerically the linear vibrations of inclined risers using the Galerkin approach. The riser is modeled as an Euler-Bernoulli beam accounting for the nonlinear mid-plane stretching and self-weight including bending moment.
- Perform a parametric investigation using 15 beam mode shapes in Galerkin method to demonstrate how the frequency values changes under different configurations, different values of applied tension and internal velocities.
- Demonstrate the structural softening behavior from the riser static deflection which affected the natural frequency values at higher configuration angles. This behavior decreases at higher configuration angles when the riser is close to the vertical configurations.

- Study how various values of applied tension the frequency values of the inclined riser shows internal resonances which can lead to complex dynamics. These resonances increase with increasing applied tension.
- Study the effect of the internal velocity on the natural frequency of the inclined riser. The effect of the static deflection was demonstrated on the higher configuration angles when the frequency values increase as the velocity is increased.