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Adaptive coupling between damage mechanics and peridynamics: a route for objective simulation of material degradation up to complete failure

Fei Han^a, Gilles Lubineau^{a,*}, Yan Azdoud^a

^a*King Abdullah University of Science and Technology (KAUST),
Physical Science and Engineering Division, COHMAS Laboratory,
Thuwal. 23955-6900. Saudi Arabia*

Abstract

The objective (mesh-independent) simulation of evolving discontinuities, such as cracks, remains a challenge. Current techniques are highly complex or involve intractable computational costs, making simulations up to complete failure difficult. We propose a framework as a new route toward solving this problem that adaptively couples local-continuum damage mechanics with peridynamics to objectively simulate all the steps that lead to material failure: damage nucleation, crack formation and propagation. Local-continuum damage mechanics successfully describes the degradation related to dispersed microdefects before the formation of a macrocrack. However, when damage localizes, it suffers spurious mesh dependency, making the simulation of macrocracks challenging. On the other hand, the peridynamic theory is promising for the simulation of fractures, as it naturally allows discontinuities in the displacement field. Here, we present a hybrid local-continuum damage/peridynamic model. Local-continuum damage mechanics is used to describe “volume” damage before localization. Once localization is detected at a point, the remaining part of the energy is dissipated through an adaptive peridynamic model capable of the transition to a “surface” degradation, typically a crack. We believe that this framework, which actually mimics the real physical process of crack formation, is the first bridge between continuum damage theories and peridynamics. Two-dimensional numerical examples are used to illustrate that an objective simulation of material failure can be achieved by this method.

Keywords: Damage, Peridynamics, Fracture, Coupling, Localization

*Corresponding author. Tel.: +966 12-808-2983. E-mail address: gilles.lubineau@kaust.edu.sa.

1. Introduction

The simulation of material degradation up to the introduction in the solution of discontinuities such as cracks, is a critical issue for the accurate simulation of structural integrity. To date, this remains to be a very challenging task that relies on two well-separated approaches: fracture mechanics and damage mechanics.

Fracture mechanics started in the 1920s with the simulation of brittle materials by Griffith [1] within the framework of linear elasticity. It was extended to elastoplasticity in the 1950s by Irwin [2], who noticed that plasticity must play a significant role in the vicinity of the crack tip, where the “surface” process (the crack) comes along with a “volume” process (in Irwin’s case, plasticity). Fracture-mechanics-based approaches were very successful in helping engineers achieve predictive designs over the last century. However, these approaches have two shortcomings: (1) they are only well suited for describing the propagation of already existing discontinuities within a structure, meaning that they fail to predict the nucleation and initiation of macroscopic cracks and (2) the explicit introduction of the discontinuous component of the solution is often a burden from the numerical point of view. Indeed, classical continuous Galerkin finite element discretization used in engineering cannot explicitly represent discontinuities. Common approaches, among others, prescribe the cracks to follow a path guided by the edges of the elements, making systematic remeshing or enhancing the numerical discretization by explicitly introducing well-suited additional shape function [3–5].

Using damage mechanics, degradation is described through its effect on some measurable properties, such as the elastic stiffness tensor. In the framework of damage mechanics, cracks are not explicitly represented, but an internal variable (the damage indicator, which can be a scalar or a tensor) is introduced to represent local modifications to the stiffness. Damage mechanics has tremendously improved since the pioneering works of Kachanov [6] and Robotnov [7] in the 1950s. Its thermodynamics framework has been well defined [8], and it has been successfully applied to a large variety of configurations, some of which are very challenging, such as composite materials [9, 10]. Although damage mechanics is usually quite easy to implement, there are some challenges: (1) macroscopic cracks are usually defined as zones of high damage, which is questionable as damage by essence does not describe macroscopic discontinuities and (2) simulations become mesh dependent when the level of damage is high, known as spurious mesh dependency. We can consider these challenges as interrelated because spurious mesh dependency can come from the fact that localization is a transition from “volume” damage, which is properly described by damage mechanics, to “surface” degradation, which cannot be adequately represented within the damage mechanics framework. This lack of consistency between the mathematical space in which the theoretical solution exists

and the numerical space that is actually used is responsible for this behavior. Many works have provided mathematical solutions to regularize the problem, by introducing damage localization stabilizers, such as non-local damage models [11], gradient-based models [12] or models with delay effects [13]. All of these approaches share a common idea: the need to introduce an additional length scale when localization starts.

Recently, the first step toward directly handling discontinuities in a unified continuum mechanics formulation has been introduced: Peridynamics redefines the mechanical problem by using integral equations rather than partial differential equations [14]. In this approach, Silling et al. obtain the equilibrium of a material point by an integral of internal forces exerted by non-adjacent points through a finite distance; in a continuous solid, these internal forces are non-local interactions. The interaction between two points is defined in terms of their connection vector, called the *bond*. When a bond is stretched beyond a critical value, it will be broken irreversibly. Consequently, peridynamics allows crack initiation and evolution to occur simultaneously at multiple sites, where spontaneous paths form inside a material without formulating a complex crack growth criterion [15]. One of the main challenges in peridynamics today is its computational cost. This has been partially resolved by allowing the use of peridynamic models only in the parts of the structure where they are needed (where the crack evolves), while the rest of the structure can be described by a classical linear elastic model based on local-continuum mechanics (*i.e.* the Cauchy stress tensor) [16–18]. Nevertheless, even when optimized, full peridynamic models elicit some limitations: (1) non-locality is introduced from the very beginning of the degradation process, which is not needed because the non-local feature should only appear when localization arises and (2) the peridynamic seed, where the initial location within the structure of the peridynamic model occurs, has to be introduced *a priori* and arbitrarily, which is not a suitable technique for general structures.

In this paper, we propose to exploit the advantages of both local-continuum damage mechanics and peridynamics to create a unified hybrid model that overcomes the above mentioned limitations: classical local-continuum damage mechanics is used over the whole structure to reduce computational time, but when localization is detected at some point, the remaining energy in the concerned zone is dissipated in a non-local fashion by supplanting the damage model with a peridynamic model. This is made possible using a coupling technique that we recently developed, so-called “morphing”, between non-local and local models [18–20]. Cracks can then appear in the peridynamic zone, which is then discretized using a discontinuous Galerkin approach. This solves most of the above shortcomings: cracks are actually represented as real discontinuities, objective simulations are guaranteed, non-locality is introduced only when needed. The locations where the peridynamics should be introduced are guided in a fully automatic way by

the damage model. We note that the more general idea of coupling local and non-local approaches has also been tried with other techniques such as the Arlequin methods [21, 22] or the thick level set model [23].

The remainder of this paper is organized as follows: Section 2 reviews the fundamentals of peridynamics and the hybrid local-continuum/peridynamic model. Section 3 is devoted to the original partition in the dissipated energy, between the part that is dissipated in a local fashion (through the local-continuum damage model), and the part that is dissipated in a non-local fashion (through the peridynamic model), and evolution laws are discussed for each of them. Then, we state the numerical algorithm of the hybrid local-continuum damage/peridynamic model in Section 4, and finally, we assess the validity of our approach through numerical examples in Section 5.

2. A review of the hybrid local-continuum/peridynamic model within an elastic framework

Here, we summarize the key features of the hybrid local-continuum/peridynamic model for pure elasticity. Let us consider an elastic body occupying an open-bounded regular domain Ω . Denote by Γ the boundary of Ω , and denote by $\underline{n}$ the outward unit normal to Γ (see Figure 1). Let Γ_u be the part of Γ , over which the displacements $\underline{u}$ are prescribed, and let Γ_T be the complementary part of Γ (*i.e.*, $\Gamma = \Gamma_T \cup \Gamma_u$), over which the external surface tractions $\underline{T}$ are prescribed. In addition to the external body forces (density) denoted by $\underline{b}$, the structure is also subjected to internal body forces (*i.e.*, the force state, $\underline{f}$), which has dimensions of force/volume² [24]. The force state, $\underline{f}(\underline{p} \rightarrow \underline{x})$, denotes an internal force to which the unit volume at a point, $\underline{x}$, is subjected along the bond by a unit volume at a non-adjacent point, $\underline{p}$.

2.1. The peridynamic bond forces

We restrict ourselves to the bond-based peridynamic model, in which the bond forces are central, such that

$$\underline{f}(\underline{p} \rightarrow \underline{x}) = \hat{f}[\underline{x}]\langle \underline{p} - \underline{x} \rangle - \hat{f}[\underline{p}]\langle \underline{x} - \underline{p} \rangle \quad (1)$$

where $\hat{f}[\underline{x}]\langle \underline{p} - \underline{x} \rangle$ (respectively $\hat{f}[\underline{p}]\langle \underline{x} - \underline{p} \rangle$) is the partial interaction due to the action of point $\underline{p}$ over point $\underline{x}$ (with respect to point $\underline{x}$ over point $\underline{p}$) [19]. The force state is thus described as the superposition of two interactions, each related to one of the bond endpoints (see Figure 1). An antisymmetric form (with respect to $\underline{x}$ and $\underline{p}$) of the interactions ensures the balance of the linear momentum [25] and the central interactions (“central” means collinear to the bond vector (*i.e.*, $\underline{\xi} = \underline{p} - \underline{x}$)) ensure the balance of

Figure 1: A hybrid local-continuum/peridynamic domain and partition of a bond force, $\underline{f}(\underline{p} \rightarrow \underline{x})$, into partial forces.

the angular momentum [24]. If small displacements and linear elasticity are assumed, a possible constitutive model is [24, 26]

$$\hat{\underline{f}}[\underline{x}]\langle \underline{p} - \underline{x} \rangle = \frac{c[\underline{x}](|\underline{\xi}|)}{2} \{u_{\underline{\xi}}(\underline{p}) - u_{\underline{\xi}}(\underline{x})\} \underline{e}_{\underline{\xi}}, \quad (2)$$

where $\underline{e}_{\underline{\xi}} = \underline{\xi}/|\underline{\xi}|$, and $u_{\underline{\xi}}(\underline{p})$ (respectively $u_{\underline{\xi}}(\underline{x})$) is the projection of the displacement at point $\underline{p}$ (with respect to point $\underline{x}$) over the bond (*i.e.*, $u_{\underline{\xi}}(\underline{p}) = \underline{u}(\underline{p}) \cdot \underline{e}_{\underline{\xi}}$ and $u_{\underline{\xi}}(\underline{x}) = \underline{u}(\underline{x}) \cdot \underline{e}_{\underline{\xi}}$). $c[\underline{x}](|\underline{\xi}|)$ is called the micromodulus in the peridynamic theory [24]. In the case of a homogeneous material (*i.e.*, $c[\underline{x}](|\underline{\xi}|) = c[\underline{p}](|\underline{\xi}|) = c^0(|\underline{\xi}|)$), the strain energy density of the peridynamic model at point $\underline{x}$ is written as [19]:

$$\phi(\underline{x}) = \frac{1}{4} \int_{H_{\delta}(\underline{x})} c^0(|\underline{\xi}|) \{u_{\underline{\xi}}(\underline{p}) - u_{\underline{\xi}}(\underline{x})\}^2 dV_{\underline{p}}, \quad (3)$$

where $H_{\delta}(\underline{x})$ denotes the neighborhood of $\underline{x}$ with a horizon, δ , over which the bond forces are neglected.

2.2. Relationships between local and non-local models under a smooth strain-field assumption

Assuming both material and strain are homogeneous over the neighborhood, $H_{\delta}(\underline{x})$, of point, $\underline{x}$, it is possible to derive the stiffness operator (the classical Hook tensor) of the

equivalent local-continuum model from the peridynamic parameters. Let us consider an infinitesimal homogeneous transformation over the neighborhood of point, $\underline{x}$, such that

$$\underline{\varepsilon}(\underline{p}) \approx \underline{\varepsilon}(\underline{x}) = \underline{\underline{\varepsilon}} \quad \text{and} \quad u_{\underline{\varepsilon}}(\underline{p}) - u_{\underline{\varepsilon}}(\underline{x}) = \frac{\underline{\xi} \cdot \underline{\underline{\varepsilon}} \cdot \underline{\xi}}{|\underline{\xi}|} \quad \forall \underline{p} \in H_{\delta}(\underline{x}), \quad (4)$$

where $\underline{\varepsilon}$ is the strain tensor. From Eq. (3), the strain energy density of an equivalent local-continuum model can be written as

$$\frac{1}{2} \underline{\underline{\varepsilon}}(\underline{x}) : \underline{\underline{K}}^0 : \underline{\underline{\varepsilon}}(\underline{x}) = \frac{1}{4} \int_{H_{\delta}(\underline{x})} c^0(|\underline{\xi}|) \{u_{\underline{\varepsilon}}(\underline{p}) - u_{\underline{\varepsilon}}(\underline{x})\}^2 dV_{\underline{p}}, \quad (5)$$

where $\underline{\underline{K}}^0$ is the stiffness tensor of the homogeneous material. Applying Eq. (4) to Eq. (5), the stiffness tensor yields

$$\underline{\underline{K}}^0 = \int_{H_{\delta}(\underline{x})} c^0(|\underline{\xi}|) \frac{\underline{\xi} \otimes \underline{\xi} \otimes \underline{\xi} \otimes \underline{\xi}}{2|\underline{\xi}|^2} dV_{\underline{p}}. \quad (6)$$

2.3. The governing equations

In summary, the governing equations of the hybrid local-continuum/peridynamic model for the linearly elastic, isotropic, homogeneous three-dimensional (3D) solid are:

□ **Kinematic admissibility and compatibility:**

$$\underline{\underline{\varepsilon}}(\underline{x}) = \frac{1}{2} (\underline{\underline{\nabla}} \cdot \underline{u}(\underline{x}) + {}^t \underline{\underline{\nabla}} \cdot \underline{u}(\underline{x})) \quad \forall \underline{x} \in \Omega, \quad (7)$$

$$\eta_{\underline{\varepsilon}}(\underline{p}, \underline{x}) = u_{\underline{\varepsilon}}(\underline{p}) - u_{\underline{\varepsilon}}(\underline{x}) \quad \forall \underline{x}, \underline{p} \in \Omega, \quad (8)$$

$$\underline{u}(\underline{x}) = \underline{\underline{u}}(\underline{x}) \quad \forall \underline{x} \in \Gamma_u. \quad (9)$$

□ **Static admissibility:**

$$\text{div} \underline{\underline{\sigma}}(\underline{x}) + \int_{H_{\delta}(\underline{x})} \left\{ \hat{f}[\underline{x}] \langle \underline{p} - \underline{x} \rangle - \hat{f}[\underline{p}] \langle \underline{x} - \underline{p} \rangle \right\} dV_{\underline{p}} = -\underline{b}(\underline{x}) \quad \forall \underline{x} \in \Omega, \quad (10)$$

$$\underline{\underline{\sigma}}(\underline{x}) \cdot \underline{n}(\underline{x}) = \underline{T}(\underline{x}) \quad \forall \underline{x} \in \Gamma_T. \quad (11)$$

□ **Constitutive equations:**

$$\underline{\underline{\sigma}}(\underline{x}) = \underline{\underline{K}}(\underline{x}) : \underline{\underline{\varepsilon}}(\underline{x}) \quad \forall \underline{x} \in \Omega, \quad (12)$$

$$\hat{f}[\underline{x}] \langle \underline{p} - \underline{x} \rangle = \frac{c[\underline{x}] (|\underline{\xi}|)}{2} \eta_{\underline{\varepsilon}}(\underline{p}, \underline{x}) \underline{e}_{\underline{\xi}} \quad \forall \underline{x}, \underline{p} \in \Omega. \quad (13)$$

$\eta_{\underline{\xi}}(\underline{p}, \underline{x})$ in Eq. (8) denotes the projection of the relative displacement at point $\underline{p}$ to point $\underline{x}$ over the bond. The left-hand side of Eq. (10) superimposes the local-continuum model (*i.e.*, the divergence of stress at the point, $\underline{x}$) with the peridynamic model (*i.e.*, the integration of the force state at the neighborhood of the point, $\underline{x}$), which denotes the hybrid model. In Eq. (12), $\underline{\underline{\underline{K}}}(\underline{x})$ is the stiffness tensor of the local-continuum model at the point, $\underline{x}$. In Eq. (13), $c[\underline{x}](|\underline{\xi}|)$ is the peridynamic micromodulus at the point $\underline{x}$. The readers can find that both $\underline{\underline{\underline{K}}}(\underline{x})$ and $c[\underline{x}](|\underline{\xi}|)$ are functions of $\underline{x}$. That is, the effective properties of stiffness and the micromodulus can vary throughout the structure. If we define $c[\underline{x}](|\underline{\xi}|)$ by a morphing function [19], α ($0 \leq \alpha \leq 1$), such that

$$c[\underline{x}](|\underline{\xi}|) = \alpha(\underline{x})c^0(|\underline{\xi}|), \quad (14)$$

then we found that a local-continuum/peridynamic partition of the hybrid model can be determined by both $\underline{\underline{\underline{K}}}(\underline{x})$ and $\alpha(\underline{x})$, which are only functions of location. Indeed, a fully peridynamic model at a point $\underline{x}$ is obtained when $\underline{\underline{\underline{K}}}(\underline{x}) = \underline{\underline{\underline{0}}}$ and $\alpha(\underline{x}') \equiv 1, \forall \underline{x}' \in H_{\delta}(\underline{x})$, a fully local-continuum model is obtained when $\underline{\underline{\underline{K}}}(\underline{x}) = \underline{\underline{\underline{K}}}^0$ and $\alpha(\underline{x}') \equiv 0, \forall \underline{x}' \in H_{\delta}(\underline{x})$, or a relationship between $\underline{\underline{\underline{K}}}(\underline{x})$ and $\alpha(\underline{x})$ is then prescribed to guarantee that both models are compatible and complementary to each other in the hybrid model. This relationship can be derived according to the assumption of homogeneous deformation (*i.e.*, Eq. (4)) and the energy equivalence, such that [19]

$$\underline{\underline{\underline{K}}}(\underline{x}) = \underline{\underline{\underline{K}}}^0 - \int_{H_{\delta}(\underline{x})} c^0(|\underline{\xi}|) \frac{\alpha(\underline{x}) + \alpha(\underline{p})}{2} \frac{\underline{\xi} \otimes \underline{\xi} \otimes \underline{\xi} \otimes \underline{\xi}}{2|\underline{\xi}|^2} dV_{\underline{p}}. \quad (15)$$

3. A hybrid dissipation of the energy between local-continuum damage mechanics and peridynamics

3.1. Local/non-local partition of the free energy and critical damage variable

To ensure that we do not dilute the main message, we assume here the most simple situation of isotropic damage; however, we should stress that the proposed approach is totally general and it can be extended to much more complex damage frameworks. The strain energy density, ϕ , at any point can be written as follows:

$$\phi(d) = \frac{1}{2} \underline{\underline{\underline{\varepsilon}}} : \underline{\underline{\underline{K}}}(d) : \underline{\underline{\underline{\varepsilon}}}, \quad (16)$$

where $\underline{\underline{\underline{K}}}(d) = (1 - d)\underline{\underline{\underline{K}}}^0$ is the damaged stiffness tensor and d is the scalar damage indicator. d ranges from 0 for the virgin structure to d_{max} , such that $0 < d_{max} \leq 1$.

d_{max} is the maximum damage value that guarantees the positiveness of the stiffness operator. In the simple isotropic damage model used here, d_{max} is a scalar value and equals 1.

However, a fundamental problem when numerically solving damage mechanics problems is that objectivity with respect to spatial or time discretization is not ensured during or after localization of damage. The localization usually happens after reaching a critical damage value, say d_{crit} , which is usually much smaller than d_{max} . Damage localization originates from the loss of positive definiteness of the tangent stiffness tensor of the material, $\underline{\underline{\mathbb{C}}}(d)$, when $d = d_{crit}$. This eventually leads to the loss of well-posedness of the initial boundary value problem [27, 28].

The positive-definiteness condition of the tangent stiffness tensor can be derived out from the material stability criterion presented by Hill [29]. We remind the reader here of the key points and limit ourselves to a time-independent material, for which the relationship between the stress rate, $\underline{\underline{\dot{\sigma}}}$, and the strain rate, $\underline{\underline{\dot{\epsilon}}}$, can be written as:

$$\underline{\underline{\dot{\sigma}}} = \underline{\underline{\mathbb{C}}}(d) : \underline{\underline{\dot{\epsilon}}}. \quad (17)$$

A sufficient condition of material stability is usually defined as the scalar product of the strain rate, $\underline{\underline{\dot{\epsilon}}}$, and the stress rate, $\underline{\underline{\dot{\sigma}}}$, being positive [27, 30]:

$$\underline{\underline{\dot{\epsilon}}} : \underline{\underline{\dot{\sigma}}} > 0. \quad (18)$$

Substituting Eq. (17) into the inequality (18) yields

$$\underline{\underline{\dot{\epsilon}}} : \underline{\underline{\mathbb{C}}}(d) : \underline{\underline{\dot{\epsilon}}} > 0. \quad (19)$$

This leads to the condition of positive definiteness of the tangent stiffness tensor, $\underline{\underline{\mathbb{C}}}(d)$, as follows:

$$\det \left(\underline{\underline{\mathbb{C}}}(d) + \underline{\underline{\mathbb{C}}}^T(d) \right) > 0, \quad (20)$$

where $\underline{\underline{\mathbb{C}}}(d)$ is generally non-symmetric [27], while the symmetric case is satisfied by Eq. (20) as well. After that, we can define the variable d_{crit} , which yields

$$\det \left(\underline{\underline{\mathbb{C}}}(d_{crit}) + \underline{\underline{\mathbb{C}}}^T(d_{crit}) \right) = 0. \quad (21)$$

In fact, d_{crit} is not known *a priori*. It varies from one material point to another because it depends on the local loading condition of each material point. In a non-linear damage mechanics simulation, each material point should then be checked at every

increment against the localization criterion to verify whether or not the d_{crit} value for this point has been reached given the loading it experiences. In this work, however, we adopt a given and sole d_{crit} that is kept constant over the whole structure to make the computations more efficient. In such a case, d_{crit} should be chosen to be small enough such that it covers all potential localization configurations that could be met in practice.

Our key idea, to prevent localization and to mimic the physical transition from “volume” to “surface” degradation, is then to split the total energy dissipated at a point between the first part that is dissipated “locally” and in a volume fashion (through local-continuum damage mechanics) and the second part, which must be dissipated “non-locally” and in a surface fashion (typically through the peridynamic model).

When the damage is less than d_{crit} , the local-continuum damage model is viable. The strain energy $\phi(d)$ at any material point can be rewritten to exhibit the role of this limiting value, d_{crit} , such that

$$\begin{aligned}\phi(d) &= \phi(d_{crit}) + \phi(1 - d_{crit} + d) \\ &= \frac{1}{2}\underline{\underline{\varepsilon}} : \underline{\underline{K}}(d_{crit}) : \underline{\underline{\varepsilon}} + \frac{1}{2}\underline{\underline{\varepsilon}} : \underline{\underline{K}}(1 - d_{crit} + d) : \underline{\underline{\varepsilon}} \\ &= \frac{1}{2}\underline{\underline{\varepsilon}} : (1 - d_{crit})\underline{\underline{K}}^0 : \underline{\underline{\varepsilon}} + \frac{1}{2}\underline{\underline{\varepsilon}} : (d_{crit} - d)\underline{\underline{K}}^0 : \underline{\underline{\varepsilon}} \\ &= \frac{1}{2}\underline{\underline{\varepsilon}} : \underline{\underline{K}}_c^0 : \underline{\underline{\varepsilon}} + \frac{1}{2}\underline{\underline{\varepsilon}} : \underline{\underline{K}}_c^d : \underline{\underline{\varepsilon}}, \quad \forall 0 \leq d \leq d_{crit},\end{aligned}\tag{22}$$

On one hand, $\phi(d_{crit})$ represents the remaining free energy that needs to be dissipated when d reaches its critical value, d_{crit} . In other words, this is the energy that we will decide later to allocate to the non-local model, which does not depend on the current value of d but depends only on the chosen critical value, d_{crit} , such that

$$\phi(d_{crit}) = \frac{1}{2}\underline{\underline{\varepsilon}} : \underline{\underline{K}}_c^0 : \underline{\underline{\varepsilon}},\tag{23}$$

where $\underline{\underline{K}}_c^0$ is the residual stiffness tensor at the critical value, d_{crit} , such that

$$\underline{\underline{K}}_c^0 = \underline{\underline{K}}(d_{crit}) = (1 - d_{crit})\underline{\underline{K}}^0\tag{24}$$

On the other hand, $\phi(1 - d_{crit} + d)$ is, for a current value of d , the remaining free energy that needs to be dissipated by the local-continuum model before we switch to a non-local description. It depends both of the critical threshold, d_{crit} , and on the current value of d , such that

$$\phi(1 - d_{crit} + d) = \frac{1}{2}\underline{\underline{\varepsilon}} : \underline{\underline{K}}_c^d : \underline{\underline{\varepsilon}}.\tag{25}$$

$\underline{\underline{K}}_c^d$ is the residual stiffness from the current damage to the critical damage and is defined as

$$\underline{\underline{K}}_c^d = \underline{\underline{K}}(1 - d_{crit} + d) = (d_{crit} - d)\underline{\underline{K}}^0 \quad (26)$$

Obviously, $\phi(1 - d_{crit} + d)$ vanishes to 0 when d reaches the critical value.

The partition from Eqs. (23) and (25) is the key to successfully apply the morphing technique here as it will be discussed in the next section.

3.2. Dissipating the local part of the free energy: a classical damage evolution law

Following the energy partitioning strategy presented in section 3.1, the energy is dissipated in a local manner as long as the local damage value, d , is less than the critical damage for localization, d_{crit} . Any classical damage evolution law can be used to dissipate the “local” part of the free energy. The only difference compared to classical existing models is that the actual value of the damage indicator, d , is bounded to be within the $[0, d_{crit}]$ interval to prevent any localization phenomenon.

All examples in this publication are obtained by assuming a damage evolution law with isotropic hardening; thus, the pseudopotential g can be written as

$$g = Y - (k_1 \times d + k_0), \quad (27)$$

where k_0 is the initial threshold for damage evolution and k_1 is the hardening parameter. Both material parameters are positive real numbers. Y is called the damage force and is defined according to the free energy density such as

$$Y = -\frac{\partial \phi(d)}{\partial d} \Big|_{\varepsilon=const} = \frac{1}{2} \underline{\underline{\varepsilon}} : \underline{\underline{K}}^0 : \underline{\underline{\varepsilon}} = \phi^0 \geq 0. \quad (28)$$

The pseudopotential is used to regulate the evolution of d by the following system of equations classically obtained by using the consistency equation:

$$\begin{cases} \dot{d} = \frac{1}{k_1} \dot{Y}, & \text{if } 0 \leq d < d_{crit} \text{ and if } g = 0, \\ \dot{d} = 0, & \text{if } d = d_{crit} \text{ or } g < 0 \text{ or } \{g = 0 \text{ and } \dot{g} < 0\}. \end{cases} \quad (29)$$

Eq. 29 is obtained by ensuring the consistency equation on the pseudopotential (*i.e.*, $\dot{g} = 0$, $\forall 0 \leq d < d_{crit}$) during the damage evolution phases.

3.3. Switching to peridynamics: the critical micromodulus and its complementary stiffness

As proposed in section 3.1, when d reaches d_{crit} at a point, $\underline{x}$, the local-continuum damage model becomes invalid and is completely replaced for the remaining stiffness/

energy by a non-local description (here, peridynamics). As we reach d_{crit} , the local-continuum model has a stiffness, K_c^0 , at point, $\underline{x}$, that will be fully replaced by a peridynamic model with micromodulus, c_{crit}^0 ; both models should bear the same energy density. Thus, following the equivalence relations developed in section 2.2, we have:

$$\underline{\underline{K}}_c^0 = \int_{H_\delta(\underline{x})} c_{crit}^0(|\underline{\xi}|) \frac{\underline{\xi} \otimes \underline{\xi} \otimes \underline{\xi} \otimes \underline{\xi}}{2|\underline{\xi}|^2} dV_{\underline{p}}. \quad (30)$$

From Eq. (24), we know that $\underline{\underline{K}}_c^0$ is independent of the damage variable, d . Moreover, comparing Eq. (30) with Eq. (6) (*i.e.*, the relationship between $\underline{\underline{K}}^0$ and $c^0(|\underline{\xi}|)$) and using Eq. (24), one obtains

$$c_{crit}^0(|\underline{\xi}|) = (1 - d_{crit})c^0(|\underline{\xi}|). \quad (31)$$

We then redefine the micromodulus, $c[\underline{x}](|\underline{\xi}|)$, in Eq. (13) based on $c_{crit}^0(|\underline{\xi}|)$, such that

$$c[\underline{x}](|\underline{\xi}|) = \alpha(\underline{x})c_{crit}^0(|\underline{\xi}|), \quad (32)$$

where $\alpha(\underline{x}') \equiv 1, \forall \underline{x}' \in H_\delta(\underline{x})$ to guarantee a fully peridynamic model at the point, $\underline{x}$.

When we introduce the peridynamic model with the micromodulus, c_{crit}^0 , using the morphing function, $\alpha(\underline{x})$, we can calculate the complementary stiffness, $\underline{\underline{K}}(\underline{x})$, in the hybrid model. According to Eqs. (16) and (22), we know that

$$\underline{\underline{K}}(d) = \underline{\underline{K}}_c^0 + \underline{\underline{K}}_c^d, \quad \forall 0 \leq d \leq d_{crit}. \quad (33)$$

When the morphing function, $\alpha(\underline{x})$, is introduced, similar to Eq. (15), one obtains that

$$\underline{\underline{K}}_c(\underline{x}) = \underline{\underline{K}}_c^0 - \int_{H_\delta(\underline{x})} c_{crit}^0(|\underline{\xi}|) \frac{\alpha(\underline{x}) + \alpha(\underline{p})}{2} \frac{\underline{\xi} \otimes \underline{\xi} \otimes \underline{\xi} \otimes \underline{\xi}}{2|\underline{\xi}|^2} dV_{\underline{p}}. \quad (34)$$

Substituting Eq. (34) into Eq. (33) yields

$$\underline{\underline{K}}(d) = \underline{\underline{K}}_c(\underline{x}) + \underline{\underline{K}}_c^d + \int_{H_\delta(\underline{x})} c_{crit}^0(|\underline{\xi}|) \frac{\alpha(\underline{x}) + \alpha(\underline{p})}{2} \frac{\underline{\xi} \otimes \underline{\xi} \otimes \underline{\xi} \otimes \underline{\xi}}{2|\underline{\xi}|^2} dV_{\underline{p}}. \quad (35)$$

Thus, the complementary stiffness in the hybrid model is

$$\begin{aligned} \underline{\underline{K}}(\underline{x}) &= \underline{\underline{K}}_c(\underline{x}) + \underline{\underline{K}}_c^d \\ &= \underline{\underline{K}}(d) - \int_{H_\delta(\underline{x})} c_{crit}^0(|\underline{\xi}|) \frac{\alpha(\underline{x}) + \alpha(\underline{p})}{2} \frac{\underline{\xi} \otimes \underline{\xi} \otimes \underline{\xi} \otimes \underline{\xi}}{2|\underline{\xi}|^2} dV_{\underline{p}}, \quad \forall 0 \leq d \leq d_{crit}, \end{aligned} \quad (36)$$

where Eq. (35) is substituted in the latter step. Then, Eq. (36) can be rewritten as

$$\underline{\underline{K}}(\underline{x}) = (1-d)\underline{\underline{K}}^0 - \int_{H_\delta(\underline{x})} c_{crit}^0(|\underline{\xi}|) \frac{\alpha(\underline{x}) + \alpha(\underline{p})}{2} \frac{\underline{\xi} \otimes \underline{\xi} \otimes \underline{\xi} \otimes \underline{\xi}}{2|\underline{\xi}|^2} dV_{\underline{p}}, \quad \forall 0 \leq d \leq d_{crit}. \quad (37)$$

3.4. Dissipating the non-local part of the free energy: a bond failure criterion

After the peridynamic model is introduced at the point, $\underline{x}$, by the morphing function, α , every bond associated with $\underline{x}$ in the peridynamic model is allowed to be broken when its stretch reaches a critical value, s_{crit} . This bond failure criterion was first presented by Silling and Askari [31].

Then, one can introduce a history-dependent scalar-valued function, μ , into the micromodulus, $c[\underline{x}](|\underline{\xi}|)$, of Eq. (13), such that

$$c[\underline{x}](|\underline{\xi}|) = \alpha(\underline{x}) c_{crit}^0(|\underline{\xi}|) \mu(t, \underline{\xi}), \quad (38)$$

where μ takes on values of either 1 or 0, such that

$$\mu(t, \underline{\xi}) = \begin{cases} 1, & \text{if } s(t', \underline{\xi}) < s_{crit} \\ 0, & \text{otherwise.} \end{cases} \quad \forall 0 \leq t' \leq t, \quad (39)$$

where t' and t denote the computational steps and s denotes the bond stretch defined by

$$s = \frac{|\underline{\xi} + \underline{\eta}| - |\underline{\xi}|}{|\underline{\xi}|}, \quad (40)$$

where $\underline{\eta}(\underline{p}, \underline{x}) = \underline{u}(\underline{p}) - \underline{u}(\underline{x})$.

The critical stretch of each bond, s_{crit} , should be an intrinsic material parameter, which determines the work, G_c , required to break all the bonds per unit fracture area in the 3D material model, as defined in [31]. Here, G_c is the fracture work per unit area, which is linked to the fracture toughness of the material. In the two-dimensional (2D) case, which is used for the numerical examples in Section 5, G_c can be calculated assuming a straight crack splits a neighborhood of a point, $\underline{x}$, in half (see Figure 2), such that

$$\begin{aligned} G_c &= \int_0^\delta \int_z^\delta \int_{-\cos^{-1}(z/|\underline{\xi}|)}^{\cos^{-1}(z/|\underline{\xi}|)} \frac{1}{2} c_{crit}^0(|\underline{\xi}|) s_{crit}^2 |\underline{\xi}|^2 |\underline{\xi}| d\phi d|\underline{\xi}| dz \\ &= s_{crit}^2 \int_0^\delta \int_z^\delta c_{crit}^0(|\underline{\xi}|) |\underline{\xi}|^3 \cos^{-1}(z/|\underline{\xi}|) d|\underline{\xi}| dz, \end{aligned} \quad (41)$$

where G_c is the work per unit length in the 2D case. Thus, the critical stretch, s_{crit} , yields

$$s_{crit} = \sqrt{\frac{G_c}{\int_0^\delta \int_z^\delta c_{crit}^0(|\underline{\xi}|) |\underline{\xi}|^3 \cos^{-1}(z/|\underline{\xi}|) d|\underline{\xi}| dz}}. \quad (42)$$

Figure 2: A schematic to evaluate the fracture energy, G_c , by integrating all the bonds crossing the crack, where $0 \leq z \leq \delta$.

4. Numerical algorithm for applications to adaptive coupling

Let us first summarize the key equations characterizing our approach. We are still facing a hybrid local-continuum/peridynamic model described by Eqs. (7)-(13). Additionally, they are associated with the constraint Eqs. (37) and (38), where the damage variable, d , is calculated through Eqs. (28) and (29), and the bond-broken switch, μ , is decided by Eqs. (39), where the bond stretch and the bond failure criterion are defined by Eq. (40) and Eq. (42). Together, these equations construct the proposed hybrid local-continuum damage/peridynamic model that is used for the objective simulation of material failure including damage and fracture.

4.1. Key points of the numerical algorithm

A detailed implementation procedure is stated as follows:

1. To determine the critical micromodulus, c_{crit}^0 , by Eqs. (24) and (30), according to d_{crit} and the stiffness tensor, $\underline{\underline{K}}^0$ (refer to [32]).
2. To evaluate the critical stretch value, s_{crit} , by Eq. (42), in terms of c_{crit}^0 and the fracture work per unit area, G_c .
3. To numerically solve Eqs. (7)-(13) with the constraint conditions (*i.e.*, Eqs. (37) and (38)) by the finite element analysis. The details are stated below with the help of the schematics (*i.e.*, Figure 3).

- (a) The morphing function is set as $\alpha(\underline{x}) \equiv 0, \forall \underline{x} \in \Omega$ at the beginning of the simulation, so the hybrid model is reduced to a local-continuum damage model over the whole structure (see Figure 3(a)).
- (b) The damages are initialized and evolved within the structure. The increment of damage is calculated by Eqs. (28) and (29). At this moment, the model is still a fully local-continuum description as no localization is detected (see Figure 3(b)).
- (c) The damage value at some point, $\underline{x}$, reaches d_{crit} . Then, the morphing function, $\alpha(\underline{x})$, is modified around $\underline{x}$, such that

$$\begin{cases} \alpha(\underline{x}') \equiv 1, & \forall \underline{x}' \in H_\delta(\underline{x}), \\ 0 < \alpha(\underline{x}') < 1, & \forall \underline{x}' \in \{H_{2\delta}(\underline{x}) \setminus H_\delta(\underline{x})\}, \\ \alpha(\underline{x}') \equiv 0, & \forall \underline{x}' \in \{\Omega \setminus H_{2\delta}(\underline{x})\}, \end{cases} \quad (43)$$

where $\{H_{2\delta}(\underline{x}) \setminus H_\delta(\underline{x})\}$ denotes a morphing zone of the hybrid model. Mathematically, a morphing zone can be any options of $\{H_h(\underline{x}) \setminus H_\delta(\underline{x}), \forall h > \delta\}$. In this paper, we choose $h = 2\delta$. Different morphing functions, α , can be chosen in the morphing zone such as a linear function and a cubic function [19, 33]. Then, the peridynamic model is introduced around the point, $\underline{x}$, by α and a hybrid model (see Figure 3(c)), in which part of the energy is described in a non-local manner.

- (d) When more and more points become damaged up to d_{crit} , the peridynamic zone increasingly extends around these points. If any two points, for example, $\underline{x}$ and $\underline{p}$, close to each other are damaged up to d_{crit} , the morphing function, α , will additionally yield the following rule in the intersection of $H_{2\delta}(\underline{x})$ and $H_{2\delta}(\underline{p})$, such that

$$\alpha(\underline{x}') = \max\{\alpha_x(\underline{x}'), \alpha_p(\underline{x}')\}, \forall \underline{x}' \in \{H_{2\delta}(\underline{x}) \cap H_{2\delta}(\underline{p})\}, \quad (44)$$

where $\alpha_x(\underline{x}')$ and $\alpha_p(\underline{x}')$ are the modified morphing functions (refer to Eq. (43)) related to the damaged points, $\underline{x}$ and $\underline{p}$, respectively.

- (e) From a computational point of view, we consider that an element cannot attain critical damage until all its quadrature points reach the critical damage values, d_{crit} . Then, this element is replaced by a discontinuous Galerkin finite element (DGFE) to allow for the development of discontinuities. Note that these introduced DGFEs can only appear in the pure peridynamic zone. The discontinuous Galerkin finite element method (DGFEM) only requires the approximated solution to be in the L^2 space, rather than in the H^1 space where it must be for the conventional continuum model. Therefore,

the DGFEM for the conventional continuum model must be associated with a flux constraint over the boundaries of the elements. However, the DGFEM can be used for the peridynamic model without any constraint because the L^2 space is precisely appropriate for a peridynamic model [34, 35].

- (f) The bond between quadrature points of the DGFE will irreversibly break when it is stretched up to s_{crit} . With the accumulation of broken bonds, the relevant elements are separated along some boundaries of elements. Then, the cracks emerge and eventually destroy the structure (see Figure 3(d)).

Figure 3: The main steps of the hybrid local-continuum damage/peridynamic simulation: (a) initial undamaged state for which the energy at every point is fully described by the local-continuum mechanics, (b) damage development prior to localization: the energy is still fully described in a local manner, (c) some damage reaches the critical value, resulting in the introduction of the non-local model and (d) the hybrid damage/fracture stage.

4.2. Iterative algorithm for assessing damage and bond failure

We propose an implicit algorithm for simulating the damage and fracture evolution. Because the local and the non-local parts of the model share the same displacement field, damage and bond failure (*i.e.*, fracture) can be coupled. However, both variables have a monotonic interdependence, which means that the increase of one induces the increase of the other and vice versa. Potentially, damage and fracture do not occur in the

same region: when damage increases in one region, the overall stiffness of the material decreases, inducing a longer stretch (*i.e.*, higher strain) on bonds in another region, which might lead to bond failure. Alternatively, if a bond is broken in one region, stress concentration increases in the other regions of the structure, which induces an increase in the damage force, eventually inducing damage. This interaction insures that the damage level and bond failure converge monotonically toward a given equilibrium point through an iterative algorithm, which grandly simplifies the way by which the algorithm is written.

Figure 4 defines the implicit algorithm used for the mutual evolution of damage and fracture. In this algorithm, at a given increment of boundary conditions, we iteratively evaluate the damage values and the broken bonds. We first test the damage by computing the current pseudopotential, update the damage and then use the new stiffness value to update the number of broken bonds by evaluating bond stretches. This is conducted iteratively until the damage and bond failure stop increasing. Next, the new damage and bond failure maps are used to define a new partition between local-continuum and peridynamic models, and then the same increment is taken again. This is done incrementally until the specified level of boundary conditions is reached. It is worth noting that this algorithm is quasi-static as we do not introduce any time-dependent process.

Figure 4: A flowchart of the iterative algorithm for the hybrid local-continuum damage/peridynamic model, where N is the number of iterations.

5. Numerical examples

The effectiveness of the proposed strategy is illustrated with three 2D examples for which we assume plane strains, so that the component of displacement, u_z , is constant (without a loss of generality, let $u_z = 0$) and the components of displacement, u_x and u_y , are functions of x and y .

5.1. Example 1: a half-circle, notched plate under traction

We consider a benchmark example, where a 2D half-circle, notched plate is subjected to a traction condition on the right side (see Figure 5).

An exponential micromodulus, c_{crit}^0 , is assumed to be $c_{crit}^0(\xi) = c^0|\xi|^2 e^{-|\xi|/l}$ (see [32, 35]), where c^0 is the constant and l is a characteristic length that is assumed to

Figure 5: The geometry and boundary conditions of 2D half-circle, notched plate (unit of length: mm).

be 0.024mm. Here, the micromodulus, $c_{crit}^0(\underline{\xi})$, includes a square of $|\underline{\xi}|$ to ensure the accurate integral in Eq. (37) when $|\underline{\xi}|$ approaches zero. The horizon, δ , is 0.12mm. The stiffness parameters in the local-continuum model, including Young's modulus and Poisson ratio, are $E = 200\text{GPa}$ and $\nu = 1/3$, respectively. Parameters, k_1 and k_0 , in the damage evolution law are $k_1 = 4500\text{MPa}$ and $k_0 = 900\text{MPa}$. The critical damage value, d_{crit} , is 0.365. The fracture work per unit area, G_c , is 0.44J/mm^2 , which determines that the critical stretch, s_{crit} , is 0.2 for bond failure, by calculating Eq. (42).

From Figure 5, we can see that the left side of the plate is fixed along the y -axis, and both top and bottom of the plate are fixed along the x -axis. The traction is applied to the right side of the plate, where $\bar{u}_x = 0.15\text{mm}$. This numerical example is implemented in a finite element framework with bilinear quadrilateral elements. A structured mesh is used in the central part of the plate whereas an unstructured mesh is used for the remaining part (see Figure 6). Additionally, the size of unstructured elements gradually increases from the boundary of the structured-element zone to the left or right boundaries of the plate to reduce the computational cost. Three sizes of structured grids are used, 0.04mm, 0.02mm and 0.01mm, to compare the effects of grid size.

Figure 6: Three sizes of structured grids were placed in the middle of plates (unit of length: mm): (a) 0.04mm, (b) 0.02mm and (c) 0.01mm. The size of unstructured elements gradually increases from the boundary of the structured-element zone to the left or right edges of the plate.

To clearly illustrate the performance of the proposed approach on preventing spurious localization, we simulate the complete failure of the plate using the local-continuum damage model and the hybrid local-continuum damage/peridynamic model. Both models are solved through the implicit solver. Because the incremental displacement is $\Delta \bar{u}_x = 0.0003\text{mm}$ in each step, the total prescribed displacement, $\bar{u}_x = 0.15\text{mm}$, is achieved by 500 iterations.

Figure 7 shows the stress about the x direction, which is calculated by averaging the normal stresses about the x direction of all elements at the right boundary of the domain. It is worth noting that all elements at the right boundary of the domain are only used for the local-continuum model, as the boundary is not damaged from the beginning to the end. When simulated with the local-continuum damage model, the damage localization happens at different steps for different-sized structured grids in the middle of plates (see Figure 7): 0.04mm at step 372, 0.02mm at step 322 and 0.01mm at step 278. The maximum stress at the step of damage localization is reduced when the mesh is refined. On the other hand, when we employ the hybrid local-continuum damage/peridynamic

model, the final break of the plate happens almost simultaneously: 0.04mm at step 428, 0.02mm at step 424 and 0.01mm at step 425. The maximum stresses are the same.

Figure 7: Stress vs. displacement steps using: (dashed red line) the local-continuum damage model with 0.04mm structured grids, (dash-double-dotted green line) with 0.02mm structured grids, (dotted blue line) with 0.01mm structured grids, (solid red line) the hybrid local-continuum damage/peridynamic model with 0.04mm structured grids, (long-dashed green line) with 0.02mm structured grids and (dash-dotted blue line) with 0.01mm structured grids. (Legend: LCD=local-continuum damage and HLP=hybrid local-continuum damage/peridynamic).

Consequently, the dissipated energy of the whole plate calculated by the local-continuum damage model is completely different for each grid size (see Figure 8(a)). As expected, the dissipated energy is reduced as the mesh is refined. While using the hybrid local-continuum damage/peridynamic model, the dissipated energy is similar for different-sized meshes (see Figure 8(b)). Moreover, the morphology of the damage localization zone simulated by the local-continuum damage model is also different for each grid size (see Figure 9). From Figures 9(a)-9(c), we can see that the color indicates the local damage calculated in every element. The colorful zone in each subfigure indicates where damage localization occurs.

Figure 8: Dissipated energy vs. displacement steps using: (a) the local-continuum damage model (solid red line) with 0.04mm structured grids, (dashed green line) with 0.02mm structured grids and (dash-dotted blue line) with 0.01mm structured grids; (b) the hybrid local-continuum damage/peridynamic model (solid red line) with 0.04mm structured grids, (dashed green line) with 0.02mm structured grids and (dash-dotted blue line) with 0.01mm structured grids. (Legend: LCD=local-continuum damage and HLP=hybrid local-continuum damage/peridynamic).

In the implementation of the hybrid local-continuum damage/peridynamic model, we introduced the peridynamic model into the critically damaged zone locally (see Figure 10(a)) before the damage localization takes place. We implement the DGFEM in the peridynamic domain (see Figure 10(b)). The critically damaged zone defined here is at least one element where the damage variables reach the critical value (*i.e.*, 0.365 in this example). As the critically damaged zone emerges, peridynamics is immediately applied into the zone where the DGFEM is used to simulate crack formation and propagation (see Figures 10(c)-10(f)). The color in Figure 10 shows the total damage value, which is the sum of the local damage value calculated by the local-continuum damage model and the non-local damage value evaluated by the peridynamic model [31]. The non-local damage value is calculated by $r \times (1 - d_{crit})$, where r is the ratio of the dissipated energy by all the already broken bonds to the entire amount of energy that could be dissipated by breaking all the bonds of a point. Figure 11 shows the final fractures of the plate for different sizes of structured grids. The shapes of the cracks for different grids are similar: a straight line on the top half of the crack and a bifurcated crack at the bottom. The branches of the crack can be seen from all subfigures. Additionally, the distributions of

damage are all the same for all grid sizes.

Figure 9: The zones of damage localization using the local-continuum damage model with differently sized structured grids in the middle of the half-circle, notched plate (unit of length: mm): (a) 0.04mm, (b) 0.02mm and (c) 0.01mm.

Figure 10: The evolution of damage and fracture from (a) to (f) in the half-circle, notched plate (unit of length: mm). The discontinuous Galerkin finite element method (represented by quadrilateral grids) is implemented in the peridynamic domain.

Figure 11: The final damage zones and fractures by the hybrid local-continuum damage/peridynamic model with differently sized structured grids in the middle of the half-circle, notched plate (unit of length: mm): (a) 0.04mm, (b) 0.02mm and (c) 0.01mm.

5.2. Example 2: three-point bending test

The second example is a three-point bending test of a single-edge notched beam [36]. The dimensions of the beam and boundary conditions are presented in Figure 12 in which the displacement condition, $\bar{u}_y = 6\text{mm}$, is prescribed at the center of the top edge of the beam. “Unstructured quadrilateral elements” are used in the simulations of a finite element framework. The grids are gradually refined from both the left and right boundaries of the beam towards the middle zone (*i.e.*, the range of x is from 115.5mm to 124.5mm) where the notch is covered (see Figure 13(a)). In the middle zone of the beam, two given sizes of unstructured mesh are applied to study the effect of the size

of the mesh. The size of the coarser mesh is about 0.5mm (see Figure 13(b)), while the size of the finer mesh is about 0.375mm (see Figure 13(c)).

Figure 12: The geometry and boundary conditions of 2D three-point bending test.

The micromodulus expression used in the peridynamic model is the same as that in Section 5.1, in which the characteristic length is $l = 0.3\text{mm}$ and the horizon is chosen to be $\delta = 1.5\text{mm}$. The Young's modulus and Poisson ratio in the local-continuum model are $E = 200\text{GPa}$ and $\nu = 1/3$, respectively. The following parameters are chosen to simulate the damage evolution: $k_1 = 400\text{MPa}$, $k_0 = 100\text{MPa}$ and $d_{crit} = 0.411$. The critical stretch, s_{crit} , is set to 0.067, in terms of the fracture work per unit area, $G_c = 5.63\text{J/mm}^2$, applied in Eq. (42). This simulation is performed in 120 iterations, so that the increment of displacement yields $\Delta\bar{u}_y = 0.05\text{mm}$ in each step.

The three-point bending test is simulated by both the local-continuum damage model and the hybrid local-continuum damage/peridynamic model with different meshes. The load-displacement curves are plotted in Figure 14(a). When the local-continuum damage model is implemented, the damage localization occurs at very different steps with the coarser and finer meshes (*i.e.*, at step 36 with the coarser mesh and at step 26 with the finer mesh) and the peak calculated with the coarser mesh is much higher than that with the finer mesh (*i.e.*, the relative difference is 28.2%). A steep decline of force occurs with both meshes immediately at the localization step. On the other hand, the hybrid local-continuum damage/peridynamic model predicts almost the same peaks of force for both coarser and finer meshes (*i.e.*, the relative difference in peaks is 1.88%). After the peaks of both curves, the force decreases smoothly as the displacement increases. Similar trends are observed for both coarser and finer meshes. However, there exists a little gap between the two declining curves after the peak point. This little gap appears because the force in the case of the coarser mesh remains almost unchanged for a few steps after the peak is formed, unlike with the finer mesh. In the subsequent steps (*i.e.*,

Figure 13: The unstructured quadrilateral meshes (unit of length: mm): (a) the size of elements gradually increases from the boundary of the middle zone (*i.e.*, $x \in [115.5, 124.5]$) towards the left or right edges of the beam. The size of elements in the middle zone is about (b) 0.5mm (coarser mesh) and (c) 0.375mm (finer mesh).

$4\text{mm} \leq u_y \leq 6\text{mm}$), both curves gradually converge. The reasons for the little gap and the convergence towards the end between both curves are explained below.

Figure 14(b) presents four load-displacement curves for different unstructured meshes in the case of the proposed hybrid model. For these four meshes, the size of the elements in the middle zone of the beam are the same (*i.e.*, 0.5mm). Figure 14(b) shows that the four curves calculated by the proposed hybrid model are consistent with each other if the perturbation of random meshes to the crack path, which causes varying forces, is ignored. The shape of these load-displacement curves is similar to the experimental responses of the three-point bending test in [37]. Zooming in on the peak region shows that the peaks corresponding to all the meshes remain almost flat for few steps (inset

to Figure 14(b)). This phenomenon is explained below.

Figure 14: The load-displacement curves of three-point bending test: (a) the local-continuum damage model (dashed red line) with coarser mesh, (dash-double-dotted blue line) with finer mesh; the hybrid local-continuum damage/peridynamic model (solid red line) with coarser mesh, (dash-dotted blue line) with finer mesh (b) the hybrid local-continuum damage/peridynamic model with four different coarser meshes. (Legend: LCD=local-continuum damage and HLP=hybrid local-continuum damage/peridynamic).

Figure 15 shows the propagation of a crack associated with the evolution of the damage by the coarser mesh in Figure 15(a)-(c) and by the finer mesh in Figure 15(d)-(f). By comparing Figure 15(a) and (d), we observe that the initial crack for both meshes appears at the same step (*i.e.*, step 50). For the finer mesh, the crack grows rapidly to about 2mm in the next step (*i.e.*, step 51) (see Figure 15(e)). However, a similar length of the crack is not reached until step 54 for the coarser mesh (see Figure 15(b)). Figures 15(c) and (f) show that the cracks with both meshes finally grow to the exact same length at the last step (*i.e.*, step 120). Both cracks are almost straight.

With the help of Figure 15, we explain below the reasons for the little gap and the convergence between both curves after the peak in Figure 14(a). The force reaches its peak at step 50 for the finer mesh. Similarly, the peak region starts from step 50 with the coarser mesh. However, the force by the finer mesh declines immediately in the next step (*i.e.*, step 51) because the crack grows rapidly in only one step (see Figure 15(e)). In contrast, the force by the coarser mesh stays in the peak region for few steps

because the crack takes a few more steps (*i.e.*, 4 steps) to grow to a similar length (see Figure 15(a) and (b)). This “few-step delay” of force results in the gap between the two curves. The underlying reason is that points near the boundary (*i.e.*, the tip of the notch) in the peridynamic model fail to hold a complete neighborhood. This incomplete neighborhood causes the finer elements near the boundary to become less stiff than the coarser elements [19], which leads to easier bond breaking. Therefore, the crack near the boundary propagates faster in the case of the finer mesh than in the case of the coarser mesh. However, this problem can be ignored when the horizon is much smaller than the scale of the structure. When the tip of the crack is farther from the boundary, both coarser and finer meshes have a complete neighborhood. Hence, the speed of crack propagation is the same, which can be verified by the two almost parallel parts of the curves in Figure 14(a). As the tip of the crack approaches the top of the beam, the crack propagation slows down and the cracks by the coarser and finer meshes reach the same length (see Figure 15(c) and (f)). Consequently, the force reduces to the same value at the end and the load-displacement curves by the coarser and finer meshes gradually converge in Figure 14(a).

5.3. Example 3: mixed mode fracture in a double-edge-notched plate

We apply the proposed method to a more difficult example, where the plate is notched in the middle of both left and right sides [38, 39]. Its geometric dimensions can be seen from Figure 16(a). The displacement conditions, $\bar{u}_y = 12\text{mm}$, are prescribed on the top and bottom of the plate, which is also subjected to the compressions, $\bar{F}_x = 1\text{GPa}$, on the upper left and lower right sides (see Figure 16(a)). The “unstructured quadrilateral elements” are used for the simulations. These grids are gradually refined from both top and bottom to the middle of the plate to accurately simulate the failure phenomena and reduce the computational cost as much as possible (see Figure 16(b)).

Most of material parameters used in this example are the same to those in section 5.1 with the exception of a characteristic length of $l = 0.4\text{mm}$ for the micromodulus, a horizon of $\delta = 2\text{mm}$ and a critical damage value of $d_{crit} = 0.475$. Accordingly, the critical stretch for bond failure is calculated by Eq. (42) in terms of the fracture work per unit area, G_c , which is 8.6J/mm^2 in this example. Then, the critical stretch yields $s_{crit} = 0.24$. This simulation is implemented by 120 iterations. In every step, the increment of displacement is $\Delta\bar{u}_y = 0.1\text{mm}$, but the compression condition is maintained constant throughout.

Figure 17 shows the deformation, damage and fracture in the middle of the plate. The figures are plotted by the contour of the total damage, which is the sum of the local and non-local damages explained in the last example. In Figure 17(a), the plate has only elastic deformation with no damage in the first step. In step 26, we can see that

Figure 15: The crack propagation associated with damage evolution in the three-point bending test by a coarser mesh (0.5mm) from (a) to (c) and by a finer mesh (0.375mm) from (d) to (f) (unit of length: mm). The discontinuous Galerkin finite element method (represented by quadrilateral grids) is implemented in the peridynamic domain.

local damage appears at the notched corners from the zoomed-in region of Figure 17(b). Note that only the pure local-continuum damage model is implemented in the structure in this step because no morphing region exist that can introduce the DGFs. In step 33 in Figure 17(c), a crack appears at one of the corners on each side of the plate along the edges of the DGFs. The DGFs appear when the peridynamic model was patched to the critically damaged zone (close to the corners). This critically damaged zone was calculated in previous steps. After that, the damage and crack quickly propagate. Then, Figure 17(d) shows that an obvious damage zone forms between two tips of cracks in step 37. However, the damage values in this damage zone will not increase further, which can be seen from the next subfigures (*i.e.*, Figure 17(e)) because two cracks pass by the damage zone and propagate farther. In addition, we found that a new damage zone gradually emerges from the boundary of the upper crack (see the

Figure 16: The 2D double-edge-notched plate: (a) the geometry and boundary conditions and (b) the unstructured quadrilateral meshes.

zoomed-in region). This “extra” damage region doesn’t appear around the tip of the lower crack but happens from the flat boundary, which may be explained by two cracks creating a neck-zone of the cracked plate where the shear stress stimulates the evolution of the damage from the boundary. Figure 17(f) shows this new damaged region and the tip of the lower crack finally touch in step 111, which may eventually result in the complete rupture of the plate. Complete separation cannot be obtained here due to the Neumann boundary conditions on the vertical edges, which would prevent convergence after separation. From these figures, we can conclude that the crack can be initialized from the critically damaged zone, after which the tip of the crack drives the evolution of damage around it. On the other hand, after the crack appears, the damage can still evolve in some regions away from the tip of the crack. The pre-existing damage zone guides the propagation of the crack in some cases, but not in others.

Figure 17: The evolution of the material failure including damage and fracture (unit of length: mm). The contours of total damage (the sum of local and non-local damages) and the fractures are shown.

6. Conclusion

All key steps leading to material failure, from damage evolution to crack propagation, can be naturally simulated based on the hybrid local-continuum damage/peridynamic model. This hybrid technique constructs a single-unified balance equation between the stiffness and the critical micromodulus. By using the morphing function in this balance equation, a peridynamic model for fracture is adaptively introduced into the local positions where the damage reaches a critical value that could lead to spurious localization.

Unlike the conventional local-continuum damage model, the proposed hybrid local-continuum damage/peridynamic model is mesh independent. A concurrent damage/fracture process is observed in numerical simulations. The interactions between dam-

age and fracture are investigated: some highly damaged zones lead to the extensions of cracks, while the tip of crack accelerates the concentration of the damage. Future researches should discuss the relevant evolution laws for damage and fracture under this hybrid framework and should focus on the interactions between damage and fracture under different loading conditions. This approach presents a new direction toward describing inner links and physical transitions from “volume” damage to “surface” degradation, which is one of the most challenging open question in integrity prediction.

References

- [1] A. A. Griffith. *The Phenomena of Rupture and Flow in Solids*, volume 221. The Royal Society, 1921.
- [2] G. R. Irwin. Analysis of stresses and strains near the end of a crack traversing a plate. *Journal of Applied Mechanics*, pages 361–364, 1957.
- [3] N. Moës, J. Dolbow, and T. Belytschko. A finite element method for crack growth without remeshing. *International Journal for Numerical Methods in Engineering*, 46(1):131–150, 1999.
- [4] T. Belytschko and T. Black. Elastic crack growth in finite elements with minimal remeshing. *International Journal for Numerical Methods in Engineering*, 45:601–620, 1999.
- [5] É. Budyn, G. Zi, N. Moës, and T. Belytschko. A method for multiple crack growth in brittle materials without remeshing. *International Journal for Numerical Methods in Engineering*, 61(10):1741–1770, 2004.
- [6] L. M. Kachanov. Rupture time under creep conditions. *Izvestia Akademii Nauk SSSR, Otdelenie tekhnicheskikh nauk*, 8:26–31, 1958.
- [7] Y. N. Rabotnov. A mechanism of a long time failure. In *Creep problems in structural members*, pages 5–7, 1959.
- [8] J. Lemaitre. *A course on damage mechanics*. Springer, 1996.
- [9] P. Ladevèze and E LeDantec. Damage modeling of the elementary ply for laminated composites. *Composites Science and Technology*, 43(3):257–267, 1992.
- [10] P. Ladevèze and G. Lubineau. On a damage mesomodel for laminates: micro-meso relationships, possibilities and limits. *Composites Science and Technology*, 61(15):2149–2158, 2001.
- [11] Z. Bazant and G. Pijaudier-Cabot. Nonlocal continuum damage, localization instability and convergence. *Journal of Applied Mechanics*, 55:287–293, 1988.
- [12] R. Peerlings, R. de Borst, W. Brekelmans, and M. Geers. Gradient-enhanced damage modelling of concrete fracture. *Mechanics of Cohesive-frictional Materials*, 3(4):323–342, 1998.

- [13] O. Allix, P. Feissel, and P. Thévenet. A delay damage mesomodel of laminates under dynamic loading: basic aspects and identification issues. *Computers & Structures*, 81(12):1177–1191, 2003.
- [14] S. A. Silling. Reformulation of elasticity theory for discontinuities and long-range forces. *Journal of the Mechanics and Physics of Solids*, 48:175–209, 2000.
- [15] B. Kilic, A. Agwai, and E. Madenci. Peridynamic theory for progressive damage prediction in center-cracked composite laminates. *Composite Structures*, 90(2):141–151, 2009.
- [16] P. Seleson, S. Beneddine, and S. Prudhomme. A force-based coupling scheme for peridynamics and classical elasticity. *Computational Materials Science*, 66:34–49, 2013.
- [17] A. Askari, Y. Azdoud, F. Han, G. Lubineau, and S. Silling. 12 - peridynamics for analysis of failure in advanced composite materials. In S. R. Hallett and P. P. Camanho, editors, *Numerical Modelling of Failure in Advanced Composite Materials*, pages 331–350. Woodhead Publishing, 2015.
- [18] Y. Azdoud, F. Han, D. Littlewood, G. Lubineau, and P. Seleson. Coupling local and non-local models. In F. Bobaru, J. T. Foster, P. H. Geubelle, and S. A. Silling, editors, *Handbook of Peridynamic Modeling*, page 600. Chapman and Hall/CRC, 2016.
- [19] G. Lubineau, Y. Azdoud, F. Han, C. Rey, and A. Askari. A morphing strategy to couple non-local to local continuum. *Journal of the Mechanics and Physics of Solids*, 60(6):1088–1102, 2012.
- [20] F. Han, G. Lubineau, Y. Azdoud, and A. Askari. A morphing approach to couple state-based peridynamics with classical continuum mechanics. *Computer Methods in Applied Mechanics and Engineering*, 2016.
- [21] F. Han and G. Lubineau. Coupling of nonlocal and local continuum models by the arlequin approach. *International Journal for Numerical Methods in Engineering*, 89(6):671–685, 2012.
- [22] H. Ben Dhia and G. Rateau. The arlequin method as a flexible engineering design tool. *International Journal for Numerical Methods in Engineering*, 62(11):1442–1462, 2005.

- [23] N. Moës, C. Stolz, and N. Chevaugeon. Coupling local and non-local damage evolutions with the thick level set model. *Advanced Modeling and Simulation in Engineering Sciences*, 1(1):1–21, 2014.
- [24] S. A. Silling and R. B. Lehoucq. Peridynamic theory of solid mechanics. *Advances in Applied Mechanics*, 44:73–168, 2010.
- [25] S. A. Silling, M. Epton, O. Weckner, J. Xu, and E. Askari. Peridynamic states and constitutive modeling. *Journal of Elasticity*, 88(2):151–184, 2007.
- [26] S. A. Silling. Linearized theory of peridynamic states. *Journal of Elasticity*, 99(1):85–111, 2010.
- [27] R. de Borst, M. Crisfield, J. Remmers, and C. Verhoosel. *Non-linear Finite Element Analysis of Solids and Structures*. Wiley, second edition, 2012.
- [28] S. Murakami. *Continuum Damage Mechanics: A Continuum Mechanics Approach to the Analysis of Damage and Fracture*. Springer, 2012.
- [29] R. Hill. A general theory of uniqueness and stability in elastic-plastic solids. *Journal of the Mechanics and Physics of Solids*, 6(3):236–249, 1958.
- [30] G. Maier and T. Hueckel. Nonassociated and coupled flow rules of elastoplasticity for rock-like materials. *International Journal of Rock Mechanics and Mining Sciences & Geomechanics Abstracts*, 16(2):77–92, 1979.
- [31] S. A. Silling and E. Askari. A meshfree method based on the peridynamic model of solid mechanics. *Computers & Structures*, 83(17-18):1526–1535, 2005.
- [32] Y. Azdoud, G. Lubineau, and F. Han. A morphing framework to couple non-local and local anisotropic continua. *International Journal of Solids and Structures*, 50(9):1332–1341, 2013.
- [33] P. T. Bauman. *Adaptive multiscale modeling of polymeric materials using goal-oriented error estimation, Arlequin coupling, and goals algorithms*. Ph.D. Thesis, The University of Texas at Austin, 2008.
- [34] X. Chen and M. Gunzburger. Continuous and discontinuous finite element methods for a peridynamics model of mechanics. *Computer Methods in Applied Mechanics and Engineering*, 200(9-12):1237–1250, 2011.

- [35] Y. Azdoud, F. Han, and G. Lubineau. The morphing method as a flexible tool for adaptive local/non-local simulation of static fracture. *Computational Mechanics*, 54(3):711–722, 2014.
- [36] Peter Grassl and Milan JirÅasek. Plastic model with non-local damage applied to concrete. *International Journal for Numerical and Analytical Methods in Geomechanics*, 30(1):71–90, 2006.
- [37] Caroline Le Bellégo, Jean François Dubé, Gilles Pijaudier-Cabot, and Bruno Gérard. Calibration of nonlocal damage model from size effect tests. *European Journal of Mechanics - A/Solids*, 22(1):33–46, 2003.
- [38] M. B. Noorumohamed, E. Schlangen, and J. G. M. Vanmier. Experimental and numerical study on the behavior of concrete subjected to biaxial tension and shear. *Advanced Cement Based Materials*, 1(1):22–37, 1993.
- [39] W. Liu and J. W. Hong. A coupling approach of discretized peridynamics with finite element method. *Computer Methods in Applied Mechanics and Engineering*, 245-246:163–75, 2012.