The Approximate Capacity Region of the Symmetric K -user Gaussian Interference Channel with Strong Interference

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Abstract—The symmetric K -user interference channel is studied with the goal of characterizing its capacity region in the strong interference regime within a constant gap. The achievable rate region of a scheme combining rate-splitting at the transmitters and interference alignment and successive decoding/computation at the receivers is derived. Next it is shown that this scheme achieves the so-called greedy-max corner points of the capacity region within a constant gap. By combining this result with previous results by Ordentlich *et al.* on the sum-capacity of the symmetric interference channel, a constant gap characterization of the capacity region for the strong interference regime is obtained. This leads to the first approximate characterization of the capacity region of the symmetric K -user IC. Furthermore, a new scheme that achieves the sum-capacity of the channel in the strong interference regime within a constant gap is also proposed, and the corresponding gap is calculated. The advantage of the new scheme is that it leads to a characterization within a constant gap without leaving an outage set contrary to the scheme by Ordentlich *et al.*

Index Terms—Interference channel; capacity region approximation; strong interference; interference alignment; lattice coding.

A WIRELESS network comprising several simultaneously communicating pairs of nodes is hindered by interference. This interference causes a degradation in the performance of the network, leading to decreased communication rates. This setback has motivated researchers to seek solutions to overcome this performance degradation. Common solutions to alleviate this performance degradation are mostly based on orthogonalization schemes. That is, the transmitters in the network agree on a policy that divides the network resources (time/frequency/space) among the communicating pairs so that receivers do not hear any interference.

In the 2-user interference channel (IC) [2] consisting of two communicating pairs of nodes, this orthogonalization scheme turns out to be optimal in terms of degrees-of-freedom (DoF), also known as multiplexing gain [3]. This is true since the 2-user IC has 1 DoF which can be achieved by time-division for example. Here, the DoF d measures the scaling (at high signal-to-noise ratio SNR) of the sum-capacity of the network in comparison to the capacity of a single point-to-point channel.

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This leads to an approximation of the sum-capacity of the network which approaches $d \log(\text{SNR})$ in the limit of high SNR.

However, orthogonalization schemes become suboptimal if we deviate slightly from this scenario by changing the sought performance metric or changing the number of communicating pairs for instance. In such cases, different approaches have to be used. For the 2-user IC for example, orthogonalization schemes do not achieve the generalized DoF (GDoF) of the network, where GDoF is an approximative metric of the sum-capacity which takes the interference-to-noise ratio INR into account¹. The GDoF of the IC is achieved by using a more involved scheme exploiting rate-splitting, and have been characterized in [4]. Furthermore, if we are interested in the capacity region of the 2-user IC besides the sum-capacity, then orthogonalization schemes become also suboptimal. In such case, the best known scheme is based on the Han-Kobayashi (HK) scheme [5], [6] which achieves rate regions very close to the capacity of the 2-user IC [7]. In fact, the HK scheme achieves the capacity region of the 2-user IC within 1 bit [4], and also its sum-capacity in the noisy interference regime² [8]–[10] where treating interference as noise becomes optimal.³ On the other hand, by extending the 2-user IC to a 3-user one (adding one more pair of users), not only orthogonalization schemes become suboptimal, but also the best known scheme for the 2-user IC (the HK scheme). In this case, interference alignment (IA) [15]–[17] becomes necessary.

Despite the major advances achieved recently in the field of network information theory, a capacity region characterization within a constant gap is still not available for the K -user IC. Nevertheless, some special cases have been studied in the literature. For instance, [18] characterized the capacity region of the K -user IC with very-strong interference, [19] characterized the capacity region of the 3-user IC with mixed strong/very-strong interference, and [20] characterized the capacity region for some classes of K -user IC's. In this paper, we pursue a capacity region characterization for the symmetric K -user IC within a constant gap. Before we highlight our contributions on this front, we put our work in its chronological context by previewing some of the available literature related to this

¹Contrary to the DoF which approximates the sum-capacity at high SNR by $d \log(\text{SNR})$ where d is a constant, GDoF leads to an approximation $d(\alpha) \log(\text{SNR})$ where $d(\alpha)$ is a function of $\alpha = \frac{\log(\text{INR})}{\log(\text{SNR})} \in [0, \infty)$ which captures the relative strength of interference.

²By considering sending private signals as a special case of the HK scheme.

³Treating interference as noise was also studied in [11]–[14].

subject.

A. Related work

It was noted that by increasing the number of users in an IC, new tools have to be used not only for achieving the capacity of the network, but even for achieving its DoF. The DoF achieving scheme for the K -user IC has been given in [21], where the optimal scheme is IA. This scheme is based on aligning interference into a subspace which can be isolated from the desired signal at the receivers. IA allows treating all $K - 1$ interferers in the K -user IC as one interferer, leading to the achievability of $1/2$ DoF per user just as in the 2-user case. From DoF point of view, it turns out that the aligned interference can be totally ignored (zero-forced).

Now if we seek the GDoF of the K -user IC, then the interference can not be simply ignored, and again new methods have to be used. Namely, in this case the aligned interference has to be decoded. An interference decoding framework for the symmetric K -user IC has been proposed in [22] to characterize the GDoF of this channel. Unfortunately, contrary to [4], a capacity characterization within a constant gap does not follow easily from [22] due to the high number of message splits required in some cases. In order to overcome this limitation, an alternative scheme based on nested-lattice codes [23] and computation [24] has been developed in [25], [26]. This scheme does not require rate-splitting, but is based on decoding two linearly independent linear combinations of two signals: the desired signal, and the effective interference signal (which consists of a superposition of all interferers). It turns out that this scheme achieves the sum-capacity of the symmetric K -user IC within a constant gap. A transmission scheme for the symmetric K -user IC based on the idea in [22] has also been proposed in [27] leading to a sum-capacity characterization within a constant gap. Besides constant gap sum-capacity characterizations, an exact sum-capacity characterization for the K -user IC with noisy interference has been obtained in [28], where treating interference as noise is optimal. Furthermore, a sum-capacity characterization has been obtained for classes of deterministic IC's in [29], [30], and for classes of Gaussian IC's in [31], [32].

The next step towards increasing our knowledge about the capacity of the K -user IC is a *capacity region* approximation/characterization. Our knowledge about this aspect is still very limited at this point (contrary to the 2-user case). One of the works in this direction is the capacity region characterization of the symmetric K -user IC in the very strong interference regime [33] corresponding to $\text{INR} \geq \text{SNR}^2$. In this characterization, lattice codes and IA were used. For regimes where interference can not be classified as very-strong, neither a capacity region characterization nor a capacity region approximation within a constant gap is available. This calls upon finding transmission schemes that can lead to such an approximate characterization, which helps towards approaching a capacity characterization for the K -user IC. In this paper, we answer this call by characterizing the capacity region of the K -user symmetric IC with strong interference $\text{SNR} < \text{INR} < \text{SNR}^2$ within a constant gap. Next, we summarize our contributions in this paper.

B. Contributions

For the purpose of characterizing the approximate capacity region of the symmetric K -user IC with strong interference, we focus on the moderate/high SNR regime. In this regime, the network is interference-limited (in contrast to noise-limited). Thus, this regime is the suitable regime to study the impact of interference in isolation from the impact of noise. A constant gap characterization in this regime is important since the constant gap becomes negligible in comparison to the capacity in the limit of high SNR.

In order to realize this characterization for the strong interference regime, we exploit the fact that the capacity outer bound obtained from the genie-aided MAC upper bound [21] has a polytope structure. Hence, using time sharing arguments, achieving its corner points within a constant gap suffices for showing the achievability of the outer bound within a constant gap. Then, we show that these corner points are indeed achievable within a constant gap for the symmetric 3-user IC⁴ with strong interference. Finally, we extend the results from the 3-user to the K -user case. To this end, our approach for proving the achievability of the outer bound within a constant gap for the 3-user case is discussed in the next paragraphs.

First, we classify the corner points of the outer bound into three classes:

- 1) **Trivial** corner points with at least one zero component,
- 2) **Greedy-max** corner points where one user is greedy, i.e., transmits at capacity, and the remaining users maximize their rate subject to this greedy constraint, and
- 3) **Sum-capacity** corner point where the sum-rate is maximized.

The first class is trivial. Namely, these corner points can be achieved by using Gaussian codes at two users and switching the third user off, and decoding both the desired signal and the interfering signal at the receivers [34], [35]. Furthermore, the achievability within a constant gap of the third class (sum-capacity corner point) was shown in [26]. Thus, it remains to consider the greedy-max corner points.

For achieving the greedy-max corner points within a constant gap, we develop a transmission scheme which combines rate-splitting [2], nested-lattice codes [23], [24], and naturally interference alignment (IA) [16], [36]. In our setting, IA is performed on the code level, i.e., by using nested-lattice code alignment as in [22], [27]. This type of IA has been used earlier to develop transmission schemes for the IC in [18], [33]. In contrast to these works where only one mode of interference alignment was needed, in our case, three modes of IA are required. These modes can be described as follows:

- II) Aligning interference to interference (hence II) similar to [33],
- DI) Aligning a desired signal to one interference signal (hence DI), and
- DII) Aligning a desired signal to two interference signals (hence DII).

⁴For readability, we start with the 3-user case, and extend results later to the K -user case.

It turns out that all these modes are required for an approximate capacity region characterization in the strong interference regime. To show the optimality of the proposed scheme within a constant gap, we perform the following steps. We divide the strong interference regime into three cases depending on the strength of interference. Namely, we divide the strong interference (SI) regime defined by the regime where $\text{SNR} < \text{INR} < \text{SNR}^2$ into the SI1 regime defined by $\text{SNR} < \text{INR} \leq \text{SNR}^{4/3}$, the SI2 regime defined by $\text{SNR}^{4/3} < \text{INR} \leq \text{SNR}^{3/2}$, and the SI3 regime defined by the remaining part of the SI regime. Then, we develop a transmission scheme which combines the three IA modes using 6 message splits at the greedy transmitter and 3 message splits at the remaining ones. Next, we show that our scheme achieves the corner points of the capacity region within a gap of $3/2$, 3 , and 2 bits per user in the regimes SI1, SI2, and SI3, respectively.

This achievability within a constant gap of the greedy-max corner points, combined with the achievability of the trivial corner points [34], and the achievability within a constant gap of the sum-capacity corner point [26], implies the achievability of the outer bound within a constant gap (by time sharing between corner points). This leads to a capacity region characterization for the symmetric 3-user IC within a constant gap in the strong interference regime. The extension to the K -user case is based on similar methods with minor adjustments.

Although the sum-capacity corner point has been shown to be achievable within a constant gap in [26], this characterization holds for the whole SI regime except for an outage set. Namely, Ordentlich *et al.* have shown in [26] that the sum-capacity in the SI regime is achievable within a gap of $4 + \frac{c}{2^c}$ bits per user ($c > 0$) for all channels with $\text{SNR} < \text{INR} < \text{SNR}^2$ except for an outage set whose size is a fraction 2^{-c} of this interval. Note that in order to reduce the size of this outage set, one has to increase c , leading to a large gap. In order to obtain a constant gap characterization which holds for the whole SI regime, we propose an alternative scheme based on rate-splitting, nested-lattice codes, interference alignment, and successive decoding/computation combined with interference cancellation. The proposed scheme achieves the sum-capacity point of the symmetric K -user IC within a gap of $\frac{1}{4} + (\kappa + \frac{1}{2}) \log(2K)$ bits per user where κ is given by $\left\lfloor \frac{\log(\text{INR})}{2(\log(\text{INR}) - \log(\text{SNR}))} \right\rfloor$. Note that since $\text{SNR} < \text{INR} < \text{SNR}^2$ in the SI regime, we can write $\text{INR} = \text{SNR}^{1+s}$ where $s \in (0, 1]$, thus, $\kappa = \left\lfloor \frac{1}{2} + \frac{1}{2s} \right\rfloor$. The proposed scheme does not require coding over two transmission blocks contrary to the scheme in [27], and it leads to a smaller gap than the one obtained in [27]. Furthermore, the gap holds for the whole SI regime without leaving an outage set contrary to [26]. The drawback of this characterization is that the gap becomes larger as INR approaches SNR which is the boundary of the SI regime.

C. Paper organization and notation

We organize the paper as follows. The system model of the symmetric K -user IC is defined in Section I, then, the main results of the paper are summarized in Section II. A capacity outer bound is given in Section III. Then, we

Fig. 1. The symmetric 3-user interference channel.

focus on the 3-user IC. We describe the alignment modes in Section IV for the linear-deterministic symmetric 3-user IC, and provide its strong interference capacity region. Next, in Section V, the capacity region of the Gaussian symmetric 3-user IC is characterized within a constant gap. An alternative transmission scheme for achieving the sum-capacity corner point of the symmetric 3-user IC within a constant gap is provided in Section VI. Finally, the results are extended to the symmetric K -user IC in Section VII, and the paper is concluded in Section VIII.

Throughout the paper, we denote scalars, vectors, and matrices by lower-case normal (x), lower-case bold ($\mathbf{x}$), and upper-case bold ($\mathbf{X}$) letters, respectively. The Gaussian capacity function $C(x)$ is used to denote $\frac{1}{2} \log(1 + x)$, and $C^+(x)$ is used to denote $\max\{0, C(x)\}$. We use $\mathbf{0}_q$ to denote the zero-vector of length q .

I. SYSTEM MODEL

Consider a symmetric Gaussian IC with $K \in \mathbb{N}$ users with the following input-output equations (see Figure 1 for a 3-user example)

$$\mathbf{y}_j = \mathbf{x}_j + \sum_{i \neq j} h \mathbf{x}_i + \mathbf{z}_j, \quad i, j \in \{1, \dots, K\},$$

where $h \in \mathbb{R}$ is the interference channel coefficient, $\mathbf{x}_j$ and $\mathbf{y}_j$ are the transmit and received signals (codewords of length N) of transmitter j and receiver j , respectively, and $\mathbf{z}_j$ is a Gaussian noise vector with independent and identically distributed components of zero mean and variance $\sigma^2 = 1$. The transmit signals have a power constraint P so that

$$\mathbb{E}[\|\mathbf{x}_j\|^2] \leq NP.$$

Transmitter j wants to send a message M_j which is a random variable uniformly distributed over the message set $\mathcal{M}_j = \{1, \dots, \lfloor 2^{NR_j} \rfloor\}$ to receiver j by using a code of length N channel uses. Thus, it encodes M_j into $\mathbf{x}_j \in \mathbb{R}^N$ and sends $\mathbf{x}_j$. After N channel observations, the receiver obtains $\mathbf{y}_j$ from which it decodes $\hat{M}_j$. An error event occurs if $M_j \neq \hat{M}_j$ for some $j \in \{1, \dots, K\}$, and has a probability $\mathbb{P}_e^{(N)}$. An $(N, R_1, \dots, R_K)$ code for the IC denotes the collection of the messages sets, the encoders, and the decoders, and induces an error probability $\mathbb{P}_e^{(N)}$. A rate tuple $\mathbf{R} = (R_1, \dots, R_K)$ is said

to be achievable if there exist a sequence of codes such that $\mathbb{P}_e^{(N)} \rightarrow 0$ as $N \rightarrow \infty$. The capacity region $\mathcal{C}$ is defined as the set of all achievable rate tuples, and the sum-capacity C_Σ is the highest achievable sum-rate $R_\Sigma = \sum_{j=1}^K R_j$. Clearly, $\mathcal{C}$ and C_Σ depend on the signal-to-noise ratio $\text{SNR} = \frac{P}{\sigma^2}$ and the interference-to-noise ratio $\text{INR} = \frac{h^2 P}{\sigma^2}$ of the channel. We do not write this dependence explicitly in the sequel for clarity.

The degrees-of-freedom (DoF) of the channel is the ratio of the sum-capacity C_Σ to $\frac{1}{2} \log(\text{SNR})$ in the limit of high signal-to-noise ratio SNR , defined as [21]

$$d_\Sigma = \lim_{\text{SNR} \rightarrow \infty} \frac{C_\Sigma}{\frac{1}{2} \log(\text{SNR})}. \quad (1)$$

The generalized DoF (GDoF) of the channel is defined as the same ratio measured at a given α defined as $\alpha = \frac{\text{INR}}{\text{SNR}}$ [4], i.e.,

$$d_\Sigma(\alpha) = \lim_{\text{SNR} \rightarrow \infty} \frac{C_\Sigma}{\frac{1}{2} \log(\text{SNR})}, \quad \text{such that } \alpha = \frac{\text{INR}}{\text{SNR}}. \quad (2)$$

In this paper we focus on the strong interference regime of the IC. The strong interference (SI) regime is defined as the regime where $\text{SNR} < \text{INR} < \text{SNR}^2$. That is, if the IC has $1 < h^2 < P$, then it is said to have strong interference. If $P \leq h^2$, then we say that the channel has very strong interference (VSI) [26]. The following section summarizes the main results of the paper.

II. MAIN RESULTS

The main results of the paper involve characterizing the capacity region and the sum-capacity of the symmetric K -user IC with SI within a constant gap. The following theorem states the first result regarding the capacity region of the channel with SI.

Theorem 1. *The capacity region of the symmetric K -user Gaussian IC with $h^2 > 1$ (SI and VSI regimes) is within a constant gap (for P sufficiently large) of the outer bound*

$$\bar{\mathcal{C}} = \left\{ \mathbf{R} \in \mathbb{R}_+^K \left| \begin{array}{l} R_j \leq C(P), \\ R_i + R_j \leq C(P + h^2 P), \\ i, j \in \{1, \dots, K\}, i \neq j \end{array} \right. \right\}. \quad (3)$$

This theorem states that the capacity outer bound $\bar{\mathcal{C}}$ is in fact achievable within a constant gap. The converse of this theorem is based on the genie-aided MAC bound [21], and is proved in Section III. The achievability within a constant gap is first shown for the symmetric 3-user IC in Section V. Although the analysis of the 3-user case is simpler than the K -user case, it still preserves the essence of the problem. Finally, the achievability is extended to the K -user case in Section VII-A.

The second result regarding the sum-capacity of the channel with SI is stated in the following theorem.

Theorem 2. *The sum-capacity of the symmetric K -user Gaussian IC with $h^2 \in (1, P)$ (SI regime) is bounded by*

$$\frac{K}{2} C(P + h^2 P) - \frac{K}{2} \Delta \leq C_\Sigma \leq \frac{K}{2} C(P + h^2 P) \quad (4)$$

for P sufficiently high, where $\Delta = \frac{1}{2} + (2\kappa + 1) \log(2K)$ and $\kappa = \left\lfloor \frac{\log(h^2 P)}{2 \log(h^2)} \right\rfloor$.

The converse of this theorem is also shown in Section III. Similar to Theorem 1, the achievability is first shown for the 3-user case in Section VI, and later extended to the K -user case in Section VII-B. Note that the main difference between Theorem 2 and the sum-capacity characterization within a constant gap given in [26] is that in our case, we do not have an outage set, contrary to [26]. Further, the characterization in Theorem 2 is tighter than the one given in [27].

In the following section, we give a review of the upper bounds that lead to an outer bound on $\mathcal{C}$ for the symmetric K -user IC with strong interference.

III. A CAPACITY OUTER BOUND

An upper bound on the achievable rates in the K -user IC which is tight from a DoF point of view is the genie-aided MAC (multiple-access channel) bound [21], which is also tight from GDoF point of view in the SI regime [22]. This bound is derived by giving the interference caused by $K - 2$ transmitters to one of the receivers as side information. Say we give $(\mathbf{x}_3, \dots, \mathbf{x}_K)$ to receiver 1 as side information. Then, this receiver is able to remove the interference from transmitters 3 to K completely, and an upper bound on $R_1 + R_2$ can be derived as in [34]. In more detail, given $(\mathbf{x}_3, \dots, \mathbf{x}_K)$, receiver 1 can construct

$$\tilde{\mathbf{y}}_1 = \mathbf{x}_1 + h\mathbf{x}_2 + \mathbf{z}_1. \quad (5)$$

Now, given that receiver 1 is able to decode $\mathbf{x}_1$, it can construct

$$\tilde{\mathbf{y}}_2 = \frac{1}{h}(\tilde{\mathbf{y}}_1 - \mathbf{x}_1) + h \sum_{j \neq 2} \mathbf{x}_j = \mathbf{x}_2 + h \sum_{j \neq 2} \mathbf{x}_j + \frac{1}{h} \mathbf{z}_1, \quad (6)$$

where $j \in \{1, \dots, K\}$, which is less noisy version of $\mathbf{y}_2$ if $h^2 > 1$. If receiver 2 can decode $\mathbf{x}_2$ from $\mathbf{y}_2$, then so does receiver 1 from the less noisy $\tilde{\mathbf{y}}_2$. Therefore, a code for the IC with SI is also a code for the MAC between transmitters 1 and 2 and receiver 1 enhanced with $(\mathbf{x}_3, \dots, \mathbf{x}_K)$. Thus, the capacity region of the resulting MAC is an outer bound on any rate pair (R_1, R_2) achievable in the K -user IC. As a result, the following upper bounds must be satisfied by any achievable rate triple in the K -user IC with strong interference:

$$R_1 \leq C(P) \quad (7)$$

$$R_2 \leq C(h^2 P) \quad (8)$$

$$R_1 + R_2 \leq C(P + h^2 P). \quad (9)$$

Similarly, one can obtain similar bounds by considering the other receiver and side information combinations, leading to the following lemma.

Lemma 1. *The capacity region $\mathcal{C}$ of the symmetric Gaussian K -user IC with $h^2 > 1$ is outer bounded by $\bar{\mathcal{C}}$ given by*

$$\bar{\mathcal{C}} = \left\{ \mathbf{R} \in \mathbb{R}_+^K \left| \begin{array}{l} R_j \leq C(P), \\ R_i + R_j \leq C(P + h^2 P), \\ i, j \in \{1, \dots, K\}, i \neq j \end{array} \right. \right\}. \quad (10)$$

Proof: The bounds follow from the genie-aided MAC bounds applied to different receivers with $K - 2$ interference signals given to the receiver as side information. ■

This lemma proves the converse of Theorem 1. Moreover, from $\bar{\mathcal{C}}$, the following sum-capacity upper bound can be obtained

$$C_{\Sigma} \leq \frac{K}{2} C(P + h^2 P), \quad (11)$$

which proves the converse of Theorem 2. Namely, this can be obtained by summing up the bounds on $R_i + R_j$ in (10) for different combinations of distinct $i, j \in \{1, \dots, K\}$. This sum-capacity upper bound provides the optimal DoF of the channel given by $K/2$ [21] in addition to its GDoF in the SI regime [22]. Furthermore, it was shown that this bound also provides the sum-capacity of the channel within a constant gap in the SI regime [26] (except for an outage set), and that it provides the capacity region in the VSI regime [33]. As we show in the sequel, in the SI regime, the whole region $\bar{\mathcal{C}}$ is achievable within a constant gap, which results in an approximate characterization of the capacity region of the channel with strong interference. We also show that the upper bound on C_{Σ} is achievable within a constant gap without an outage set as in [26] by using an alternative transmission scheme.

Next, we focus on the achievability of Theorem 1. We use a progressive approach starting with the linear-deterministic (LD) IC, and extend the obtained insights to the Gaussian case later on. We also start with the 3-user case where we illustrate the main idea, and extend the results to the K -user case afterwards (Section VII-A). In the next section, we characterize the capacity region of the symmetric 3-user LD IC in the strong interference regime.

IV. CAPACITY OF THE SYMMETRIC 3-USER LD IC WITH STRONG INTERFERENCE

The linear-deterministic model is a high-SNR approximative model for Gaussian networks introduced by Avestimehr *et al.* in [37]. This LD approximation was applied to the 2-user IC by Bresler *et al.* in [38]. Here, we apply it to the symmetric 3-user IC and characterize the capacity region of the resulting LD IC in the SI regime.

We can write the input-output equations of the symmetric 3-user LD IC with SI as follows

$$\mathbf{y}_j = \mathbf{S}^{n-m} \mathbf{x}_j \oplus \sum_{i \neq j} \mathbf{x}_i, \quad i, j \in \{1, 2, 3\}, \quad (12)$$

where $\mathbf{x}_j$ and $\mathbf{y}_j$ are binary vectors of length $n > m$ denoting the transmit and received signals of j th transmitter and receiver, respectively. The integers m and n denote the strength of the direct and the interference channels, respectively, and can be obtained from the Gaussian IC as

$$m = \left\lceil \frac{1}{2} \log(P) \right\rceil \quad \text{and} \quad n = \left\lceil \frac{1}{2} \log(h^2 P) \right\rceil. \quad (13)$$

Here, $\mathbf{S}$ is a down-shift matrix, and additions are modulo-2. A sample symmetric 3-user LD IC is shown in Figure 2. The following theorem presents the main result of this section.

Fig. 2. A symmetric 3-user LD IC with $n = 3$ and $m = 2$. The circles denotes components (bits) of the input and output vectors $\mathbf{x}_j$ and $\mathbf{y}_j$, and the lines denote noiseless bit-pipes. Note that each receiver sees more interference bits than desired bits, which implies that the channel has strong interference.

Theorem 3. The capacity region $\mathcal{C}_d$ of the symmetric 3-user LD IC with $n > m$ is given by

$$\mathcal{C}_d = \left\{ (R_1, R_2, R_3) \in \mathbb{R}_+^3 \mid \begin{array}{l} R_j \leq m \\ R_i + R_j \leq n \\ i, j \in \{1, 2, 3\}, i \neq j \end{array} \right\}.$$

This theorem indicates that the genie-aided MAC bounds suffice to characterize the capacity region of the symmetric 3-user LD IC with SI. This is a similar observation as for the 2-user IC where the MAC bounds provide the capacity region of the IC with SI both in the LD case [38] and in the Gaussian case [34]. The proof of this theorem is given in the following subsections.

A. Converse

Let us express the outer bound of Lemma 1 for the LD case. Using similar arguments as in Section III, given $\mathbf{x}_3$, receiver 1 can decode both $\mathbf{x}_1$ and $\mathbf{x}_2$ if interference is strong, i.e., $n > m$. Thus, a code for the LD IC in this case is also a code for the LD MAC formed by transmitters 1 and 2 and receiver 1, given $\mathbf{x}_3$. The capacity region of this LD MAC is given by [37]

$$R_1 \leq m, \quad (14)$$

$$R_1 + R_2 \leq n. \quad (15)$$

Therefore, any achievable rate in this IC must satisfy those two rate constraints (an alternative derivation is given in Appendix A). We can derive similar bounds by considering different receivers and side information, which leads to

$$R_j \leq m, \quad (16)$$

$$R_i + R_j \leq n, \quad (17)$$

for $i \neq j$. This coincides with $\mathcal{C}_d$ and ends the proof of the converse of Theorem 3. Next we show the achievability of this rate region.

B. Achievability

Notice that the region $\mathcal{C}_d$ given in Theorem 3 is a polytope. Hence, proving the achievability its corners suffices for proving the achievability of the whole region. Let us start by characterizing the corner points of $\mathcal{C}_d$.

Fig. 3. The capacity region of the symmetric 3-user LD IC with strong and very strong interference. The vertical and horizontal planes correspond to the single user bounds, while the slanted ones correspond to the genie-aided MAC bounds.

1) *Corner points of $\mathcal{C}_d$* : The general structure of $\mathcal{C}_d$ is shown in Figure 3 for the SI regime ($m < n < 2m$) and the very SI (VSI) regime ($2m \leq n$). We divide the corner points of $\mathcal{C}_d$ into three classes:

- **Trivial** corner points are the corner points of $\mathcal{C}_d$ that have at least one zero component. These points are denoted by T in Figures 3(a) and 3(b), where the subscript of T indicates the non-zero rates. The origin O can also be considered as a trivial corner point.
- **Greedy-max** corner points are the corner points where one user is greedy, i.e., transmitting at his capacity ($R_i = m$), and the other users maximize their rates subject to this greedy constraint ($R_j = n - m$). These points are denoted by G in Figure 3(a), where the subscript of G indicates the greedy user.
- **Sum-capacity** corner point is the point S_1 in Figure 3(a) whose coordinates are $\frac{1}{2}(n, n, n)$, and the point S_2 in 3(b) whose coordinates are (m, m, m) . At these points, the sum-capacity is maximum.

Remark 1. The sum-capacity point S_2 can also be interpreted as a Greedy-max corner point.

The first class of corner points is trivial because it describes cases where at least one user is inactive. This transforms the 3-user IC into a 2-user IC with SI/VSI or a point-to-point channel, for which the capacity is known [38]. Thus, the trivial corner points are clearly achievable. The focus here is on the non-trivial corner points. Next we consider the points G_i which exist if $m < n < 2m$.

2) *Greedy-max points*: Note that the rate bounds in Theorem 3 are the same as the optimal bounds of the 2-user LD IC with strong interference⁵ [38]. That is, we only have single user bounds, and pair-wise bounds. This indicates that the two interferers in the 3-user IC should be treated as one single interferer. This conclusion has been exploited earlier in [21], [22], [39] where interference alignment (IA) has been used to ‘steer’ the interference in a way that basically gives the receiver the illusion that only one interferer exists. This is the main idea that will be used in the transmission schemes that will follow. The main new ingredient is the new alignment

⁵This statement is also true for the rate bounds in Lemma 1 which coincide with the optimal bounds of the 2-user Gaussian IC with strong interference [34], [35]

modes required for achieving the greedy-max corner points, as we shall see.

Recall that the greedy-max corner points (G_i in Figure 3(a)) correspond to maximizing the rates of two users subject to a greedy third user which transmits at a rate equal to the capacity of his desired channel. Due to symmetry, we focus on the achievability of the point $G_1 = (m, n - m, n - m)$; the other points can be achieved similarly.

The transmission scheme required for achieving the greedy-max points varies between three different regimes:

- SI1) $m < n \leq \frac{4m}{3}$,
- SI2) $\frac{4m}{3} < n \leq \frac{3m}{2}$, and
- SI3) $\frac{3m}{2} < n < 2m$.

The main ingredients of the transmission schemes in these regimes are rate-splitting, IA, computation, and interference cancellation. The number of splits and the alignment modes vary between these three regimes.

In general, $\mathbf{x}_1$ consists of 6 splits, and $\mathbf{x}_2$ and $\mathbf{x}_3$ consist of 3 splits each (plus 2 copy splits). We will use a general numbering of the splits which indicates their decoding order. In the cases where some splits are not required, these splits will be removed.

Three modes of alignment will be used in the following schemes. These modes are denoted

- II) which corresponds to aligning two interference signals (interference-interference),
- DI) which corresponds to aligning a desired signal with one interference signal (desired-interference), and
- DII) which corresponds to aligning a desired signal with both interferers (desired-interference-interference).

Next, let us study the regime SI1.

a) *SI1*: A graphical illustration of the splitting scheme in the SI1 regime is shown in Figure 4. Let us start analysing this sketch starting from receiver 2 (Figure 4(b)). Since the greedy transmitter 1 requires a rate of m bits, transmitter 2 can only send $n - m$ bits interference free on the lower-most positions of $\mathbf{x}_2$, indicated by $\hat{\mathbf{u}}_{21}$. Similarly transmitter 3 sends $\hat{\mathbf{u}}_{31}$ on the same positions. This causes interference at receiver 1 (Figure 4(a)), and hence, to cancel this interference, transmitters 2 and 3 send copy bits⁶ (copies of $\hat{\mathbf{u}}_{21}$ and $\hat{\mathbf{u}}_{31}$) on the top-most positions of $\mathbf{x}_2$ and $\mathbf{x}_3$, denoted $\mathbf{u}_{21}$ and $\mathbf{u}_{31}$, respectively.⁷ At this point, receiver 1 can decode the aggregate interference $\mathbf{u}_{21} \oplus \mathbf{u}_{31}$, then subtract $\hat{\mathbf{u}}_{21} \oplus \hat{\mathbf{u}}_{31}$ (as in [22]), and finally, decode all m bits of $\mathbf{x}_1$. Receiver 2 can decode its desired signal directly from the lower-most positions of $\mathbf{y}_2$. However, in order to make the correspondence between this scheme for the LD IC and the one for the Gaussian IC (to be explained later in Section V) clear, we use a successive decoding approach where the receivers have to decode from top to bottom. Thus receiver 2 decodes the aggregate interference $\mathbf{u}_{11} \oplus \mathbf{u}_{31}$, then the sum of the interference $\mathbf{u}_{14}$ from transmitter 1 and the desired signal $\mathbf{u}_{21}$ from transmitter 2,⁸ then the sum $\mathbf{u}_{16} \oplus \hat{\mathbf{u}}_{31}$, and finally, the

⁶Using the terminology of [22], a copy bit is a repetition of a bit at a different component of the transmit signal vector $\mathbf{x}_i$.

⁷Note that if $\hat{\mathbf{u}}$ is a copy of $\mathbf{u}$, then $\mathbf{u}$ is also a copy of $\hat{\mathbf{u}}$. We use this wording interchangeably throughout the paper.

⁸with $\mathbf{0}_{4m-3n}$ appended to $\mathbf{u}_{21}$ to make the two vectors equal in length.

Fig. 4. Received signals at receivers 1 and 2 in the SI1 regime. Receiver 1 decodes $\mathbf{u}_{21} \oplus \mathbf{u}_{31}$ first, then it removes the contribution of $\hat{\mathbf{u}}_{21} \oplus \hat{\mathbf{u}}_{31}$, and then it decodes $\mathbf{x}_1$. Receiver 2 decodes all bits of $\mathbf{y}_2$ from top to bottom thus recovering $\hat{\mathbf{u}}_{21}$.

desired signal $\hat{\mathbf{u}}_{21}$. Thus, receiver 2 obtains the desired $n - m$ bits.

Note that in this regime, the alignment scheme requires using three splits for $\mathbf{x}_1$, and one split for $\mathbf{x}_2$ and $\mathbf{x}_3$ (plus one copy split), with different treatment for different splits at the receivers. Moreover, we see two modes of alignment in this regime, namely, modes II corresponding to the alignment of the pairs $(\mathbf{u}_{21}, \mathbf{u}_{31})$, $(\mathbf{u}_{11}, \mathbf{u}_{31})$, and $(\mathbf{u}_{16}, \hat{\mathbf{u}}_{31})$, and DI corresponding to the alignment of $\mathbf{u}_{21}$ and the top-most $n - m$ bits of $\mathbf{u}_{14}$. Notice that since by the end of the decoding procedure, each receiver obtains its desired signal, then receiver 2 (and similarly receiver 3) can calculate $\mathbf{u}_{14}$, and thus, $\mathbf{u}_{14}$ can be interpreted as a common signal. In conclusion, user 1 achieves m bits, and users 2 and 3 achieve $n - m$ bits each, which achieves the greedy-max point G_1 .

b) *SI2*: In the SI2 regime, splitting gets more involved as shown in Figure 5. Once again transmitter 2 sends its bits on the lower-most $n - m$ positions of $\mathbf{x}_2$ (Figure 5(a)). Then, in order to allow interference cancellation at receiver 1, these bits are repeated on the top-most positions of $\mathbf{x}_2$, and similarly at transmitter 3. This allows receiver 1 to recover all m bits from transmitter 1. Let us examine receiver 2 (Figure 5(b)). This construction leaves a gap of $3m - 2n$ zeros in $\mathbf{x}_2$ and $\mathbf{x}_3$ in between the two copies of the signals as shown in the figure. These are the $3m - 2n$ positions where the DI alignment mode at receivers 2 and 3 can be used. Therefore, transmitter 1 aligns a split of its signal, denoted $\mathbf{u}_{14}$ with the top-most $3m - 2n$ bits of $\mathbf{x}_2$. For this reason, we have to split the two copies of the signal in $\mathbf{x}_2$ into two splits, one of length $3m - 2n$ denoted $\mathbf{u}_{21}$, and the other of length $3m - 4n$ denoted $\mathbf{u}_{22}$. The copies are denoted $\hat{\mathbf{u}}_{21}$ and $\hat{\mathbf{u}}_{22}$. Similar splitting is used by transmitter 3. Now in order to match this splitting, transmitter 1 splits the remaining bits of $\mathbf{x}_1$ into $\mathbf{u}_{11}$, $\mathbf{u}_{12}$, $\mathbf{u}_{15}$, and $\mathbf{u}_{16}$ of lengths $3m - 2n$, $3n - 4m$, $3n - 4m$, and $3m - 2n$, to align with $\mathbf{u}_{31}$, $\mathbf{u}_{32}$, $\hat{\mathbf{u}}_{32} \oplus \mathbf{u}_{22}$, and $\hat{\mathbf{u}}_{31}$ at receiver 2, respectively (Figure 5(b)).

In this regime (SI2), we use the third alignment mode, i.e., the DII mode, in addition to II and DI. This mode is used in the alignment of $\mathbf{u}_{22}$, $\mathbf{u}_{15}$, and $\hat{\mathbf{u}}_{32}$ at receiver 2 as shown in Figure 5(b). Moreover, we used 5 splits for $\mathbf{x}_1$ and 2 splits (plus two copy splits) for $\mathbf{x}_2$ and $\mathbf{x}_3$. Using this splitting and alignment procedure, all receivers are able to obtain their desired signals, which proves the achievability of G_1 in this regime.

Fig. 5. Received signals at receivers 1 and 2 in the SI2 regime. Receiver 1 decodes $\mathbf{u}_{21} \oplus \mathbf{u}_{31}$ and $\mathbf{u}_{22} \oplus \mathbf{u}_{32}$ first, then it removes the copies $\hat{\mathbf{u}}_{21} \oplus \hat{\mathbf{u}}_{31}$ and $\hat{\mathbf{u}}_{22} \oplus \hat{\mathbf{u}}_{32}$, and finally it decodes $\mathbf{x}_1$. Receiver 2 decodes all bits of $\mathbf{y}_2$ from top to bottom thus recovering $\hat{\mathbf{u}}_{21}$ and $\hat{\mathbf{u}}_{22}$.

Fig. 6. Received signals at receivers 1 and 2 in the SI3 regime. Receiver 1 decodes $\mathbf{u}_{22} \oplus \mathbf{u}_{32}$ and $\mathbf{u}_{23} \oplus \mathbf{u}_{33}$, then it removes the contribution of $\hat{\mathbf{u}}_{22} \oplus \hat{\mathbf{u}}_{32}$, and finally it decodes $\mathbf{x}_1$. Receiver 2 decodes all bits of $\mathbf{y}_2$ from top to bottom thus recovering $\mathbf{u}_{23}$ and $\hat{\mathbf{u}}_{22}$.

c) *SI3*: Finally, in the SI3 regime, transmitter 2 sends its $n - m$ bits on the lower-most positions of $\mathbf{x}_2$. Only the lower-most $2m - n$ positions of $\mathbf{x}_2$ interfere with $\mathbf{x}_1$ as shown in Figure 6(a). Thus, transmitter 2 repeats the lower-most $2m - n$ bits of $\mathbf{x}_2$ denoted $\hat{\mathbf{u}}_{22}$ on the top-most $2m - n$ bits of $\mathbf{x}_2$ denoted $\mathbf{u}_{22}$. The remaining $2n - 3m$ positions are occupied by another split $\mathbf{u}_{23}$. Note that this latter split is not copied since it does not cause any interference at receiver 1. Similar splitting is used by transmitter 3. Transmitter 1 splits its signal to match $\mathbf{x}_3$ at receiver 2 (Figure 6(b)) into $\mathbf{u}_{12}$, $\mathbf{u}_{13}$, and $\mathbf{u}_{15}$ of lengths $2m - n$, $2n - 3m$, and $2m - n$, respectively. Receiver 1 decodes the aggregate interference, removes the copy bits, and recovers m bits of $\mathbf{x}_1$. Receiver 2 decodes all bits from top to bottom, recovering $n - m$ bits of $\mathbf{x}_2$. Receiver 3 proceeds similarly. This achieves the desired corner point in this regime.

This ends our proof of the achievability of the greedy-max corner points in the SI regime. It remains to show the achievability of the sum-capacity corner points S_1 and S_2 . These corner points can be achieved by using the alignment scheme of Jafar and Vishwanath in [22]. We repeat their scheme here for completeness.

3) *Sum-capacity point*: To achieve the sum-capacity corner points S_1 and S_2 , each transmitter sends a binary vector structured as follows

$$\mathbf{x}_j^T = [\mathbf{u}_{j1}^T \quad \mathbf{u}_{j2}^T \quad \hat{\mathbf{u}}_{j1}^T \quad \mathbf{0}_{n-m}^T]^T, \quad (18)$$

Fig. 7. A pictorial representation of $\mathbf{y}_1$ for a symmetric LD IC. Each rectangle designates a chunk of bits of the transmit signals.

where $\mathbf{u}_{j1}$ and $\mathbf{u}_{j2}$ are binary vectors of length⁹ $k = \max\{0, \frac{2m-n}{2}\}$ and $\min\{n-m, m\}$, respectively, and $\hat{\mathbf{u}}_{j1}$ is a flipped (top-bottom) copy of $\mathbf{u}_{j1}$, i.e.,

$$\hat{\mathbf{u}}_{j1}^T = [u_{j1}(k) \ \cdots \ u_{j1}(2) \ u_{j1}(1)]^T, \quad (19)$$

and $u_{j1}(\ell)$ is the ℓ -th component of $\mathbf{u}_{j1}$. The received signal at receiver 1 for instance has the structure shown in Figure 7.

Receiver 1 starts by decoding the interference-free desired bits from the bottom of $\mathbf{y}_1$ (bits of $\hat{\mathbf{u}}_{11}$), and then removing the copies from the top ($\mathbf{u}_{11}$). Then it decodes the aggregate interference from the top of $\mathbf{y}_1$ (bits of $\mathbf{u}_{21} \oplus \mathbf{u}_{31}$), and it removes their contribution from the bottom ($\hat{\mathbf{u}}_{21} \oplus \hat{\mathbf{u}}_{31}$). It proceeds in this ‘peeling’ procedure until all bits of $\mathbf{u}_{11}$ and $\mathbf{u}_{21} \oplus \mathbf{u}_{31}$ are decoded. Finally, it decodes both $\mathbf{u}_{12}$ and $\mathbf{u}_{22} \oplus \mathbf{u}_{32}$. Similar decoding is used by receivers 2 and 3. Note that the decoding outcome is analogous to that obtained in the 2-user IC with SI and VSI [34], [35], [38] with the single difference that the decoded interference comprises the sum of two interfering signals.

This scheme achieves $k + n - m = \frac{n}{2}$ bits per user if $m < n < 2m$, and m bits per user if $2m \leq n$, hence achieving the sum-capacity corner points S_1 and S_2 in Figure 3.

At this point, the achievability of all corner points of $\mathcal{C}_d$ is shown. Using time-sharing between the corner points, any point in $\mathcal{C}_d$ can be achieved, which proves the achievability of Theorem 3. Now that the splitting of the signals and the alignment modes have been explained for the symmetric 3-user LD IC, we switch to the approximate capacity region characterization of the Gaussian case.

V. APPROXIMATE CAPACITY REGION OF THE SYMMETRIC GAUSSIAN 3-USER IC WITH STRONG INTERFERENCE

The goal of this section is to characterize the capacity region $\mathcal{C}$ of the symmetric Gaussian 3-user IC within a constant gap in the strong interference regime. The main result of this section is summarized in the following theorem.

Theorem 4. *The capacity region $\mathcal{C}$ of the symmetric Gaussian 3-user IC with $h^2 > 1$ (SI and VSI regimes) is within a*

⁹We assume $2m - n$ is divisible by 2. If not, then we consider two channel uses. Two instances of a deterministic IC with given $n = a$ and $m = b$, $a, b \in \mathbb{N}$, can be modeled as an IC with $n = 2a$ and $m = 2b$ as stated in [40]. This extended IC has even $2m - n = 4b - 2a$ which is divisible by 2.

constant gap of the outer bound $\bar{\mathcal{C}}$ given in Lemma 1 for sufficiently large P .

It is worth at this point to remark that in [33], the outer bound $\bar{\mathcal{C}}$ was shown to be achievable if $h^2 \geq \frac{(1+P)^2}{P}$ by using lattice codes and alignment. Moreover, the corner point S_2 has been shown to be achievable within a gap of one bit per user if $h^2 \geq P$ in [26]. Thus, Theorem 4 is proved for the regime $h^2 \geq P$ which can be classified as a very-strong interference regime. However, the strong interference regime $1 < h^2 < P$ remained open. Next we focus on this regime.

Notice that the outer bound $\bar{\mathcal{C}}$ has the same structure as the one shown in Figure 3. To achieve the outer bound $\bar{\mathcal{C}}$ within a constant gap, it is enough to prove the achievability of the corner points within a constant gap. Using similar arguments as for the LD IC, the trivial corner points can be achieved easily, and therefore we focus on the non-trivial corner points.

The sum-capacity corner point S_1 defined as

$$S_1 = \frac{1}{2}(C(P + h^2P), C(P + h^2P), C(P + h^2P)) \quad (20)$$

is achievable within a constant gap by using the lattice-coding scheme described by Ordentlich *et al.* in [26]. In more detail, Ordentlich *et al.* have shown that the rate $R_1 = R_2 = R_3 = R_{sym}$ given by

$$R_{sym} \leq \frac{1}{2}C^+(h^2P - 1) - \frac{c}{2} - 3 \quad (21)$$

is achievable for all $h^2 \in (1, P)$, $P > 1$, except for an outage set whose measure is a fraction 2^{-c} of this interval for any $c > 0$. By comparing this achievable rate with (20), we can see that S_1 is achievable in the SI regime within a constant gap of $\frac{c}{2} + 4$ per user.

Thus, it remains to show the achievability of the greedy-max corner points G_i within a constant gap to complete the proof of Theorem 4. In order to achieve the greedy-max corner points, we extend the scheme described in Section IV-B2 to the Gaussian IC. Due to symmetry, we consider the the greedy-max point G_1 which is in this case given by

$$G_1 = \left(C(P), C\left(\frac{h^2P}{1+P}\right), C\left(\frac{h^2P}{1+P}\right) \right). \quad (22)$$

The following lemma states the achievability of G_1 within a constant gap.

Lemma 2. *For a symmetric Gaussian 3-user IC with $1 < h^2 < P$ with sufficiently large P , the rate triple $G_1 - (\delta, \delta, \delta)$ is achievable, where G_1 is defined in (22), and*

$$\delta = \begin{cases} \frac{3}{2}, & h^2 \in (1, \sqrt[3]{P}] \\ 3, & h^2 \in (\sqrt[3]{P}, \sqrt{P}] \\ 2, & h^2 \in (\sqrt{P}, P) \end{cases} \quad (23)$$

With this lemma, the proof of Theorem 4 is completed. It remains to propose a transmission scheme which proves Lemma 2. This is done in the following subsections.

A. Transmission scheme

We propose a generic transmission scheme based on the intuition discussed in Section IV for the LD IC. This generic

scheme combines all splitting and alignment schemes necessary for the three sub-regimes of the SI regime as those considered in Section IV-B2 (SI1, SI2, and SI3), which are to be defined later for the Gaussian IC. All in all, the proposed scheme for the Gaussian IC uses 6 splits at the greedy transmitter and 3 at the other two, and uses all alignment modes (II, DI, and DII). Later on, we show that the proposed scheme achieves the greedy-max corner point for each sub-regime within a constant gap.

1) *Message splitting*: The greedy transmitter (transmitter 1) splits its message $m_1 \in \mathcal{M}_1$ into six parts, m_{11} to m_{16} , with rates R_{11} to R_{16} where $R_1 = \sum_{i=1}^6 R_{1i}$. Transmitter $j \in \{2, 3\}$ splits its messages m_j into three splits, m_{j1} , m_{j2} , and m_{j3} so that $R_j = \sum_{i=1}^3 R_{ji}$.

2) *Encoding*: The message split m_{ji} is encoded by using an N -dimensional nested-lattice code (a brief introduction to nested-lattice codes is given in Appendix B) with fine lattice Λ_{ji}^f , coarse lattice Λ_{ji}^c , nesting ratio R_{ji} , and unit power. Such a nested-lattice code is denoted $(\Lambda_{ji}^f, \Lambda_{ji}^c)$. First, m_{ji} is mapped to a codeword $\mathbf{u}_{ji} \in \Lambda_{ji}^f \cap \mathcal{V}(\Lambda_{ji}^c)$ where $\mathcal{V}(\Lambda_{ji}^c)$ denotes the fundamental Voronoi region of Λ_{ji}^c . Then, this codeword is dithered and reduced modulo- Λ_{ji}^c to obtain $(\mathbf{u}_{ji} - \mathbf{d}_{ji}) \bmod \Lambda_{ji}^c$. This is then scaled using a power allocation parameter P_{ji} to

$$\mathbf{x}_{ji} = \sqrt{P_{ji}} [(\mathbf{u}_{ji} - \mathbf{d}_{ji}) \bmod \Lambda_{ji}^c].$$

Now as we have done in Section IV for the LD IC, we need to copy $\mathbf{x}_{21}$ and $\mathbf{x}_{22}$ for the purpose of interference cancellation at receiver 1. The two copies of each signal have different powers (corresponding to different positions in $\mathbf{x}_2$ in the LD IC as shown in Figure 5 for instance). Similarly, we need to copy $\mathbf{x}_{31}$ and $\mathbf{x}_{32}$. Thus, transmitter $j \in \{2, 3\}$ constructs

$$\hat{\mathbf{x}}_{ji} = \sqrt{\frac{\hat{P}_{ji}}{P_{ji}}} \mathbf{x}_{ji} \quad i \in \{1, 2\}, \quad (24)$$

where $\hat{P}_{ji}$ is the power of $\hat{\mathbf{x}}_{ji}$. Finally, transmitters 1, 2, and 3 send

$$\mathbf{x}_1 = \sum_{i=1}^6 \mathbf{x}_{1i} \quad (25)$$

$$\mathbf{x}_j = \sum_{i=1}^3 \mathbf{x}_{ji} + \sum_{i=1}^2 \hat{\mathbf{x}}_{ji}, \quad j \in \{2, 3\}, \quad (26)$$

which should satisfy the power constraint, thus

$$\sum_{i=1}^6 P_{1i} \leq P \quad (27)$$

$$\sum_{i=1}^3 P_{ji} + \sum_{i=1}^2 \hat{P}_{ji} \leq P, \quad j \in \{2, 3\}. \quad (28)$$

3) *Alignment*: Alignment in the Gaussian case refers to lattice alignment. That is, signals are aligned by encoding them using the same nested-lattice code and making their power at the receiver equal. This allows the decodability of the sum of the two corresponding codewords modulo coarse lattice (see Appendix B). For instance, if we set $(\Lambda_{21}^f, \Lambda_{21}^c) = (\Lambda_{31}^f, \Lambda_{31}^c)$,

Fig. 8. Received signal levels at receivers 1 and 2 showing the alignment pattern. The vertical axis indicates the power levels, where $\bar{P}_{ji}$ indicates the sum of P_{ji} and the power level just below $\bar{P}_{ji}$. For instance, $\bar{P}_{15} = P_{15} + P_{16}$, $\bar{P}_{12} = P_{12} + h^2 \hat{P}_{21}$ and $\bar{P}_{23} = P_{23} + \hat{P}_{22} + \hat{P}_{21}$. This figure indicates the restrictions that the power allocation must satisfy in order to guarantee alignment. For instance, to align the three signals $\mathbf{x}_{12}$, $\hat{\mathbf{x}}_{22}$, and $\hat{\mathbf{x}}_{32}$ at receiver 1, we need to fix $P_{12} = h^2 \hat{P}_{22} = h^2 \hat{P}_{32} = \bar{P}_{12} - h^2 \hat{P}_{21}$.

and $P_{21} = P_{31}$, then the lattice codewords $\mathbf{u}_{21}$ and $\mathbf{u}_{31}$ corresponding to the signals $\mathbf{x}_{21}$ and $\mathbf{x}_{31}$ align at receiver 1 with powers $h^2 P_{21} = h^2 P_{31}$, allowing receiver 1 to decode $(\mathbf{u}_{21} + \mathbf{u}_{31}) \bmod \Lambda_{21}^c$.

The required signal level alignment is depicted in Figure 8. In order to facilitate aggregate interference decoding and cancellation at receiver 1, we need to align splits of the interference signals $\mathbf{x}_2$ and $\mathbf{x}_3$ with the same index, i.e., $\mathbf{x}_{21}$ with $\mathbf{x}_{31}$, $\mathbf{x}_{22}$ with $\mathbf{x}_{32}$ and so on (II alignment mode). To facilitate aggregate interference decoding and cancellation at receiver 2, we need to align $\mathbf{x}_{11}$, $\mathbf{x}_{12}$, $\mathbf{x}_{13}$, and $\mathbf{x}_{16}$ with $\mathbf{x}_{31}$, $\mathbf{x}_{32}$, $\mathbf{x}_{33}$, and $\hat{\mathbf{x}}_{31}$, respectively, which is the II alignment mode. Furthermore, we need to align $\mathbf{x}_{14}$ with $\mathbf{x}_{21}$ (DI mode), and $\mathbf{x}_{15}$ with $\mathbf{x}_{22}$ and $\hat{\mathbf{x}}_{32}$ (DII mode). Similar alignment constraints are required at receiver 3. From these alignment constraints, we obtain

$$(\Lambda_{j1}^f, \Lambda_{j1}^c) = (\Lambda_{11}^f, \Lambda_{11}^c) = (\Lambda_{14}^f, \Lambda_{14}^c) = (\Lambda_{16}^f, \Lambda_{16}^c), \quad (29)$$

$$(\Lambda_{j2}^f, \Lambda_{j2}^c) = (\Lambda_{12}^f, \Lambda_{12}^c) = (\Lambda_{15}^f, \Lambda_{15}^c), \quad (30)$$

$$(\Lambda_{j3}^f, \Lambda_{j3}^c) = (\Lambda_{13}^f, \Lambda_{13}^c), \quad (31)$$

for $j = 2, 3$, and

$$P_{11} = P_{21} = P_{31} = h^2 P_{14} \triangleq P_a, \quad (32)$$

$$P_{12} = P_{22} = P_{32} = h^2 P_{15} = h^2 \hat{P}_{22} = h^2 \hat{P}_{32} \triangleq P_b, \quad (33)$$

$$P_{13} = P_{23} = P_{33} \triangleq P_c, \quad (34)$$

$$P_{16} = \hat{P}_{21} = \hat{P}_{31} \triangleq P_d. \quad (35)$$

4) *Decoding*: The decoding is done successively at the receivers. At each decoding step, the contribution of the previously decoded signals is removed, and a signal is decoded while treating the remaining signals as noise. We start with receiver 1.

Receiver 1 receives the signal

$$\mathbf{y}_1 = \mathbf{x}_1 + h\mathbf{x}_2 + h\mathbf{x}_3 + \mathbf{z}_1 \quad (36)$$

$$= \mathbf{x}_1 + h(\mathbf{x}_{21} + \mathbf{x}_{31}) + h(\mathbf{x}_{22} + \mathbf{x}_{32}) + h(\mathbf{x}_{23} + \mathbf{x}_{33}) \\ + h(\hat{\mathbf{x}}_{21} + \hat{\mathbf{x}}_{31}) + h(\hat{\mathbf{x}}_{22} + \hat{\mathbf{x}}_{32}) + \mathbf{z}_1. \quad (37)$$

It starts with decoding the aggregate interference given by the quantities in brackets in (37) successively [41]. It decodes $(\mathbf{u}_{21} + \mathbf{u}_{31}) \bmod \Lambda_{21}^c$ first, then it reconstructs $\mathbf{x}_{21} + \mathbf{x}_{31}$ and removes its contribution from $\mathbf{y}_1$ (see Appendix B for more details). In the same time, it removes the contribution of $\hat{\mathbf{x}}_{21} + \hat{\mathbf{x}}_{31}$ which is equal to $\sqrt{\frac{P_d}{P_a}}(\mathbf{x}_{21} + \mathbf{x}_{31})$ (cf. (24), and (32)-(35)). Next, it decodes $(\mathbf{u}_{22} + \mathbf{u}_{32}) \bmod \Lambda_{22}^c$ and removes the contribution of $\mathbf{x}_{22} + \mathbf{x}_{32}$ and $\hat{\mathbf{x}}_{22} + \hat{\mathbf{x}}_{32}$ from $\mathbf{y}_1$. Finally, it decodes $(\mathbf{u}_{23} + \mathbf{u}_{33}) \bmod \Lambda_{23}^c$ and removes the contribution of $\mathbf{x}_{23} + \mathbf{x}_{33}$ from $\mathbf{y}_1$. At this point, receiver 1 has the interference free signal $\mathbf{x}_1 + \mathbf{z}_1$ from which it decodes $\mathbf{u}_{11}$ to $\mathbf{u}_{16}$ successively to obtain its desired signal. The rate constraints imposed by this decoding procedure are (38)-(40) at the top of next page and (41)-(46) below.

$$R_{11} \leq C \left(\frac{P_a}{1 + \frac{1}{h^2}P_a + (1 + \frac{1}{h^2})P_b + P_c + P_d} \right) \quad (41)$$

$$R_{12} \leq C \left(\frac{P_b}{1 + \frac{1}{h^2}P_a + \frac{1}{h^2}P_b + P_c + P_d} \right) \quad (42)$$

$$R_{13} \leq C \left(\frac{P_c}{1 + \frac{1}{h^2}P_a + \frac{1}{h^2}P_b + P_d} \right) \quad (43)$$

$$R_{14} \leq C \left(\frac{\frac{1}{h^2}P_a}{1 + \frac{1}{h^2}P_b + P_d} \right) \quad (44)$$

$$R_{15} \leq C \left(\frac{\frac{1}{h^2}P_b}{1 + P_d} \right) \quad (45)$$

$$R_{16} \leq C(P_d). \quad (46)$$

Receiver 2 obtains the signal

$$\mathbf{y}_2 = \mathbf{x}_2 + h\mathbf{x}_1 + h\mathbf{x}_3 + \mathbf{z}_2 \quad (47)$$

$$= h(\mathbf{x}_{11} + \mathbf{x}_{31}) + h(\mathbf{x}_{12} + \mathbf{x}_{32}) + h(\mathbf{x}_{13} + \mathbf{x}_{33}) \\ + (\mathbf{x}_{21} + h\mathbf{x}_{14}) + (\mathbf{x}_{22} + h\mathbf{x}_{15} + h\hat{\mathbf{x}}_{32}) \\ + h(\mathbf{x}_{16} + \hat{\mathbf{x}}_{31}) + \mathbf{x}_{23} + \hat{\mathbf{x}}_{22} + \hat{\mathbf{x}}_{21} + \mathbf{z}_2. \quad (48)$$

It decodes the signals in the order shown in (48). That is, it decodes $(\mathbf{u}_{11} + \mathbf{u}_{31}) \bmod \Lambda_{11}^c$ first, then $(\mathbf{u}_{12} + \mathbf{u}_{32}) \bmod \Lambda_{12}^c$, then $(\mathbf{u}_{13} + \mathbf{u}_{33}) \bmod \Lambda_{13}^c$ (II alignment mode), next it decodes $(\mathbf{u}_{21} + \mathbf{u}_{14}) \bmod \Lambda_{21}^c$ from $\mathbf{x}_{21} + h\mathbf{x}_{14}$ (DI alignment mode) followed by $(\mathbf{u}_{22} + \mathbf{u}_{15} + \mathbf{u}_{32}) \bmod \Lambda_{22}^c$ from $\mathbf{x}_{22} + h\mathbf{x}_{15} + h\hat{\mathbf{x}}_{32}$ (DII alignment mode), then it decodes $(\mathbf{u}_{16} + \hat{\mathbf{u}}_{31}) \bmod \Lambda_{16}^c$, and finally, it decodes the desired signals $\mathbf{u}_{23}$, $\hat{\mathbf{u}}_{22}$, and $\hat{\mathbf{u}}_{21}$. After each decoding, the contribution of the decoded signal is removed from $\mathbf{y}_2$. Similar decoding is performed by receiver 3. This leads to the rate constraints (49)-(52) at the top of next page and (53)-(57) below.

$$R_{15}, R_{22}, R_{32} \leq C^+ \left(\frac{P_b}{1 + \frac{1}{h^2}P_b + P_c + (1 + 2h^2)P_d} - \frac{2}{3} \right) \quad (53)$$

$$R_{16}, R_{21}, R_{31} \leq C^+ \left(\frac{h^2P_d}{1 + \frac{1}{h^2}P_b + P_c + P_d} - \frac{1}{2} \right) \quad (54)$$

$$R_{23}, R_{33} \leq C \left(\frac{P_c}{1 + \frac{1}{h^2}P_b + P_d} \right) \quad (55)$$

$$R_{22}, R_{32} \leq C \left(\frac{\frac{1}{h^2}P_b}{1 + P_d} \right) \quad (56)$$

$$R_{21}, R_{31} \leq C(P_d). \quad (57)$$

After this decoding, each receiver obtains its desired signal. We obtain the following theorem.

Theorem 5. *The rate triple $\hat{G}_1 = (R_1, R_2, R_3)$ given by $R_1 = \sum_{i=1}^6 R_{1i}$, $R_2 = \sum_{i=1}^3 R_{2i}$, and $R_3 = \sum_{i=1}^3 R_{3i}$, where the rates R_{ij} satisfy (38)-(46) and (49)-(57) is achievable.*

Now, we have to adjust P_a , P_b , P_c , and P_d in order to maximize the achievable rates. It turns out that the point $\hat{G}_1$ can be made close to the greedy-max corner point G_1 (22) by adjusting the power allocation parameters appropriately as shown in Figure 9. This figure shows the inner bound achievable by using our scheme designed for approaching G_1 . Note that by rotating the inner region, we obtain the rate regions achievable by modifying our scheme to approach G_2 and G_3 . These two other regions are not plotted here for clarity. The points G_1 and $\hat{G}_1$ are marked for comparison, where the difference between the two points can be seen (< 1 bit per user in this case). In the following subsection, we show that this difference is bounded by a constant independent of the channel parameters, which proves the achievability of G_1 within a constant gap.

B. Gap to the greedy-max corner

In order to show that our scheme achieves rates that are within a constant gap to the greedy-max corner points for any P , we need to find a good power allocation policy. For this purpose, we use the insights gained from the LD IC in Section IV. We convert the bit allocation policy for the LD IC into a power allocation policy for the Gaussian IC. Then, we show that the given power allocation yields rates that are close to the greedy-max corner points. As for the LD IC, we split the SI regime into 3 cases, which simplifies the analysis:

- SI1) $h^2 \in (1, \sqrt[3]{P}]$,
- SI2) $h^2 \in (\sqrt[3]{P}, \sqrt{P}]$, and
- SI3) $h^2 \in (\sqrt{P}, 1 + P)$.

1) *SI1*: In this case, only three signals are sent by transmitter 1, and only two by transmitters 2 and 3 (see Figure 4(a)). Namely, transmitter 1 sends $\mathbf{x}_1 = \mathbf{x}_{11} + \mathbf{x}_{14} + \mathbf{x}_{16}$, transmitter $j \in \{2, 3\}$ sends $\mathbf{x}_j = \mathbf{x}_{j1} + \hat{\mathbf{x}}_{j1}$. Therefore, the remaining signals have zero power, i.e., $P_b = P_c = 0$. Now let us start converting the bit allocation of the LD IC into a power allocation. The signal $\mathbf{u}_{16}$ in Figure 4(a) has rate $n - m$. This rate can be approximated as

$$n - m = \left\lceil \frac{1}{2} \log(h^2 P) \right\rceil - \left\lceil \frac{1}{2} \log(P) \right\rceil \approx \frac{1}{2} \log(h^2). \quad (58)$$

That is, in the Gaussian IC, while decoding $\mathbf{x}_{16}$ interference-free, a rate $R_{16} \approx \frac{1}{2} \log(h^2)$ should be achieved. This is possible by setting $P_{16} = h^2$. Thus, we have $P_d = h^2$, and it

$$R_{21}, R_{31} \leq C^+ \left(\frac{h^2 P_a}{1 + (1 + \frac{1}{h^2}) P_a + (3 + \frac{1}{h^2} + 2h^2) P_b + (1 + 2h^2) P_c + (1 + 2h^2) P_d} - \frac{1}{2} \right) \quad (38)$$

$$R_{22}, R_{32} \leq C^+ \left(\frac{h^2 P_b}{1 + (1 + \frac{1}{h^2}) P_a + (3 + \frac{1}{h^2}) P_b + (1 + 2h^2) P_c + P_d} - \frac{1}{2} \right) \quad (39)$$

$$R_{23}, R_{33} \leq C^+ \left(\frac{h^2 P_c}{1 + (1 + \frac{1}{h^2}) P_a + (1 + \frac{1}{h^2}) P_b + P_c + P_d} - \frac{1}{2} \right) \quad (40)$$

$$R_{11}, R_{21}, R_{31} \leq C^+ \left(\frac{h^2 P_a}{1 + 2P_a + (3 + \frac{1}{h^2} + 2h^2) P_b + (1 + 2h^2) P_c + (1 + 2h^2) P_d} - \frac{1}{2} \right) \quad (49)$$

$$R_{12}, R_{22}, R_{32} \leq C^+ \left(\frac{h^2 P_b}{1 + 2P_a + (3 + \frac{1}{h^2}) P_b + (1 + 2h^2) P_c + (1 + 2h^2) P_d} - \frac{1}{2} \right) \quad (50)$$

$$R_{13}, R_{23}, R_{33} \leq C^+ \left(\frac{h^2 P_c}{1 + 2P_a + (3 + \frac{1}{h^2}) P_b + P_c + (1 + 2h^2) P_d} - \frac{1}{2} \right) \quad (51)$$

$$R_{14}, R_{21}, R_{31} \leq C^+ \left(\frac{P_a}{1 + (3 + \frac{1}{h^2}) P_b + P_c + (1 + 2h^2) P_d} - \frac{1}{2} \right) \quad (52)$$

remains to find P_a . We allocate the remaining power $P - h^2$ to the remaining two signals $\mathbf{x}_{11}$ and $\mathbf{x}_{14}$ whose powers are P_a and $\frac{P_a}{h^2}$ as given in (32). Therefore, $P_a(1 + \frac{1}{h^2}) = P - h^2$ leading to $P_a = \frac{h^2}{1+h^2}(P - h^2)$.

Remark 2. Note that with this power allocation, transmitters 2 and 3 do not transmit at full power since $P_{21} + \hat{P}_{21} = P_a + P_d < P$. The scheme can be improved by using the remaining power to send an extra signal. However, this improvement is marginal and not required for the purpose of this paper.

Now, we need to plug these power allocation parameters into the rate constraints of our scheme. We start with R_{11} which is constrained by (41) and (49). Since under the given power allocation, the first constraint (41) is larger than the second (49), we obtain from (49)

$$R_{11} \leq C^+ \left(\frac{h^4 P - h^6}{1 + 2h^2 + h^4 + 2h^6 + 2h^2 P} - \frac{1}{2} \right) \quad (59)$$

$$= C \left(\frac{h^4 P - h^6}{1 + 2h^2 + h^4 + 2h^6 + 2h^2 P} - \frac{1}{2} \right), \quad (60)$$

where the last step holds by assuming a moderately large P so that the rate constraint for R_{11} is positive. Similarly, the rate constraints on R_{14} and R_{16} , i.e., (44), (46), (52), and (54), become

$$R_{14} \leq C \left(\frac{h^2 P - h^4}{1 + 2h^2 + 3h^4 + 2h^6} - \frac{1}{2} \right), \quad (61)$$

$$R_{16} \leq C \left(\frac{h^4}{1 + h^2} - \frac{1}{2} \right). \quad (62)$$

By adding the right-hand sides of (60)-(62), and after some manipulations, we can show that

$$R_1 \leq \frac{1}{2} \log(1 + P) - \frac{3}{2} \quad (63)$$

is achievable (see details in Appendix E).

Now consider the rate constraint for the second user, i.e., R_{21} . By substituting the power allocation parameters in the rate constraints (38), (49), (52), (54), and (57), and comparing them, and after some manipulations, we can show that the rate

$$R_2 \leq \frac{1}{2} \log \left(1 + \frac{h^2 P}{1 + P} \right) - \frac{1}{2} \log(7) \quad (64)$$

is achievable as shown in Appendix E. The achievable rate R_3 is bounded similarly to R_2 . The gap for R_1 , R_2 , and R_3 is less than $3/2$, and hence, the rate triple $(R_1, R_2, R_3) = G_1 - (\frac{3}{2}, \frac{3}{2}, \frac{3}{2})$ is achievable, for a gap of $3/2$ bits per user, where G_1 is given in (22). By symmetry, the other greedy-max corner points G_2 and G_3 can also be achieved within the same gap. This proves Lemma 2 for the SI1 regime.

2) SI2: Now we consider the SI2 regime given by $\sqrt[3]{P} < h^2 \leq \sqrt{P}$. In this case, five signals are sent by transmitter 1, and four by transmitters 2 and 3 (see Figure 5(a)). Transmitter 1 sends $\mathbf{x}_1 = \mathbf{x}_{11} + \mathbf{x}_{12} + \mathbf{x}_{14} + \mathbf{x}_{15} + \mathbf{x}_{16}$ and transmitter $j \in \{2, 3\}$ sends $\mathbf{x}_j = \mathbf{x}_{j1} + \mathbf{x}_{j2} + \hat{\mathbf{x}}_{j1} + \hat{\mathbf{x}}_{j2}$. Since the transmitters do not send $\mathbf{x}_{j3}$, we set $P_c = 0$. Moreover, similar to the SI1 regime, we need to set $P_{16} + P_{15} = h^2$. Therefore, from (33) and (35) we get

$$P_d + \frac{P_b}{h^2} = h^2. \quad (65)$$

We allocate the remaining power $P - h^2$ to the remaining signals, i.e., $P_{11} + P_{12} + P_{14} = P - h^2$, from which we obtain (see (32)-(35))

$$P_a \left(1 + \frac{1}{h^2} \right) + P_b = P - h^2. \quad (66)$$

We need one more equations to complete a system of linear equations from which we can obtain P_a , P_b , and P_d . For this purpose, we observe that the sum $\mathbf{x}_{12} + \dots + \mathbf{x}_{16}$ must have

(a) Rate region outer and inner bounds.

(b) An (R_1, R_2) view of the regions. Due to symmetry, the (R_1, R_3) view is similar.

(c) An (R_2, R_3) view of the regions.

Fig. 9. Capacity inner and outer bounds for a symmetric IC with $\alpha = \frac{\log(\text{INR})}{\log(\text{SNR})} = 1.25$ and $\text{SNR} = 40\text{dB}$. The outer region is the capacity outer bound. The green inner region is the rate region achieved by our transmission scheme designed for approaching G_1 . The point $\hat{G}_1$ is also marked for comparison. Note that the gap between G_1 and $\hat{G}_1$ is less than 1 bit per user in this example.

the same power as $h(\hat{\mathbf{x}}_{21} + \hat{\mathbf{x}}_{22})$ (equal power level $\bar{P}_{12}$ as shown in Figure 8) which is equal to $h^2(\hat{P}_{21} + \hat{P}_{22}) = h^4$ (by (33), (35), and (65)), i.e., $P_{12} + \dots + P_{16} = h^4$ which yields

$$\frac{P_a}{h^2} + P_b \left(1 + \frac{1}{h^2}\right) + P_d = h^4. \quad (67)$$

Now, by solving (65)-(67), we obtain

$$P_a = P - h^4, \quad P_b = h^4 - \frac{P}{h^2}, \quad P_d = \frac{P}{h^4}. \quad (68)$$

Note that $P_b > 0$ due to the condition of the SI2 regime ($\sqrt[3]{P} < h^2$).

By substituting these power allocation parameters into the rate constraints, we can show that the following rate can be achieved by the first user

$$R_1 \leq \frac{1}{2} \log(1 + P) - 2 - \frac{1}{2} \log(3). \quad (69)$$

For user 2, after substituting the given power allocation parameters into the rate constraints, and after some manipulation, we can show that the rate

$$R_2 \leq \frac{1}{2} \log \left(1 + \frac{h^2 P}{1 + P}\right) - \frac{3}{2} - \frac{1}{2} \log(3) \quad (70)$$

is achievable (details are given in Appendix E). Similar rate constraint holds for user 3. As a result, Since the gap for

R_1, R_2 , and R_3 is less than 3, the rate triple $(R_1, R_2, R_3) = G_1 - (3, 3, 3)$ is achievable, for a gap of 3 bits per user to the greedy-max corner point G_1 defined in (22). The other greedy-max corner points G_2 and G_3 can be achieved within the same gap similarly. This proves Lemma 2 for the SI2 regime.

3) *SI3*: In this case we have $\sqrt{P} < h^2 \leq P + 1$. Here, each transmitter sends three signals only. Namely, transmitter 1 sends $\mathbf{x}_1 = \mathbf{x}_{12} + \mathbf{x}_{13} + \mathbf{x}_{15}$, while transmitter $j \in \{2, 3\}$ sends $\mathbf{x}_j = \mathbf{x}_{j2} + \mathbf{x}_{j3} + \hat{\mathbf{x}}_{j2}$. This accounts for three message splits at transmitter 1, and two at each of transmitters 2 and 3. Therefore, we set $P_a = P_d = 0$ (cf. (32)-(35)). In order to find a power allocation which achieves rates close to the greedy-max corner points, we observe Figure 6(a). Using similar arguments as for the SI1 and SI2 regimes, we conclude that we need to set $P_{13} + P_{15} = h^2$ and $P_{12} = P - h^2$. By using (33) and (34), we get

$$P_c + \frac{P_b}{h^2} = h^2, \quad P_b = P - h^2. \quad (71)$$

Solving for P_c we obtain

$$P_c = h^2 - \frac{P}{h^2} + 1. \quad (72)$$

Next, we plug this power allocation into the rate constraints of our scheme. After some manipulations as shown in Appendix E, we can show that the rates

$$R_1 \leq \frac{1}{2} \log(1 + P) - \frac{1}{2} \log(12) \quad (73)$$

$$R_2, R_3 \leq \frac{1}{2} \log \left(1 + \frac{h^2 P}{1 + P}\right) - \frac{1}{2} \log(12) \quad (74)$$

are achievable. Consequently, the gap is less than 2 bits per user, and thus, the rate triple $(R_1, R_2, R_3) = G_1 - (2, 2, 2)$ is achievable. The corner points G_2 and G_3 can be achieved within the same gap similarly. This completes the proof of Lemma 2 for the SI3 regime.

With this, Lemma 2 has been proved, and therefore, the achievability of the greedy-max corner points within a constant gap. Consequently, the outer bound $\bar{\mathcal{C}}$ provides an approximate characterization of the capacity region of the symmetric Gaussian 3-user IC with strong interference within a constant gap as stated in Theorem 4.

Note that while the gap corresponding to the greedy-max corner points is a universal constant (107) valid for all h^2 in the SI regime, the gap to corresponding to the sum-capacity corner point S_1 (21) is valid outside an outage set whose size is inversely proportional to the gap [26]. In more detail, the gap in the latter case is $\frac{c}{2} + 4$, and holds for the interval $h^2 \in (1, P)$ excluding an outage set whose size is a fraction 2^{-c} , $c > 0$, of this interval. Reducing the size of this outage set leads to increasing the gap. Thus, in order to obtain a constant gap for the whole range of $h^2 \in (1, P)$, we need to tolerate a large gap. Next, we propose an alternative scheme using a superposition of several nested-lattice codes with alignment and successive decoding. This scheme achieves the GDoF of the IC with strong interference, and the sum-capacity within a constant gap. However, this gap grows as h^2 approaches 1.

VI. APPROXIMATE SUM-CAPACITY OF THE SYMMETRIC GAUSSIAN 3-USER IC WITH STRONG INTERFERENCE

The sum-capacity of the 3-user IC has been treated by Jafar and Vishwanath in [22] from a GDoF point of view (see (2)). The main idea of their GDoF achieving scheme, described originally in their paper for the Gaussian IC, was explained in Section IV-B3 for the LD IC. Although the GDoF of the K -user IC was characterized in [22], the gap between the achievable sum-rate and the upper bound was not calculated. As will be discussed in the following, it turns out that the scheme in [22] needs to be modified to achieve the sum-capacity within a constant gap. Next, we propose a scheme which achieves the sum-capacity of the symmetric 3-user IC with strong interference within a constant gap, and we calculate the corresponding gap.

Our scheme is based on rate-splitting, nested-lattice codes, interference alignment, successive decoding, and interference cancellation. The rate-splitting and alignment patterns we use are very similar to those proposed in [22] with some modifications that are necessary to enable successive decoding. In more detail, the scheme of [22] splits the signal into several Q -bits with different powers. In the decoding procedure, some signals with lower powers are decoded before some other signals with higher power. In our successive decoding framework (as the one used in Section V-A), this would lead to zero rate constraints¹⁰ ($R \leq 0$) which is clearly not desired. Thus, we introduce some modifications to the scheme which enable successive decoding, and further, achieve the sum-capacity within a constant gap. The next theorem presents the main result of this section.

Theorem 6. *For sufficiently high P , the rate triple (R, R, R) with a symmetric rate R given by*

$$R = \frac{1}{2}C(P + h^2P) - \frac{1}{4} - \frac{2\kappa + 1}{2}\log(6), \quad (75)$$

is achievable in the symmetric Gaussian 3-user IC with strong interference $h^2 \in (1, P)$, where $\kappa = \lfloor \frac{\log(h^2P)}{2\log(h^2)} \rfloor$.

Proof: We present a sketch of the proof here considering a special example, while the general proof is given in Appendix D. In particular, we consider a symmetric IC with $\frac{\log(h^2P)}{2\log(h^2)} = \frac{5}{2}$ and thus $\kappa = 2$. Due to symmetry, we only discuss the scheme from the perspective of receiver 1.

For approaching the sum-capacity of the channel, each transmitter sends $2\kappa + 1$ messages. After encoding these messages, the transmitter generates two copies of each codeword. Then, the transmitter sends a superposition of the $2\kappa + 1$ signals plus their $2\kappa + 1$ copies. Thus, for this particular example, each transmitter sends the sum of 10 signals, accounting for $2\kappa + 1 = 5$ messages. Namely, as shown in Figure 10(a), transmitters 1 for instance sends the messages $\mathbf{u}_{1,1}, \mathbf{u}_{1,2}, \mathbf{u}_{1,3}, \mathbf{v}_{1,1}, \mathbf{v}_{1,2}$, encoded in 5 signals $\mathbf{x}_{1,u[1]}, \mathbf{x}_{1,v[1]}, \mathbf{x}_{1,u[2]}, \mathbf{x}_{1,v[2]}, \mathbf{x}_{1,u[3]}$, and copies of these signals (sent at lower powers) denoted $\hat{\mathbf{x}}_{1,u[1]}, \hat{\mathbf{x}}_{1,v[1]}, \hat{\mathbf{x}}_{1,u[2]}, \hat{\mathbf{x}}_{1,v[2]},$ and $\hat{\mathbf{x}}_{1,u[3]}$, respectively. This splitting pattern is justified in Appendix C.

¹⁰Since the ratio between the signal power and the interference plus noise power is less than 1.

(a) Alignment pattern at receiver 1.

Fig. 10. The received signals at the receivers showing the power level of each signal. The signal splits with the same color carry the same information since they are copies of each other.

Figure 10(a) shows the power levels of the transmitted signals as received at receiver 1. The power of each signal is the difference between its power level and the power level of lower signal. For example, the power of $\hat{\mathbf{x}}_{1,u[3]}$ is equal to its power level $\sqrt{P/h^{10}}$ minus 0, and the power of $\hat{\mathbf{x}}_{1,u[2]}$ is $P/h^8 - \sqrt{P/h^{10}} = 1 - \sqrt{P/h^{10}}$ since $\frac{\log(h^2P)}{2\log(h^2)} = \frac{5}{2}$. Notice that due to $\kappa = \lfloor \frac{\log(h^2P)}{2\log(h^2)} \rfloor$, we have $\sqrt{P/h^{10}} < 1$ and $\sqrt{P/h^6} > 1$, and hence, the power level of $\hat{\mathbf{x}}_{1,u[3]}$ is below the power of noise, and the power level of $\hat{\mathbf{x}}_{1,v[2]}$ is above the power of noise.

To facilitate aggregate interference decoding, all signals are nested-lattice codewords. That is,

$$\mathbf{x}_{j,u[i]} = \sqrt{P_{u,i}}[(\mathbf{u}_{j,i} - \mathbf{d}_{j,i}) \bmod \Lambda_u^c] \quad (76)$$

$$\mathbf{x}_{j,v[i]} = \sqrt{P_{v,i}}[(\mathbf{v}_{j,i} - \mathbf{e}_{j,i}) \bmod \Lambda_v^c], \quad (77)$$

where $P_{u,i}$ and $P_{v,i}$ are power allocation parameters, $\mathbf{u}_{j,i}$ and $\mathbf{v}_{j,i}$ are nested lattice codewords, $\mathbf{d}_{j,i}$ and $\mathbf{e}_{j,i}$ are random dithers, and Λ_u^c and Λ_v^c are the coarse lattices corresponding to these nested-lattice codes (more details in Appendix B). Furthermore, to facilitate interference cancellation, the copied

signals have alternating signs. In other words

$$\hat{\mathbf{x}}_{j,u[i]} = (-1)^i \sqrt{\frac{\hat{P}_{u,i}}{P_{u,i}}} \mathbf{x}_{j,u[i]} \quad (78)$$

$$\hat{\mathbf{x}}_{j,v[i]} = (-1)^i \sqrt{\frac{\hat{P}_{v,i}}{P_{v,i}}} \mathbf{x}_{j,v[i]}, \quad (79)$$

where $\hat{P}_{u,i}$ and $\hat{P}_{v,i}$ denote the powers of $\hat{\mathbf{x}}_{j,u[i]}$, and $\hat{\mathbf{x}}_{j,v[i]}$, respectively. This alternation of signs is necessary to guarantee that interference cancellation does not result in desired signal cancellation as well.

Denote the rates of these messages as $R_{u,i}$ and $R_{v,i}$, respectively. Receiver 1 starts by decoding the sum of the interference signals $\mathbf{u}_{2,1} + \mathbf{u}_{3,1}$ (from $\mathbf{x}_{2,u[1]} + \mathbf{x}_{3,u[1]}$) while treating the remaining signals (of power $P + 2\sqrt{h^{10}P} = P + 2h^4\sqrt{h^2P}$) as noise. This leads to the rate constraint

$$R_{u,1} \leq C^+ \left(\frac{h^2P}{1 + P + 2h^4\sqrt{h^2P}} - \frac{1}{2} \right) \quad (80)$$

$$\approx \frac{1}{2} \log(h^2P) - \frac{1}{2} \log(h^4\sqrt{h^2P}), \quad (81)$$

at sufficiently high P where the approximation is within a constant of the exact expression¹¹ (note that $\kappa = \lfloor \frac{\log(h^2P)}{2\log(h^2)} \rfloor$ implies that $h^4\sqrt{h^2P} > P$). This rate constraint is equal to the difference between the power levels of $\mathbf{x}_{2,u[1]}$ and the lower signal $\mathbf{x}_{2,v[1]}$ on a logarithmic scale. This correspondence can be used to easily obtain the remaining rate constraints. Next, the contribution of $\mathbf{x}_{2,u[1]} + \mathbf{x}_{3,u[1]}$ is subtracted from the received signal, and $\mathbf{v}_{2,1} + \mathbf{v}_{3,1}$ is decoded from $\mathbf{x}_{2,v[1]} + \mathbf{x}_{3,v[1]}$. The resulting rate constraint is given by $R_{v,1} \leq C^+ \left(\frac{h^4\sqrt{h^2P}}{1+3P} - \frac{1}{2} \right)$ which can be approximated at sufficiently high P by the difference of the power levels of $\mathbf{x}_{2,v[1]}$ and $\mathbf{x}_{2,u[2]}$ on a logarithmic scale, i.e.,

$$R_{v,1} \leq \frac{1}{2} \log(h^4\sqrt{h^2P}) - \frac{1}{2} \log(P). \quad (82)$$

Note that these two decoding steps involve the interference signals that use the II alignment mode.

Next, receiver 1 decodes the signals that are aligned using the DII alignment mode. It starts with $\mathbf{u}_{1,1} + \mathbf{u}_{2,2} + \mathbf{u}_{3,2}$, which it decodes from $\mathbf{x}_{1,u[1]} + h\mathbf{x}_{2,u[2]} + h\mathbf{x}_{3,u[2]}$, then subtracts its contribution. The same procedure is performed to decode $\mathbf{v}_{1,1} + \mathbf{v}_{2,2} + \mathbf{v}_{3,2}$ from $\mathbf{x}_{1,v[1]} + h\mathbf{x}_{2,v[2]} + h\mathbf{x}_{3,v[2]}$. Finally, receiver 1 decodes $\mathbf{u}_{1,2} + \mathbf{u}_{2,3} + \mathbf{u}_{3,3}$ from $\mathbf{x}_{1,u[2]} + h\mathbf{x}_{2,u[3]} + h\mathbf{x}_{3,u[3]}$. The resulting rate constraints, approximated at sufficiently high P , are then

$$R_{u,1}, R_{u,2} \leq \frac{1}{2} \log(P) - \frac{1}{2} \log(h^2\sqrt{h^2P}) \quad (83)$$

$$R_{v,1}, R_{v,2} \leq \frac{1}{2} \log(h^2\sqrt{h^2P}) - \frac{1}{2} \log(P/h^2) \quad (84)$$

$$R_{u,2}, R_{u,3} \leq \frac{1}{2} \log(P/h^2) - \frac{1}{2} \log(\sqrt{h^2P}). \quad (85)$$

At this point, the received signal is reduced to the one shown in Figure 10(b).

Now, receiver 1 can perform interference cancellation as follows. It scales $\mathbf{x}_{2,u[1]} + \mathbf{x}_{3,u[1]}$ down by $\sqrt{\frac{\hat{P}_{u,1}}{P_{u,1}}}$ to obtain $\hat{\mathbf{x}}_{2,u[1]} + \hat{\mathbf{x}}_{3,u[1]}$ which is then subtracted from the received signal. Similarly, receiver 1 removes the contribution of $\hat{\mathbf{x}}_{2,v[1]} + \hat{\mathbf{x}}_{3,v[1]}$.

Receiver 1 then scales $\mathbf{x}_{1,u[1]} + h\mathbf{x}_{2,u[2]} + h\mathbf{x}_{3,u[2]}$ down by $\sqrt{\frac{\hat{P}_{u,1}}{P_{u,1}}} = \sqrt{\frac{\hat{P}_{u,2}}{P_{u,2}}}$ (cf. Figure 10(a)) to obtain $\mathbf{s}_1 = \hat{\mathbf{x}}_{1,u[1]} - h\hat{\mathbf{x}}_{2,u[2]} - h\hat{\mathbf{x}}_{3,u[2]}$. Note that the received signal at receiver 1 at this point is the sum of $\mathbf{s}_2 = \hat{\mathbf{x}}_{1,u[1]} + h\hat{\mathbf{x}}_{2,u[2]} + h\hat{\mathbf{x}}_{3,u[2]}$ plus some other signals and noise. The sum of $\mathbf{s}_1$ and $\mathbf{s}_2$ is $\mathbf{s}_1 + \mathbf{s}_2 = 2\hat{\mathbf{x}}_{1,u[1]}$. Thus, by adding $\mathbf{s}_1$ to the received signal, the interference from $\hat{\mathbf{x}}_{2,u[2]} + \hat{\mathbf{x}}_{3,u[2]}$ is canceled, and the contribution of $\hat{\mathbf{x}}_{1,u[1]}$ is doubled. Here we can see the usefulness of (78), which allows us to cancel the interference without canceling the desired signal. Without this sign alternation, $\mathbf{s}_1$ would have been $\hat{\mathbf{x}}_{1,u[1]} + h\hat{\mathbf{x}}_{2,u[2]} + h\hat{\mathbf{x}}_{3,u[2]}$ instead. In this case, canceling the contribution of the interference $\hat{\mathbf{x}}_{2,u[2]} + \hat{\mathbf{x}}_{3,u[2]}$ would have also canceled the desired signal $\hat{\mathbf{x}}_{1,u[1]}$. This sign alternation guarantees that this does not happen.

The same procedure is performed to cancel the interference from $\hat{\mathbf{x}}_{2,v[2]} + \hat{\mathbf{x}}_{3,v[2]}$, which also doubles the contribution of $\hat{\mathbf{x}}_{1,v[1]}$. Finally, receiver 1 cancels the interference of $\hat{\mathbf{x}}_{2,u[3]} + \hat{\mathbf{x}}_{3,u[3]}$ which lifts $\hat{\mathbf{x}}_{1,u[2]}$ to a higher power level, namely, $\hat{\mathbf{x}}_{1,u[2]}$ is lifted to a power of $\frac{\hat{P}_{u,3}P_{u,2}}{P_{u,3}}$ which is equal to the power level of $\hat{\mathbf{x}}_{1,v[2]}$ (the accumulated power of $\hat{\mathbf{x}}_{1,u[3]}$, $\hat{\mathbf{x}}_{1,u[2]}$, and $\hat{\mathbf{x}}_{1,v[2]}$). The reduced received signal at this point is shown in Figure 10(c).

Now, receiver 1 starts decoding the desired signals successively. It starts with $\mathbf{v}_{1,2}$ and $\mathbf{u}_{1,3}$ which can be decoded if

$$R_{v,2} \leq \frac{1}{2} \log(\sqrt{h^2P}) - \frac{1}{2} \log(P/h^4) \quad (86)$$

$$R_{u,3} \leq \frac{1}{2} \log(P/h^4) - \frac{1}{2} \log(\sqrt{P/h^2}) \quad (87)$$

for sufficiently high P . Then, receiver 1 removes the contribution of $\mathbf{v}_{1,2}$, $\mathbf{u}_{1,3}$, $\hat{\mathbf{v}}_{1,2}$, and $\hat{\mathbf{u}}_{1,3}$. Next, the signals $\hat{\mathbf{v}}_{1,1}$, $\hat{\mathbf{u}}_{1,1}$, and $\hat{\mathbf{u}}_{1,2}$ are decoded successively, leading to the rate constraints

$$R_{v,1} \leq \frac{1}{2} \log(\sqrt{P/h^2}) - \frac{1}{2} \log(P/h^6) \quad (88)$$

$$R_{u,1} \leq \frac{1}{2} \log(P/h^6) - \frac{1}{2} \log(\sqrt{P/h^2}) \quad (89)$$

$$R_{u,2} \leq \frac{1}{2} \log(\sqrt{P/h^6}), \quad (90)$$

for sufficiently high P . By collecting all the rate constraints (81)-(90), we obtain

$$R_{v,2} \leq \frac{1}{2} \log(\sqrt{h^2P}) - \frac{1}{2} \log(P/h^4) \quad (91)$$

$$R_{u,3} \leq \frac{1}{2} \log(P/h^4) - \frac{1}{2} \log(\sqrt{P/h^2}) \quad (92)$$

$$R_{v,1} \leq \frac{1}{2} \log(\sqrt{P/h^2}) - \frac{1}{2} \log(P/h^6) \quad (93)$$

$$R_{u,1} \leq \frac{1}{2} \log(P/h^6) - \frac{1}{2} \log(\sqrt{P/h^2}) \quad (94)$$

$$R_{u,2} \leq \frac{1}{2} \log(\sqrt{P/h^6}). \quad (95)$$

¹¹This approximation is made more precise in Appendix D

By adding these rates, we conclude that the rate given by

$$R = \frac{1}{2} \log(\sqrt{h^2 P}) \quad (96)$$

is achievable for each user. This rate is approximately equal to (75) at sufficiently high P .

Note that for each message split, we get a loss term introduced by manipulating the rate constraints. However, this loss term is constant. In total, we have $2\kappa + 1$ message split, which leads $2\kappa + 1$ additive constants (gap) in (75). A more rigorous calculation of the achievable rates and gap is provided in Appendix D. ■

The following corollaries are consequences of Theorem 6. Corollary 1 states the achievability of the sum-capacity corner point S_1 within a constant gap using the proposed scheme, and corollary 2 states the achievability of the GDoF of the symmetric 3-user IC with strong interference using our scheme (as an alternative to the scheme in [22]).

Corollary 1. *The sum-capacity corner point*

$$S_1 = \frac{1}{2} (C(P + h^2 P), C(P + h^2 P), C(P + h^2 P)) \quad (97)$$

of the capacity region outer bound $\bar{C}$ of the symmetric Gaussian 3-user IC with strong interference $h^2 \in (1, P)$ is achievable within a gap of $\frac{1}{4} + \frac{2\kappa+1}{2} \log(6)$ bits per user, where $\kappa = \left\lfloor \frac{\log(h^2 P)}{2 \log(h^2)} \right\rfloor$.

Proof: The proof is immediately obtained by comparing (75) and (97). ■

Corollary 2. *The proposed scheme achieves the GDoF (defined in (2)) of the symmetric 3-user IC with strong interference, namely, $d_\Sigma(\alpha) = \frac{3}{2}\alpha$ for $\alpha \in (1, 2)$ where $\alpha = \frac{\log(h^2 P)}{\log(P)}$.*

Proof: The proof follows by calculating the achievable sum-rate $R_\Sigma = R_1 + R_2 + R_3 = 3R$ where R is given in (75), dividing by $\frac{1}{2} \log(P)$, and letting $P \rightarrow \infty$ with $\alpha = \frac{\log(h^2 P)}{\log(P)}$, which leads to the achievable GDoF $d_\Sigma(\alpha) = \frac{3}{2}\alpha$. The converse follows from [22] (see Lemma 1). ■

We have hence obtained an approximation of the sum-capacity of the symmetric Gaussian 3-user IC with strong interference. Our approximation covers the whole range $h^2 \in (1, P)$ in contrast to the approximation in [26] which excludes an outage set. However, the two approximations have the following in common: The approximation gap is not a universal constant. Namely, the approximation gap in [26] is a function of a parameter which controls the size of the outage set. Reducing the size of the outage set can be attained at the expense of increasing the gap. On the other hand, the approximation gap in Corollary 1 is a function of a parameter which depends on the ratio of h^2 to the power P on a logarithmic scale ($\kappa = \left\lfloor \frac{1}{2} + \frac{\log(P)}{2 \log(h^2)} \right\rfloor$). This gap is not relevant from GDoF point of view for $\alpha > 1$ since

$$\kappa = \left\lfloor \frac{1}{2} + \frac{1 \log(P)}{2 \log(h^2)} \right\rfloor = \left\lfloor \frac{1}{2} + \frac{1}{2} \frac{1}{\alpha - 1} \right\rfloor \quad (98)$$

and hence $\lim_{P \rightarrow \infty} \frac{\kappa}{\frac{1}{2} \log(P)} = 0$. However, it is relevant in terms of sum-capacity since κ increases as h^2 approaches 1.

In the next corollary, we upper bound the gap in Corollary 1 by a constant for a range of $h^2 \in (1, P)$.

Corollary 3. *If $h^2 > \sqrt[a]{P}$ for some $a > 1$, then the gap in Corollary 1 can be upper bounded by $\frac{1}{4} + (1 + \frac{a}{2}) \log(6)$.*

Proof: If $h^2 > \sqrt[a]{P}$, then κ can be upper bounded as follows

$$\kappa = \left\lfloor \frac{\log(h^2 P)}{2 \log(h^2)} \right\rfloor \leq \frac{\log(h^2 P)}{2 \log(h^2)} < \frac{\log(h^{2+2a})}{2 \log(h^2)} = \frac{1+a}{2},$$

where in the second inequality, we used $h^2 > \sqrt[a]{P}$. Thus, the gap $\frac{1}{4} + \frac{2\kappa+1}{2} \log(6) < \frac{1}{4} + \frac{a+2}{2} \log(6)$ which concludes the proof. ■

Thus, for $h^2 > \sqrt[a]{P}$, the gap is upper bounded by a constant dependent on a . The gap for $h^2 \in \mathcal{S} = (1, \sqrt[a]{P})$ is not bounded by $\frac{1}{4} + \frac{a+2}{2} \log(6)$. But a similar bound as in Corollary 3 can be derived by breaking the interval $(1, \sqrt[a]{P})$ into the union of intervals of the form $(\sqrt[a]{P}^{a_t}, \sqrt[a]{P}^{a_{t+1}}]$ for some $a_t > a_{t-1}$ with $a_0 = a$ and $t = 1, 2, \dots$. For each such interval, the gap can be upper bounded by $\frac{1}{4} + \frac{a_t+2}{2} \log(6)$ similar to Corollary 3. Interestingly, the size of set $\mathcal{S}$ decreases exponentially as we increase a , while the gap increases linearly in a . This is similar to the relation in [26] between the gap and the size of the outage set, with the main difference that in our case we have no outage. Note that this increasing gap as $h^2 \rightarrow 1$ ($\alpha \rightarrow 1$) also arises in the two time-slot coding scheme proposed in [27].

Figure 11 shows the achievable symmetric rate R (corresponding to an achievable rate tuple of the form (R, R, R)) given by our scheme and the single-layer lattice coding scheme¹² of Ordentlich *et al.* [26] plotted as a function of α . The capacity upper bound is also plotted for comparison. The gap between the rate achievable by our scheme and the upper bound is small at large α (VSI), and grows as α approaches 1. However, the gap maintains a ‘roughly’ constant value for a wide range of α , contrary to the gap of Ordentlich *et al.* which alternates between smaller and larger values (corresponding to outage events). The performance of our scheme deteriorates as α approaches 1 since κ is large in this case. However, in this regime, the performance of TDMA is relatively good, and thus, switching to TDMA is a better alternative.

If we treat the SI regime as we have done for the greedy-max corner points in Section V, i.e., by splitting the SI regime into SI1, SI2, and SI3, we can state the following for the latter two regimes.

Corollary 4. *The capacity region of the symmetric Gaussian 3-user IC is within a constant gap of 6 bits per user of the outer bound $\bar{C}$ in the SI2 regime ($h^2 \in (\sqrt[3]{P}, \sqrt{P})$), and a gap of 7 bits per user in the SI3 regime ($h^2 \in (\sqrt{P}, P)$).*

Proof: The statement follows by replacing a in Corollary 3 by 2 and 3, respectively for the regimes SI2 and SI3 to obtain the gaps corresponding to the sum-capacity corner point, and observing that the obtained gaps are larger than those of the greedy-max corner points obtained in Lemma 2. ■

¹²Two schemes have been proposed in [26]: a single-layer lattice code scheme, and a lattice Han-Kobayashi scheme. We plot the achievable rate of the first scheme only since this is the scheme used for constant gap characterization for the IC with strong interference in [26, Section VI.B].

Fig. 11. Achievable symmetric rates for the symmetric 3-user IC at an SNR of 35dB plotted as a function of $\alpha = \frac{\log(\text{INR})}{\log(\text{SNR})}$. The achievable rate by Ordentlich *et al.* is given in [26, Equation (64)]. The achievable rate of our scheme is derived by using the exact rate expressions from Appendix D without approximation (similar to (168)). The upper bound is plotted for comparison.

Unfortunately, while the gap corresponding to the greedy-max corner points is small in these sub-regimes as shown in Lemma 2 (3 and 2 bits per user in the SI2 and SI3 regimes, respectively), the gap corresponding to the sum-capacity corner point is larger due to the splitting approach used in this case.

VII. EXTENSION TO THE SYMMETRIC K -USER IC

At this point, we can extend the results obtained for the symmetric 3-user IC to the symmetric K -user case. We show that, with slight changes, the scheme proposed in Section V achieves the capacity region of the symmetric K -user IC with SI within a constant gap, thus proving Theorem 1. We also show that, with slight changes, our proposed scheme in Section VI achieves the sum-capacity of the symmetric K -user IC within a constant gap, thus proving Theorem 2. In both cases (capacity region and sum-capacity), the transmission schemes remain essentially the same; the only difference appears in the rate constraints, and therefore also in the calculated gap.

A. Capacity Region in the SI Regime

First, we need to show that for the K -user case, it also suffices to show the achievability of the greedy-max corner points within a constant gap to prove the achievability of the whole outer bound $\mathcal{C}$ within a constant gap in the SI regime. First we argue that it is enough to consider corner points with non-zero components. Consider for instance the extension from three to four users. Among the corner points of the outer bound corresponding to the 4-user IC, there are corner points with at least one zero component. For these corner points, the symmetric 4-user IC reduces to a symmetric 3-user IC. Since the corner points of the symmetric 3-user IC are achievable within a constant gap in the SI regime, then the corner points of the symmetric 4-user IC with at least one zero component are also achievable within a constant gap in the SI regime. Thus, it remains to prove the achievability within a constant gap of the corner points with non-zero components. These corner points are the greedy-max corner points (the 4-user IC has 4 such corner points) and the sum-capacity corner point. The sum-capacity corner point is achievable within a constant gap by the scheme in [26].¹³ Thus, it remains to prove the achievability of the greedy-max corner points within a constant gap. Similar arguments can be used for the symmetric K -user IC: If the corner points of the symmetric $(K-1)$ -user IC are achievable within a constant gap, then to prove the achievability of the corner points of the symmetric K -user case, it suffices to consider the greedy-max corner points.

In the symmetric K -user case, the greedy-max corner points are given by the tuple

$$G_i = (G_{i1}, \dots, G_{iK}), \quad (99)$$

$$G_{i\ell} = \begin{cases} C(P), & \text{if } \ell = i \\ C\left(\frac{h^2 P}{1+P}\right), & \text{if } \ell \neq i \end{cases} \quad (100)$$

To achieve G_1 for instance, the transmit signal of the greedy user (user 1) remains as described in (25), and the transmit signals of the remaining users remains as described in (26), for $j \in \{1, \dots, K\}$, $j \neq 1$.

At the receivers, while decoding the aligned interference, the receivers decode the sum of $K-1$ interferers in the II alignment mode and the sum of one desired signal and $K-1$ interferers in the DII alignment mode (the DI mode remains the same). The impact of this decoding on the rate constraints is only through a constant.

Consider for instance the II aligned signals. For the 3-user case, while decoding $(\mathbf{u}_{21} + \mathbf{u}_{31}) \bmod \Lambda_{21}^c$ at receiver 1, the interference-plus-noise power is $I_3 = 1 + P_1 + 2h^2(P_2 - P_a)$ (cf. (38)) where P_a is the power of the signal to be decoded, P_1 is the power of the transmitted signals of user 1, and P_2 is the transmitted power of user 2, which is equal to the transmitted power of user 3 due to the symmetry of the channel (cf. Figure 8). For the K -user case, while decoding $(\mathbf{u}_{21} + \mathbf{u}_{31} + \dots + \mathbf{u}_{K1}) \bmod \Lambda_{21}^c$, the interference-plus-noise power becomes $I_K = 1 + P_1 + (K-1)h^2(P_2 - P_a)$. This

¹³Or the extension of the scheme in Section VI to the K -user case discussed in Section VII-B.

interference-plus-noise ratio can be bounded for $K \geq 3$ by

$$1 + P_1 + (K-1)h^2(P_2 - P_a) \leq \frac{K-1}{2} + \frac{K-1}{2}P_1 + (K-1)h^2(P_2 - P_a) \quad (101)$$

$$= \frac{K-1}{2}I_3. \quad (102)$$

Therefore, the signal-to-interference-plus-noise ratio of the II aligned signals of the K -user case satisfies $\text{SINR}_{II,K} \geq L_{II}\text{SINR}_{II,3}$, where $L_{II} = \frac{2}{K-1}$ and $\text{SINR}_{II,3}$ is the SINR of the 3-user case. Now, the rate constraint for decoding II aligned signals is given by $R \leq \frac{1}{2} \log\left(\frac{1}{2} + \text{SINR}_{II,3}\right) = R_{II,3}$ in the 3-user case (cf. (38)). The rate constraint in the K -user case becomes $R \leq \frac{1}{2} \log\left(\frac{1}{K-1} + \text{SINR}_{II,K}\right) = R_{II,K}$ (cf. Appendix B). Note that $R_{II,3}$ and $R_{II,K}$ can be related by using the following simple calculation

$$R_{II,K} = \frac{1}{2} \log\left(\frac{1}{K-1} + \text{SINR}_{II,K}\right) \quad (103)$$

$$\geq \frac{1}{2} \log\left(\frac{1}{K-1} + L_{II}\text{SINR}_{II,3}\right) \quad (104)$$

$$= \frac{1}{2} \log\left(L_{II}\left(\frac{1}{2} + \text{SINR}_{II,3}\right)\right) \quad (105)$$

$$= \frac{1}{2} \log(L_{II}) + R_{II,3}. \quad (106)$$

Using similar steps for the DII aligned signals, we can write $\text{SINR}_{DII,K} \geq L_{DII}\text{SINR}_{DII,3}$, and $R_{DII,K} \geq \frac{1}{2} \log(L_{DII}) + R_{DII,3}$, where $L_{DII} = \frac{3}{K}$. The quantities $\frac{1}{2} \log(L_{II})$ and $\frac{1}{2} \log(L_{DII})$ (both negative) can be interpreted as rate-loss induced by going from the 3-user to the K -user case. Similarly, the rate-loss for the DI mode can be shown to be $L_{DI} = L_{DII}$. Since $L_{DII} \leq L_{II} \leq 1$ for $K \geq 3$, we assume a worst case performance given by a rate-loss of $\frac{1}{2} \log(L_{DII})$ per message split.

Consequently, for each message split, we have a rate-loss of $\frac{1}{2} \log(L_{DII})$. Next, we examine the largest number of splits per user in order to consider the worst case. We have 3 splits in the SI1 regime (cf. Sec. V-B1), 5 in the SI2 regime (cf. Sec. V-B2), and 3 in the SI3 regime (cf. Sec. V-B3). Therefore, we have a rate-loss of $\frac{3}{2} \log(L_{DII})$ in the SI1 regime, $\frac{5}{2} \log(L_{DII})$ in the SI2 regime, and $\frac{3}{2} \log(L_{DII})$ in the SI3 regime. As a result, we obtain the following corollary.

Corollary 5. *For a symmetric K -user Gaussian IC with $1 < h^2 < P$ with sufficiently large P , the greedy-max corner point G_i is achievable within a gap of δ bits per user, where G_i is defined in (99), and*

$$\delta = \begin{cases} \frac{3}{2} \log\left(\frac{2}{3}\right) + \frac{3}{2} \log(K), & h^2 \in (1, \sqrt[3]{P}] \\ 2 \log\left(\frac{2}{3}\right) + \frac{5}{2} \log(K), & h^2 \in (\sqrt[3]{P}, \sqrt{P}] \\ \log\left(\frac{2}{3}\right) + \frac{3}{2} \log(K), & h^2 \in (\sqrt{P}, P) \end{cases} \quad (107)$$

Proof: Consider first the SI1 regime and the greedy-max corner point G_1 . According to the discussion above, in this case, each user can achieve the rate achievable in the 3-user case plus 3 times the rate-loss $\frac{1}{2} \log(L_{DII})$ since in this case, we have a maximum of 3 message splits per user. In the 3-user case, user 1 can achieve $C(P) - \frac{3}{2}$ and user $j \neq 1$ can

achieve $C\left(\frac{h^2 P}{1+P}\right) - \frac{3}{2}$ as shown in Lemma 2. Thus, in the K -user case, user 1 can achieve

$$R_1 \leq C(P) - \frac{3}{2} + \frac{3}{2} \log(L_{DII}) = C(P) - \frac{3}{2} \log\left(\frac{2}{3}\right) - \frac{3}{2} \log(K), \quad (108)$$

and similarly, user $j \neq 1$ can achieve

$$R_j \leq C\left(\frac{h^2 P}{1+P}\right) - \frac{3}{2} + \frac{3}{2} \log(L_{DII}) = C\left(\frac{h^2 P}{1+P}\right) - \frac{3}{2} \log\left(\frac{2}{3}\right) - \frac{3}{2} \log(K). \quad (109)$$

As a result, in the K -user case, G_1 is achievable within a gap of $\frac{3}{2} \log\left(\frac{2}{3}\right) + \frac{3}{2} \log(K)$ bits per user. Similarly, using (69) and (70), the gap for the SI2 case can be bounded by $2 \log\left(\frac{2}{3}\right) + \frac{5}{2} \log(K)$ bits per user, and using (73) and (73), the gap for the SI3 case can be bounded by $\log\left(\frac{2}{3}\right) + \frac{3}{2} \log(K)$ bits per user. ■

With the achievability of the greedy-max corner points within a constant proved, and knowing that the trivial corner points are achievable, and that the sum-capacity corner point is achievable within a constant gap [26], we conclude that the outer bound $\bar{C}$ given in Theorem 1 is achievable within a constant gap. Thus, the proof of Theorem 1 is completed. Note that by substituting $K = 3$ in Corollary 5, we obtain Lemma 2.

B. Sum-capacity in the SI Regime

Here, we extend Theorem 6 to the symmetric K -user IC with SI. The transmit signals remain the same as given by (151) in Appendix D. The decoding also remains essentially the same, except that we decode the sum of $K-1$ and K signals when decoding the II and the DII aligned signals, respectively (instead of 2 and 3 signals in the 3-user case). Using similar arguments as for the greedy-max corner points, we argue that the rate constraints decrease by $\frac{1}{2} \log(L_{DII})$ for each message split. Since each user sends $2\kappa + 1$ messages (cf. Sec. D-A), we have a total rate-loss of $\frac{2\kappa+1}{2} \log(L_{DII})$ bits per user. Therefore, we obtain the following corollary.

Corollary 6. *In the symmetric Gaussian K -user IC with strong interference $h^2 \in (1, P)$, the symmetric rate R given by*

$$R = \frac{1}{2}C(P + h^2 P) - \frac{1}{4} - \frac{2\kappa+1}{2} \log(2K), \quad (110)$$

is achievable, where $\kappa = \left\lfloor \frac{\log(h^2 P)}{2 \log(h^2)} \right\rfloor$.

Proof: We start by repeating the achievable rate of the 3-user case given in Theorem 6 as $\frac{1}{2}C(P + h^2 P) - \frac{1}{4} - \frac{2\kappa+1}{2} \log(6)$. By extending this achievable rate to the K -user case, we induce a rate-loss of $\frac{2\kappa+1}{2} \log(L_{DII})$, leading to the achievable rate

$$R = \frac{1}{2}C(P + h^2 P) - \frac{1}{4} - \frac{2\kappa+1}{2} (\log(6) - \log(L_{DII})) = \frac{1}{2}C(P + h^2 P) - \frac{1}{4} - \frac{2\kappa+1}{2} \log(2K),$$

Fig. 12. Achievable symmetric rates for a symmetric K -user IC at an SNR of 80dB plotted as a function of $\alpha = \frac{\log(\text{INR})}{\log(\text{SNR})}$ for $K = 5$ and $K = 25$. The plotted achievable rates are derived by using the exact rate expressions from Appendix D without approximation (similar to (168)).

which proves the corollary. ■

This completes the proof of Theorem 2. Note that by substituting $K = 3$ in Corollary 6, we obtain Theorem 6. Figure 12 shows the achievable symmetric rate of our scheme for $K \in \{3, 9, 27\}$ versus α for a channel with an SNR of 80dB. This figure shows the slow increase of the gap (logarithmic behavior) as a function of K .

VIII. CONCLUSION

We have studied the capacity region of the symmetric K -user IC in the strong interference regime. We have proposed transmission schemes for the channel which achieve its capacity within a constant gap. Interestingly, similar to the 2-user IC where the MAC bound provides the capacity region in the strong interference regime, the extension of the MAC bound to the K -user case, denoted the genie-aided MAC bound, provides the capacity region of the symmetric K -user IC with strong interference within a constant gap. To show this, we proved that our transmission schemes can achieve rate tuples which are within a constant of the corner points of an outer bound obtained from the genie-aided MAC bounds. Since the genie-aided MAC bound has a polytope structure, it suffices to show that its corner points can be approached, since any other rate tuple in the outer bound can be approached by time sharing between different corner points.

In conclusion, we have characterized the capacity region of the symmetric K -user IC with strong interference within a constant gap. Moreover, we have proposed a new transmission scheme which achieves the sum-capacity of the channel within a constant gap. This proposed scheme is an alternative to the one proposed in [26] which achieves the sum-capacity within a constant gap in the strong interference regime except for an outage set. Although our proposed scheme is similar in spirit to the GDoF achieving scheme in [22], it differs from [22] in that our scheme uses nested-lattice codes, interference alignment, successive decoding/computation, and interference cancellation. This allows us to express the achievable sum-rate

explicitly and to calculate the gap to the outer bound. While this gap is small for high interference-to-noise ratio INR, it grows as INR approaches SNR which is the border of the strong interference regime.

The paper also settles the problem of the achievability within a constant gap of the genie-aided MAC bounds in the symmetric K -user IC with strong interference, leading to a constant gap characterization of the capacity region of the K -user case.

It remains to solve the problem of the sum-capacity corner point which has not been entirely settled so far. Finding a characterization of this corner point within a gap which is a universal constant is an interesting problem. Furthermore, as the characterization of the capacity of the 2-user IC is exact in the strong interference regime, it is interesting to find an exact capacity characterization for the K -user case, or to find a tighter approximation. Another interesting direction for further research involves studying game-theoretic aspects in the K -user IC as in [42], [43], and IC's with limited channel-state information at the transmitters [44]–[46].

APPENDIX A CONVERSE OF THEOREM 3

Denote the length- N codeword sent by transmitter i by $\mathbf{x}_i^N$, also denote the received signal at receiver j by $\mathbf{y}_j^N$. Let us enhance the channel by giving $\mathbf{x}_3^N$ and $(\mathbf{y}_1^N, \mathbf{x}_1^N, \mathbf{x}_3^N)$ as side information to receivers 1 and 2, respectively. Then, using Fano's inequality, we have

$$N(R_1 - \varepsilon_{1N}) \leq I(\mathbf{x}_1^N; \mathbf{y}_1^N, \mathbf{x}_3^N), \quad (111)$$

$$N(R_2 - \varepsilon_{2N}) \leq I(\mathbf{x}_2^N; \mathbf{y}_2^N, \mathbf{y}_1^N, \mathbf{x}_1^N, \mathbf{x}_3^N), \quad (112)$$

where $\varepsilon_{1N} \rightarrow 0$ and $\varepsilon_{2N} \rightarrow 0$ as $N \rightarrow \infty$.

Using the chain rule, and since $\mathbf{x}_1^N$, $\mathbf{x}_2^N$, and $\mathbf{x}_3^N$ are independent, we get

$$N(R_1 - \varepsilon_{1N}) \leq I(\mathbf{x}_1^N; \mathbf{y}_1^N | \mathbf{x}_3^N), \quad (113)$$

$$N(R_2 - \varepsilon_{2N}) \leq I(\mathbf{x}_2^N; \mathbf{y}_2^N, \mathbf{y}_1^N | \mathbf{x}_1^N, \mathbf{x}_3^N). \quad (114)$$

We proceed with the last inequality as follows

$$N(R_2 - \varepsilon_{2N}) \leq I(\mathbf{x}_2^N; \mathbf{y}_1^N | \mathbf{x}_1^N, \mathbf{x}_3^N) + I(\mathbf{x}_2^N; \mathbf{y}_2^N | \mathbf{y}_1^N, \mathbf{x}_1^N, \mathbf{x}_3^N). \quad (115)$$

But given $\mathbf{x}_1^N$ and $\mathbf{x}_3^N$, $\mathbf{x}_2^N \rightarrow \mathbf{y}_1^N \rightarrow \mathbf{y}_2^N$ forms a Markov chain, i.e., $\mathbf{x}_2^N$ is independent of $\mathbf{y}_2^N$ given $\mathbf{y}_1^N$. Therefore, $I(\mathbf{x}_2^N; \mathbf{y}_2^N | \mathbf{y}_1^N, \mathbf{x}_1^N, \mathbf{x}_3^N) = 0$ and

$$N(R_2 - \varepsilon_{2N}) \leq I(\mathbf{x}_2^N; \mathbf{y}_1^N | \mathbf{x}_1^N, \mathbf{x}_3^N). \quad (116)$$

By adding (113) and (116), we get

$$N(R_1 + R_2 - \varepsilon_{1N} - \varepsilon_{2N}) \leq I(\mathbf{x}_1^N; \mathbf{y}_1^N | \mathbf{x}_3^N) + I(\mathbf{x}_2^N; \mathbf{y}_1^N | \mathbf{x}_1^N, \mathbf{x}_3^N) \quad (117)$$

$$= I(\mathbf{x}_1^N, \mathbf{x}_2^N; \mathbf{y}_1^N | \mathbf{x}_3^N) \quad (118)$$

$$= H(\mathbf{y}_1^N | \mathbf{x}_3^N) - H(\mathbf{y}_1^N | \mathbf{x}_1^N, \mathbf{x}_2^N, \mathbf{x}_3^N) \quad (119)$$

$$= H(\mathbf{y}_1^N | \mathbf{x}_3^N) \quad (120)$$

$$\leq H(\mathbf{y}_1^N). \quad (121)$$

Now using the chain rule, we can write $H(\mathbf{y}_1^N) = \sum_{k=1}^N H(\mathbf{y}_1(k)|\mathbf{y}_1(1), \dots, \mathbf{y}_1(k-1)) \leq \sum_{k=1}^N H(\mathbf{y}_1(k))$. But $H(\mathbf{y}_1(k)) \leq n$ since $\mathbf{y}_1(k)$ is a binary vector of length n . Thus, $H(\mathbf{y}_1^N) \leq N \cdot n$. Thus, we have

$$N(R_1 + R_2 - \varepsilon_{1N} - \varepsilon_{2N}) \leq N \cdot n. \quad (122)$$

By dividing by N , and letting $N \rightarrow \infty$, we get the desired bound $R_1 + R_2 \leq n$. The bound $R_1 \leq m$ follows by applying similar steps to $N(R_1 - \varepsilon_{1N}) \leq I(\mathbf{x}_1^N; \mathbf{y}_1^N, \mathbf{x}_2^N, \mathbf{x}_3^N)$ and noting that given $\mathbf{x}_2$ and $\mathbf{x}_3$, only m components of $\mathbf{y}_1$ are random.

APPENDIX B

NESTED-LATTICE CODE PROCESSING

In this appendix, we provide a brief introduction about nested-lattice codes (more details can be found in [24]).

A. Nested-lattices

An N -dimensional lattice Λ^f is a subset of $\mathbb{R}^N$ such that

$$\mathbf{u}_1, \mathbf{u}_2 \in \Lambda^f \Rightarrow \mathbf{u}_1 + \mathbf{u}_2 \in \Lambda^f, \quad (123)$$

i.e., it is an additive subgroup of $\mathbb{R}^N$. The fundamental Voronoi region of Λ^f denoted $\mathcal{V}(\Lambda^f)$ is the set of all points $\mathbf{x} \in \mathbb{R}^N$ such that $Q_{\Lambda^f}(\mathbf{x}) = \mathbf{0}_N$, where $Q_{\Lambda^f}(\mathbf{x})$ is a lattice quantizer which maps $\mathbf{x} \in \mathbb{R}^N$ to the closest $\lambda \in \Lambda^f$ in terms of Euclidean distance. The remainder of this quantization is defined as

$$\mathbf{x} \bmod \Lambda^f = \mathbf{x} - Q_{\Lambda^f}(\mathbf{x}). \quad (124)$$

The $\bmod \Lambda^f$ operation satisfies the following property [24]

$$[\mathbf{x} \bmod \Lambda^f + \mathbf{y}] \bmod \Lambda^f = [\mathbf{x} + \mathbf{y}] \bmod \Lambda^f \quad (125)$$

for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$. A lattice Λ^c is said to be nested in Λ^f if $\Lambda^c \subset \Lambda^f$. In this context, Λ^c is referred to as the coarse lattice and Λ^f as the fine lattice.

B. Nested-lattice codes

A nested-lattice code is constructed by using two lattices Λ^f and $\Lambda^c \subset \Lambda^f$. We denote such a nested lattice code by the pair (Λ^f, Λ^c) . The codewords of the code are chosen as the fine lattice points $\mathbf{u} \in \Lambda^f$ that lie in $\mathcal{V}(\Lambda^c)$, i.e., the codewords are $\mathbf{u} \in \Lambda^f \cap \mathcal{V}(\Lambda^c)$ (Λ^c plays the role of a shaping lattice). The power of the code is determined by the volume of $\mathcal{V}(\Lambda^c)$ and the rate of the code is defined by the nesting ratio, i.e., the number of fine lattice points in $\Lambda^f \cap \mathcal{V}(\Lambda^c)$.

In order to encode a message $m \in \{1, \dots, \lfloor 2^{NR} \rfloor\}$ (of rate R) using a nested-lattice code, we use a pair of nested lattices providing a code (Λ^f, Λ^c) with rate R and power 1. We map each index in the message set to a point in the nested lattice, i.e., m is mapped to $\mathbf{u}(m) \in \Lambda^f \cap \mathcal{V}(\Lambda^c)$. Then, we subtract a random dither vector $\mathbf{d}$, uniformly distributed over $\mathcal{V}(\Lambda^c)$ from $\mathbf{u}$, and reduce the result modulo Λ^c to get¹⁴ $(\mathbf{u} - \mathbf{d}) \bmod \Lambda^c$. Finally, the signal $\mathbf{x}$ given by

$$\mathbf{x} = \sqrt{P}[(\mathbf{u} - \mathbf{d}) \bmod \Lambda^c], \quad (126)$$

¹⁴We drop the message index m henceforth for clarity.

is transmitted where P is the power constraint of the transmitter.

Nested-lattice codes have been show in [23] to achieve the capacity of the Gaussian point-to-point (P2P) channel. That is, a rate of

$$R \leq C_{P2P} = C\left(\frac{P}{\sigma^2}\right) \quad (127)$$

is achievable over a Gaussian P2P channel with transmit power P and noise variance σ^2 . Next we explain the decodability of combinations of nested-lattice codes.

C. Decoding combinations of nested-lattice codes

Consider K nodes, with messages $m_i, i = 1 \dots, K$, where all m_i have rate R . Let these nodes use the same nested-lattice codebook (Λ^f, Λ^c) with power 1, to encode their messages into codewords $\mathbf{u}_i$ with length N . The nodes then construct their transmit signals $\mathbf{x}_i$ as

$$\mathbf{x}_i = \sqrt{P_i}[(\mathbf{u}_i - \mathbf{d}_i) \bmod \Lambda^c], \quad (128)$$

where $\mathbf{d}_i$ is an N -dimensional dither vector uniformly distributed over $\mathcal{V}(\Lambda^c)$, and P_i is the transmit power of node i which satisfies a power constraint P , i.e., $P_i \leq P$. A receiver observes

$$\mathbf{y} = \sum_{i=1}^K \mathbf{x}_i + \mathbf{z} \quad (129)$$

where $\mathbf{z}$ is an additive white Gaussian noise vector with i.i.d. components with zero mean and variance σ^2 . The receiver node knows all $\mathbf{d}_i$ and wants to decode $(\sum_{i=1}^K \mathbf{u}_i) \bmod \Lambda^c$. This decoding is possible as shown in the following lemma.

Lemma 3 ([24]). *The sum $(\sum_{i=1}^K \mathbf{u}_i) \bmod \Lambda^c$ can be decoded reliably from $\mathbf{y} = \sum_{i=1}^K \mathbf{x}_i + \mathbf{z}$ where $\mathbf{x}_i$ is defined in (128) as long as*

$$R \leq C\left(\frac{P}{\sigma^2} - \frac{K-1}{K}\right). \quad (130)$$

Proof: Set $P_i = P$ in (128) for all $i = 1, \dots, K$. Then, normalize the channel noise by dividing $\mathbf{y}$ by σ to get $\mathbf{y}' = \sum_{i=1}^K \mathbf{x}'_i + \mathbf{z}'$ where the noise $\mathbf{z}'$ has unit variance and $\mathbf{x}'_i = \frac{1}{\sigma} \mathbf{x}_i$ and has power P/σ^2 . Then, by setting $\mathbf{h} = \mathbf{a} = \mathbf{1}_K^T$ in [24, Theorem 2], the desired rate is obtained. ■

Next, we consider the decodability of the sum of lattice codewords from a weighted sum of signals $\mathbf{x}_i$. This kind of decoding is needed in the context of this paper. Suppose that the receiver observes

$$\mathbf{y} = \mathbf{x}_1 + h \sum_{i=2}^K \mathbf{x}_i + \mathbf{z}, \quad (131)$$

for some h such that $h^2 \geq 1$. In this case, the receiver can decode $(\sum_{i=1}^K \mathbf{u}_i) \bmod \Lambda^c$ as shown in the following lemma.

Lemma 4. *The sum $(\sum_{i=1}^K \mathbf{u}_i) \bmod \Lambda^c$ can be decoded reliably from $\mathbf{y} = \mathbf{x}_1 + h \sum_{i=2}^K \mathbf{x}_i + \mathbf{z}$ where $h^2 > 1$ as long as*

$$R \leq C\left(\frac{P}{\sigma^2} - \frac{K-1}{K}\right). \quad (132)$$

Proof: The main idea is to align the signals $\mathbf{x}_1$ and $\mathbf{x}_i$, $i = 2, \dots, K$, such that they have the same power at the receiver. To this end, set $P_1 = P$ and $P_i = \frac{P}{h^2}$ in (128) for $i = 2, \dots, K$. Then, the received signal $\mathbf{y}$ can be written as

$$\mathbf{y} = \mathbf{x}_1 + h \sum_{i=2}^K \mathbf{x}_i + \mathbf{z} \quad (133)$$

$$= \sqrt{P}[(\mathbf{u}_1 - \mathbf{d}_1) \bmod \Lambda^c] + h \sum_{i=2}^K \sqrt{\frac{P}{h^2}}[(\mathbf{u}_i - \mathbf{d}_i) \bmod \Lambda^c] + \mathbf{z} \quad (134)$$

$$= \sqrt{P}[(\mathbf{u}_1 - \mathbf{d}_1) \bmod \Lambda^c] + \sum_{i=2}^K \sqrt{P}[(\mathbf{u}_i - \mathbf{d}_i) \bmod \Lambda^c] + \mathbf{z} \quad (135)$$

$$= \sum_{i=1}^K \mathbf{x}'_i + \mathbf{z}, \quad (136)$$

where $\mathbf{x}'_i = \sqrt{P}[(\mathbf{u}_i - \mathbf{d}_i) \bmod \Lambda^c]$. Now by applying Lemma 3, the statement follows. ■

D. Successive computation

For the proposed transmission schemes in this paper, successive computation is required. Successive computation is similar to successive decoding mostly known as the capacity achieving scheme in the MAC [47]. In successive decoding, a received signal $\mathbf{y}$ consisting of a noisy superposition of several signals $\mathbf{x}_i$, $i = 1, \dots, K$, is treated as follows. One signal $\mathbf{x}_i$ is decoded at a time while treating the remaining signals $\mathbf{x}_{i+1}, \dots, \mathbf{x}_K$ as noise. Then, the contribution of the decoded signal is removed from $\mathbf{y}$, and the process is repeated again until all $\mathbf{x}_i$ are decoded. Successive computation is similar except that instead of decoding a signal, a function of several signals is computed. However, in computation, one does not compute the sum of transmitted signals directly, but the modulo sum of transmitted codewords (Lemmas 3 and 4). Thus, the contribution of the computed sum can not be removed from the received signal in a straightforward fashion.

This problem has been solved in [41]. Namely, [41, Lemma 1] states that the real sum of signals can be reconstructed from the decoded modulo sum of codewords as long as the computation rate is satisfied and is positive. Consider Lemma 3 for example. Here, [41, Lemma 1] states that after decoding $\bar{\mathbf{u}} = (\sum_{i=1}^K \mathbf{u}_i) \bmod \Lambda^c$, the sum $\bar{\mathbf{x}} = \sum_{i=1}^K \mathbf{x}_i$ can be reconstructed with an error probability that tends to zero as $N \rightarrow \infty$ as long as the rate constraint (130) is satisfied and is positive. This reconstruction is performed as follows. After decoding $\bar{\mathbf{u}} = (\sum_{i=1}^K \mathbf{u}_i) \bmod \Lambda^c$, the receiver subtracts the dithers from $\bar{\mathbf{u}}$ and reduces the whole modulo Λ^c to obtain

$$\begin{aligned} \left[\bar{\mathbf{u}} - \sum_{i=1}^K \mathbf{d}_i \right] \bmod \Lambda^c &= \left[\sum_{i=1}^K (\mathbf{u}_i - \mathbf{d}_i) \right] \bmod \Lambda^c \quad (137) \\ &= \left[\sum_{i=1}^K (\mathbf{u}_i - \mathbf{d}_i) \bmod \Lambda^c \right] \bmod \Lambda^c. \end{aligned}$$

We proceed by using (125), and (128) with $P_i = P$ to obtain

$$\left[\bar{\mathbf{u}} - \sum_{i=1}^K \mathbf{d}_i \right] \bmod \Lambda^c = \left[\frac{1}{\sqrt{P}} \sum_{i=1}^K \mathbf{x}_i \right] \bmod \Lambda^c \quad (138)$$

$$= \left[\frac{\bar{\mathbf{x}}}{\sqrt{P}} \right] \bmod \Lambda^c. \quad (139)$$

Thus, the receiver has obtained the quantization error of $\frac{\bar{\mathbf{x}}}{\sqrt{P}}$ when quantizing with respect to Λ^c . Then the receiver computes $Q_{\Lambda^c} \left(\alpha \frac{\mathbf{y}}{\sqrt{P}} - \left[\frac{\bar{\mathbf{x}}}{\sqrt{P}} \right] \bmod \Lambda^c \right)$ where α is the MMSE coefficient given by $\alpha = \frac{KP}{\sigma^2 + KP}$ [24]. This leads to

$$\begin{aligned} Q_{\Lambda^c} \left(\alpha \frac{\mathbf{y}}{\sqrt{P}} - \left[\frac{\bar{\mathbf{x}}}{\sqrt{P}} \right] \bmod \Lambda^c \right) \\ = Q_{\Lambda^c} \left(Q_{\Lambda^c} \left(\frac{\bar{\mathbf{x}}}{\sqrt{P}} \right) + \alpha \frac{\mathbf{z}}{\sqrt{P}} + (\alpha - 1) \frac{\bar{\mathbf{x}}}{\sqrt{P}} \right) \quad (140) \end{aligned}$$

$$= Q_{\Lambda^c} \left(\frac{\bar{\mathbf{x}}}{\sqrt{P}} \right) \quad (141)$$

where the last step follows since if the right-hand-side of (130) is positive, then the variance of $\alpha \frac{\mathbf{z}}{\sqrt{P}} + (\alpha - 1) \frac{\bar{\mathbf{x}}}{\sqrt{P}}$ is smaller than 1, and hence, the noise does not drive $Q_{\Lambda^c} \left(\frac{\bar{\mathbf{x}}}{\sqrt{P}} \right)$ closer to any other $\mathbf{r} \in \Lambda^c$ almost surely as $N \rightarrow \infty$. Thus, the receiver obtains the quantization of $\frac{\bar{\mathbf{x}}}{\sqrt{P}}$ with respect to the coarse lattice. The receiver then adds the quantized version (141) to the quantization error (139) to obtain a scaled version of the desired signal

$$\begin{aligned} Q_{\Lambda^c} \left(\frac{\bar{\mathbf{x}}}{\sqrt{P}} \right) + \left[\frac{\bar{\mathbf{x}}}{\sqrt{P}} \right] \bmod \Lambda^c \\ = Q_{\Lambda^c} \left(\frac{\bar{\mathbf{x}}}{\sqrt{P}} \right) + \frac{\bar{\mathbf{x}}}{\sqrt{P}} - Q_{\Lambda^c} \left(\frac{\bar{\mathbf{x}}}{\sqrt{P}} \right) \quad (142) \end{aligned}$$

$$= \frac{\bar{\mathbf{x}}}{\sqrt{P}} \quad (143)$$

where we used (124). Thus, by decoding the sum of codewords modulo coarse lattice, the sum of signals can be constructed. This can be also applied to Lemma 4 where it can be shown that the sum $\mathbf{x}_1 + h \sum_{i=2}^K \mathbf{x}_i$ can be reconstructed from $(\sum_{i=1}^K \mathbf{u}_i) \bmod \Lambda^c$ as long as (132) is satisfied.

APPENDIX C

SPLITTING AND ALIGNMENT PATTERN FOR ACHIEVING THE APPROXIMATE SUM-CAPACITY OF THE SYMMETRIC 3-USER IC

In this appendix, we explain the splitting and alignment strategy used to prove Theorem 6. We do this with the aid of the LD model. The same analysis can be carried out for the Gaussian case.

Let us consider a symmetric 3-user LD IC with strong interference $n > m$, and further with $\frac{n}{2(n-m)} = \frac{5}{2}$. For this IC, we need to show that each user can achieve a rate of $n/2$ bits.

Let us represent the received signals at receiver 1 given by

$$\mathbf{y}_1 = \mathbf{S}^{n-m} \mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3$$

Fig. 13. A graphical illustration of the transmitted signals (vectors) $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ as received at receiver 1 showing the evolution of the splitting and alignment strategy.

as shown in Figure 13(a). In this figure, user $j \in \{1, 2, 3\}$ sends a vector $\mathbf{t}_j$ of $n/2$ bits on his top-most levels¹⁵. We can see that receiver 1 receives interference on levels $\frac{n}{2}, \dots, m$. To recover these bits which are corrupted by interference, the transmitters can split $\mathbf{t}_j$ into $\mathbf{w}_j$ and $\mathbf{v}_j$, and repeat $\mathbf{w}_j$ on lower levels. However, this leads to interference on lower levels too as shown in Figure 13(b). Namely, in this figure, $\hat{\mathbf{w}}_2$ and $\hat{\mathbf{w}}_3$ interfere with $\mathbf{v}_1$ and $\hat{\mathbf{w}}_1$. Note that, $n - m$ bits of

¹⁵Since the considered IC is symmetric where a symmetric scheme can achieve its sum-capacity, we stick to a symmetric signalling scheme.

the interference caused by $\hat{\mathbf{w}}_2$ and $\hat{\mathbf{w}}_3$ can be removed after decoding their sum from the top-most $n - m$ bits. Due to this, the transmitters split $\mathbf{w}_j$ to $\mathbf{u}_j$ and $\mathbf{u}_{j,2}$, and only repeat $\mathbf{u}_j$ on lower levels as shown by $\hat{\mathbf{u}}_j$ in Figure 13(c).

Now, receiver 1 is able to decode $\mathbf{v}_1$ and $\hat{\mathbf{u}}_1$, but not $\mathbf{u}_{1,2}$. To ensure the decodability of $\mathbf{u}_{j,2}$, transmitter j can repeat it on a lower level (below level $\frac{4m-3n}{2}$). However, by doing so, we cause interference to $\hat{\mathbf{u}}_1$ as shown in Figure 13(d). To avoid this, the transmitters do not repeat $\mathbf{u}_{j,2}$, but repeat the part of $\mathbf{v}_j$ which interferes with $\mathbf{u}_{k,2}$ of the other users. Thus, $\mathbf{v}_j$ is split into two signals $\mathbf{v}_{j,2}$ and $\mathbf{u}_{j,3}$ as shown in Figure 13(e), where $\mathbf{u}_{j,3}$ is repeated on lower levels as $\hat{\mathbf{u}}_{j,3} = \mathbf{u}_{j,3}$.

Using this strategy, receiver 1 can decode $\mathbf{u}_2 \oplus \mathbf{u}_3$, then the sum of $\mathbf{u}_1 \oplus \begin{bmatrix} \mathbf{u}_{2,2} \oplus \mathbf{u}_{3,2} \\ \mathbf{v}_{2,2} \oplus \mathbf{v}_{3,2} \end{bmatrix}$, then the sum of $\mathbf{u}_{1,2} \oplus \mathbf{u}_{2,3} \oplus \mathbf{u}_{3,3}$. Then, using $\mathbf{u}_2 \oplus \mathbf{u}_3$, receiver 1 cancels the interference from $\hat{\mathbf{u}}_2 \oplus \hat{\mathbf{u}}_3$ and decodes $\mathbf{v}_{1,2}$ and $\mathbf{u}_{1,3}$. Then, it decodes $\hat{\mathbf{u}}_1$, and finally, it uses $\mathbf{u}_{1,2} \oplus \mathbf{u}_{2,3} \oplus \mathbf{u}_{3,3}$ to cancel the interference of $\hat{\mathbf{u}}_{2,3} \oplus \hat{\mathbf{u}}_{3,3}$, effectively replacing this interference by the desired signal $\mathbf{u}_{1,2}$ in the process. Then, receiver 1 decodes $\mathbf{u}_{1,2}$. As a result, receiver 1 was able to decode $\mathbf{v}_{1,2}$, $\mathbf{u}_{1,3}$, $\hat{\mathbf{u}}_1$, and $\mathbf{u}_{1,2}$ which accounts for $n/2$ bits as desired. Receiver 2 and 3 can also decode the same number of bits similarly. Finally, in order to match the splitting of the 3 transmitters, we split $\mathbf{u}_1$ (and consequently $\hat{\mathbf{u}}_1$) into two parts each of them aligned with one of $\mathbf{u}_{2,2}$ and $\mathbf{v}_{2,2}$. The final splitting pattern is shown in Figure 13(f).

Note that the ordering of some signals is reversed in comparison to their copies. For instance, $\mathbf{u}_{j,1}$ is above $\mathbf{v}_{j,1}$ while $\hat{\mathbf{u}}_{j,1}$ is below $\hat{\mathbf{v}}_{j,1}$. This is done due to the fact that $\mathbf{u}_{1,1}$ has to align with $\mathbf{u}_{2,2}$ of size $\ell_1 = \frac{4m-3n}{2}$, which results in $\mathbf{v}_{1,1}$ having a size of $\ell_2 = \frac{5n-4m}{2}$. The result is that $\mathbf{u}_{2,3}$ has size ℓ_1 which is equal to the size of $\mathbf{u}_{1,3}$. Thus, $\mathbf{u}_{1,3}$ and $\hat{\mathbf{v}}_{2,1}$ have different sizes and can not be aligned. Therefore, $\mathbf{u}_{1,3}$ is aligned with $\hat{\mathbf{u}}_{2,1}$ resulting in a reversed order of $\hat{\mathbf{u}}_{j,1}$ and $\hat{\mathbf{v}}_{j,1}$.

As can be seen in Figure 13(f), the signals $\mathbf{u}_{j,2}$ and $\mathbf{v}_{j,2}$ are not repeated at lower levels. Although repeating these signals is not needed, repeating them would lead to a more consistent splitting and alignment strategy. However, this repetition is harmful in the LD case. In more detail, by repeating $\mathbf{v}_{2,2}$ and $\mathbf{v}_{3,2}$ for instance, aligned with $\hat{\mathbf{v}}_{1,1}$ as shown in Figure 14(a), interference from $\hat{\mathbf{v}}_{2,2} \oplus \hat{\mathbf{v}}_{3,2}$ can not be removed any more without removing the desired signal. That is, if the receiver decodes $\mathbf{v}_{1,1} \oplus \mathbf{v}_{2,2} \oplus \mathbf{v}_{3,2}$ and adds this signal to $\hat{\mathbf{v}}_{1,1} \oplus \hat{\mathbf{v}}_{2,2} \oplus \hat{\mathbf{v}}_{3,2}$, the result is zero and the desired signal is lost. In the Gaussian IC however, this repetition is possible if we invert the sign of the signal, i.e., repeat $\mathbf{x}_{2,v[2]}$ (which is the transmit signal corresponding to the codeword $\mathbf{v}_{2,2}$) as $\hat{\mathbf{x}}_{2,v[2]} = -a\mathbf{x}_{2,v[2]}$ where a is a power allocation (scaling) parameter. In this case, after decoding $\mathbf{x}_{1,v[1]} + h\mathbf{x}_{2,v[2]} + h\mathbf{x}_{3,v[2]}$, the receiver scales it by a to obtain $a\mathbf{x}_{1,v[1]} + ah\mathbf{x}_{2,v[2]} + ah\mathbf{x}_{3,v[2]}$, and then adds it to the received signal containing $a\mathbf{x}_{1,v[1]} - ah\mathbf{x}_{2,v[2]} - ah\mathbf{x}_{3,v[2]}$ yielding $2a\mathbf{x}_{1,v[1]}$. This does not cancel the desired signal, but doubles its magnitude.

To translate this level assignment to power allocation in the Gaussian IC, we use the following method. Consider for

instance the power level $\frac{4m-3n}{2}$. Using the definition of m and n given in (13) by $m = \lceil \frac{1}{2} \log(P) \rceil$ and $n = \lceil \frac{1}{2} \log(h^2 P) \rceil$, we get

$$\frac{4m-3n}{2} = \frac{4\lceil \frac{1}{2} \log(P) \rceil - 3\lceil \frac{1}{2} \log(h^2 P) \rceil}{2} \quad (144)$$

$$\approx \frac{2\log(P) - \frac{3}{2}\log(h^2 P)}{2} \quad (145)$$

$$= \frac{1}{2} \log(\sqrt{P/h^6}). \quad (146)$$

Thus, the power corresponding to this level is $\sqrt{P/h^6}$ as shown in Figure 14(b). The remaining power levels can be obtained similarly. Finally, the power of each signal is given by the difference between its power level and the lower power level. For instance, the power of $\mathbf{x}_{1,u[1]}$ is $P - \sqrt{h^6 P}$.

APPENDIX D

APPROXIMATE SUM-CAPACITY OF THE SYMMETRIC 3-USER IC WITH STRONG INTERFERENCE: PROOF OF THEOREM 6

The scheme we describe next is an extension for the scheme given in Section VI to a general value of κ with a more rigorous calculation of the achievable rates. We start by describing the message splitting procedure.

A. Message splitting

Each user splits his message into $2\kappa + 1$ parts for some $\kappa \in \mathbb{N}$. User j splits m_j into $m_{j,u[1]}, \dots, m_{j,u[\kappa+1]}$, and $m_{j,v[1]}, \dots, m_{j,v[\kappa]}$, where the rate of $m_{j,u[i]}$ is $R_{u,i}$ and the rate of $m_{j,v[i]}$ is $R_{v,i}$, $i = 0, \dots, \kappa$. Thus

$$R_j = \sum_{i=1}^{\kappa+1} R_{u,i} + \sum_{i=1}^{\kappa} R_{v,i}. \quad (147)$$

For the purpose of this section, we set

$$\kappa = \left\lfloor \frac{\log(h^2 P)}{2 \log(h^2)} \right\rfloor. \quad (148)$$

B. Encoding

Next, each of the $m_{j,u[i]}$ messages is encoded using a nested-lattice code denoted $(\Lambda_u^f, \Lambda_u^c)$ where Λ_u^f denotes the fine lattice and Λ_u^c denotes the coarse lattice. The used nested-lattice code has power 1 and can support a rate up to $\max_i R_{u,i}$. Similarly, each of the $m_{j,v[i]}$ messages is encoded using a nested-lattice code denoted $(\Lambda_v^f, \Lambda_v^c)$ with power 1 and rate $\max_i R_{v,i}$.

Denote the nested-lattice codeword corresponding to $m_{j,u[i]}$ as $\mathbf{u}_{j,i}$, and the nested-lattice codeword corresponding to $m_{j,v[i]}$ as $\mathbf{v}_{j,i}$. Then, each of these codewords is dithered and scaled as follows

$$\mathbf{x}_{j,u[i]} = \sqrt{P_{u,i}} [(\mathbf{u}_{j,i} - \mathbf{d}_{j,i}) \bmod \Lambda_u^c] \quad (149)$$

$$\mathbf{x}_{j,v[i]} = \sqrt{P_{v,i}} [(\mathbf{v}_{j,i} - \mathbf{e}_{j,i}) \bmod \Lambda_v^c], \quad (150)$$

where $\mathbf{d}_{j,i}$ and $\mathbf{e}_{j,i}$ are random dithers, and $P_{u,i}$ and $P_{v,i}$ denote the powers allocated to $\mathbf{x}_{j,u[i]}$ and $\mathbf{x}_{j,v[i]}$, respectively.

(a) In the LD IC, repeating $\mathbf{v}_{j,2}$ as $\hat{\mathbf{v}}_{j,2} = \mathbf{v}_{j,2}$ does not enable interference cancellation without cancelling the desired signal.

(b) In the Gaussian IC, repeating $\mathbf{x}_{j,v[2]}$ as $-\frac{\mathbf{x}_{j,v[2]}}{h^4}$ enables interference cancellation without cancelling the desired signal.

Fig. 14. Signal splitting and alignment strategy: from the LD IC to the Gaussian IC.

Then, user j constructs the transmit signal $\mathbf{x}_j$ as follows

$$\mathbf{x}_j = \sum_{i=1}^{\kappa+1} (\mathbf{x}_{j,u[i]} + \hat{\mathbf{x}}_{j,u[i]}) + \sum_{i=1}^{\kappa} (\mathbf{x}_{j,v[i]} + \hat{\mathbf{x}}_{j,v[i]}), \quad (151)$$

where $\hat{\mathbf{x}}_{j,u[i]}$ and $\hat{\mathbf{x}}_{j,v[i]}$ are scaled versions of $\mathbf{x}_{j,u[i]}$ and $\mathbf{x}_{j,v[i]}$ such that

$$\hat{\mathbf{x}}_{j,u[i]} = (-1)^i \sqrt{\frac{\hat{P}_{u,i}}{P_{u,i}}} \mathbf{x}_{j,u[i]} \quad (152)$$

$$\hat{\mathbf{x}}_{j,v[i]} = (-1)^i \sqrt{\frac{\hat{P}_{v,i}}{P_{v,i}}} \mathbf{x}_{j,v[i]}, \quad (153)$$

where $\hat{P}_{u,i}$ and $\hat{P}_{v,i}$ denote the powers allocated to the repeated $\mathbf{x}_{j,u[i]}$ and $\mathbf{x}_{j,v[i]}$ signals, respectively. Here, the alter-

nating signs of these signals is used to insure that interference cancellation does not lead to desired signal cancellation as we shall see later in the decoding section. In order to satisfy the power constraint at the transmitters, we need

$$\sum_{i=1}^{\kappa+1} (P_{u,i} + \hat{P}_{u,i}) + \sum_{i=1}^{\kappa} (P_{v,i} + \hat{P}_{v,i}) \leq P. \quad (154)$$

C. Power allocation and alignment

We need to fix the powers of the signals in such a way that a desired alignment pattern is obtained. The goal is to adjust the powers so that by decoding $\mathbf{x}_{2,u[1]} + \mathbf{x}_{3,u[1]}$ and $\mathbf{x}_{2,v[1]} + \mathbf{x}_{3,v[1]}$, we can start an interference decoding and cancellation chain that ends with decoding all desired signals interference free. To this end, we set

$$P_{u,j} = \left(\frac{1}{h^2}\right)^{j-1} P - (h^2)^{\kappa-j+1} \sqrt{\frac{P}{h^2}}, \quad (155)$$

$$P_{v,i} = (h^2)^{\kappa-i+1} \sqrt{\frac{P}{h^2}} - \left(\frac{1}{h^2}\right)^i P, \quad (156)$$

$$\hat{P}_{v,i} = (h^2)^{1-i} \sqrt{\frac{P}{h^2}} - \left(\frac{1}{h^2}\right)^{\kappa+i} P, \quad (157)$$

$$\hat{P}_{u,i} = \left(\frac{1}{h^2}\right)^{\kappa+i} P - (h^2)^{-i} \sqrt{\frac{P}{h^2}}, \quad (158)$$

$$\hat{P}_{u,\kappa+1} = (h^2)^{-\kappa} \sqrt{\frac{P}{h^2}}. \quad (159)$$

with $j = 1, \dots, \kappa + 1$ and $i = 1, \dots, \kappa$. Note that this power allocation satisfies the power constraint with equality. With this power allocation, we have alignment between the three signals $\mathbf{x}_{1,u[i]}$, $\mathbf{x}_{2,u[i+1]}$, and $\mathbf{x}_{3,u[i+1]}$ at receiver 1 for $i = 1, \dots, \kappa$, i.e., these signals are received at the same power at receiver 1 (cf. Figure 15). Similarly, each of the following signal triples

$$\mathbf{x}_{1,v[i]}, \mathbf{x}_{2,v[i+1]}, \mathbf{x}_{3,v[i+1]}, \quad i = 0, \dots, \kappa - 1, \quad (160)$$

$$\mathbf{x}_{1,v[\kappa]}, \hat{\mathbf{x}}_{2,v[1]}, \hat{\mathbf{x}}_{3,v[1]}, \quad (161)$$

$$\mathbf{x}_{1,u[\kappa+1]}, \hat{\mathbf{x}}_{2,u[1]}, \hat{\mathbf{x}}_{3,u[1]}, \quad (162)$$

$$\hat{\mathbf{x}}_{1,v[i]}, \hat{\mathbf{x}}_{2,v[i+1]}, \hat{\mathbf{x}}_{3,v[i+1]}, \quad i = 0, \dots, \kappa - 1, \quad (163)$$

$$\hat{\mathbf{x}}_{1,u[i]}, \hat{\mathbf{x}}_{2,u[i+1]}, \hat{\mathbf{x}}_{3,u[i+1]}, \quad i = 0, \dots, \kappa - 1, \quad (164)$$

are aligned at receiver 1. This can be easily verified by using the power allocation in (155)-(159) to show that all signals in each triple arrive at the same power at receiver 1. Similar alignment holds at receivers 2 and 3 by symmetry.

With this power allocation, the received signal at receiver 1

can be written as

$$\begin{aligned} \mathbf{y}_1 &= \mathbf{x}_1 + h(\mathbf{x}_2 + \mathbf{x}_3) + \mathbf{z}_1 \\ &= h(\mathbf{x}_{2,u[1]} + \mathbf{x}_{3,u[1]}) + h(\mathbf{x}_{2,v[1]} + \mathbf{x}_{3,v[1]}) \\ &\quad + \sum_{i=1}^{\kappa} (\mathbf{x}_{1,u[i]} + h\mathbf{x}_{2,u[i+1]} + h\mathbf{x}_{3,u[i+1]}) \\ &\quad + \sum_{i=1}^{\kappa-1} (\mathbf{x}_{1,v[i]} + h\mathbf{x}_{2,v[i+1]} + h\mathbf{x}_{3,v[i+1]}) \\ &\quad + (\mathbf{x}_{1,v[\kappa]} + h\hat{\mathbf{x}}_{2,v[1]} + h\hat{\mathbf{x}}_{3,v[1]}) \\ &\quad + (\mathbf{x}_{1,u[\kappa+1]} + h\hat{\mathbf{x}}_{2,u[1]} + h\hat{\mathbf{x}}_{3,u[1]}) \\ &\quad + \sum_{i=1}^{\kappa-1} (\hat{\mathbf{x}}_{1,v[i]} + h\hat{\mathbf{x}}_{2,v[i+1]} + h\hat{\mathbf{x}}_{3,v[i+1]}) \\ &\quad + \sum_{i=1}^{\kappa-1} (\hat{\mathbf{x}}_{1,u[i]} + h\hat{\mathbf{x}}_{2,u[i+1]} + h\hat{\mathbf{x}}_{3,u[i+1]}) \\ &\quad + (\hat{\mathbf{x}}_{1,v[\kappa]} + h\hat{\mathbf{x}}_{2,u[\kappa+1]} + h\hat{\mathbf{x}}_{3,u[\kappa+1]}) \\ &\quad + \hat{\mathbf{x}}_{1,u[\kappa]} + \hat{\mathbf{x}}_{1,u[\kappa+1]} + \mathbf{z}_1, \end{aligned} \quad (166)$$

where the signals collected in brackets are received with the same power at the receiver (cf. Figure 15(a)).

D. Decoding

We discuss the decoding at receiver 1 only since the decoding procedure at the other receivers is similar. The decoding is split into three stages. First, the interference is decoded. Second, the repeated signals are cancelled. Finally, the desired signals are decoded.

1) *Interference decoding*: The first stage in the decoding procedure involves interference decoding. The decoding starts with the first bracket in (166), i.e., $h(\mathbf{x}_{2,u[1]} + \mathbf{x}_{3,u[1]})$. The receiver decodes the sum (mod Λ_u^c) of the nested-lattice code-words corresponding to this sum, i.e., $(\mathbf{u}_{2,1} + \mathbf{u}_{3,1}) \bmod \Lambda_u^c$. All the remaining signals are treated as noise. Thus the noise power is given by η given in (167) at the top of next page. Thus, decoding $(\mathbf{u}_{2,1} + \mathbf{u}_{3,1}) \bmod \Lambda_u^c$ is possible if

$$R_{u,1} \leq C^+ \left(\frac{h^2 P_{u,1}}{\eta} - \frac{1}{2} \right). \quad (168)$$

Now we manipulate the right hand side of (168) as follows

$$C^+ \left(\frac{h^2 P_{u,1}}{\eta} - \frac{1}{2} \right)$$

$$\stackrel{(a)}{=} \left[\frac{1}{2} \log \left(\frac{1}{2} + \frac{h^2 P - h^{2\kappa} \sqrt{h^2 P}}{1 + P + 2h^{2\kappa} \sqrt{h^2 P}} \right) \right]^+ \quad (169)$$

$$= \left[\frac{1}{2} \log \left(\frac{1 + P(1 + 2h^2)}{2(1 + P + 2h^{2\kappa} \sqrt{h^2 P})} \right) \right]^+ \quad (170)$$

$$= \left[C(P + 2h^2 P) - C(P + 2h^{2\kappa} \sqrt{h^2 P}) - \frac{1}{2} \right]^+ \quad (171)$$

$$\stackrel{(b)}{=} C(P + 2h^2 P) - C(P + 2h^{2\kappa} \sqrt{h^2 P}) - \frac{1}{2}, \quad (172)$$

where (a) follows since $\hat{P}_{u,1} + \sum_{i=2}^{\kappa+1} (P_{u,i} + \hat{P}_{u,i}) + \sum_{i=1}^{\kappa} (P_{v,i} + \hat{P}_{v,i}) = h^{2\kappa} \sqrt{\frac{P}{h^2}}$ by (155)-(159), and where

Fig. 15. Alignment pattern of the transmitted signals at receiver 1. Each rectangle represents part of the transmit signal. The transmit signal consists of the sum of these parts.

$$\begin{aligned} \eta &= 1 + \sum_{i=1}^{\kappa+1} (P_{u,i} + \hat{P}_{u,i}) + \sum_{i=1}^{\kappa} (P_{v,i} + \hat{P}_{v,i}) + 2h^2 \left(\hat{P}_{u,1} + \sum_{i=2}^{\kappa+1} (P_{u,i} + \hat{P}_{u,i}) + \sum_{i=1}^{\kappa} (P_{v,i} + \hat{P}_{v,i}) \right) \\ &= 1 + P + 2h^2 \left(\hat{P}_{u,1} + \sum_{i=2}^{\kappa+1} (P_{u,i} + \hat{P}_{u,i}) + \sum_{i=1}^{\kappa} (P_{v,i} + \hat{P}_{v,i}) \right). \end{aligned} \quad (167)$$

(b) follows by using (148) to show that $h^2P > h^{2\kappa}\sqrt{h^2P}$ which means that (171) can be made positive by increasing P sufficiently. Thus, we obtain

$$R_{u,1} \leq C(P + 2h^2P) - C(P + 2h^{2\kappa}\sqrt{h^2P}) - \frac{1}{2}. \quad (173)$$

Next, receiver 1 constructs $\mathbf{x}_{2,u[1]} + \mathbf{x}_{3,u[1]}$ from the decoded sum $(\mathbf{u}_{2,1} + \mathbf{u}_{3,1}) \bmod \Lambda_u^c$ (as described in Appendix B), and subtracts the contribution of these signals from the received signal. Then, receiver 1 decodes the sum $(\mathbf{v}_{2,1} + \mathbf{v}_{3,1}) \bmod \Lambda_v^c$, and subtracts the contribution of $\mathbf{x}_{2,v[1]} + \mathbf{x}_{3,v[1]}$ from the received signal. The decoding constraint for R_v can be written similar to above leading to

$$R_{v,1} \leq C(P + 2h^{2\kappa}\sqrt{h^2P}) - C(3P) - \frac{1}{2}. \quad (174)$$

So far, the receiver decoded the II aligned signals. Now, it starts decoding the DII aligned signals. It starts with the aligned signals $\mathbf{x}_{1,u[1]} + h\mathbf{x}_{2,u[2]} + h\mathbf{x}_{3,u[2]}$. In other words, the sum $(\mathbf{u}_{1,1} + \mathbf{u}_{2,2} + \mathbf{u}_{3,2}) \bmod \Lambda_u^c$ is decoded, and then the

contribution of $\mathbf{x}_{1,u[1]} + h\mathbf{x}_{2,u[2]} + h\mathbf{x}_{3,u[2]}$ is removed from the received signal. This decoding is possible if

$$R_{u,1}, R_{u,2} \leq C(3P) - C(3h^{2(\kappa-1)}\sqrt{h^2P}) - \frac{1}{2}\log(3). \quad (175)$$

Then, the sum $(\mathbf{v}_{1,1} + \mathbf{v}_{2,2} + \mathbf{v}_{3,2}) \bmod \Lambda_v^c$ is decoded. This decoding is possible if

$$R_{v,1}, R_{v,2} \leq C(3h^{2(\kappa-1)}\sqrt{h^2P}) - (3h^{-2}P) - \frac{1}{2}\log(3). \quad (176)$$

The receiver continues with this procedure by decoding $(\mathbf{u}_{1,i} + \mathbf{u}_{2,i+1} + \mathbf{u}_{3,i+1}) \bmod \Lambda_u^c$ then $(\mathbf{v}_{1,i} + \mathbf{v}_{2,i+1} + \mathbf{v}_{3,i+1}) \bmod \Lambda_v^c$ alternatively, from $i = 2$ up to $i = \kappa - 1$, ending this decoding chain with $(\mathbf{u}_{1,\kappa} + \mathbf{u}_{2,\kappa+1} + \mathbf{u}_{3,\kappa+1}) \bmod \Lambda_u^c$. With this, all the interference sums from users 2 and 3 has been decoded (apart from the repeated interference $\hat{\mathbf{x}}_{j,u[i]}$ and $\hat{\mathbf{x}}_{j,v[i]}$), where each decoded sum is attached to one desired signal. Before we remove the repeated interference ($\hat{\mathbf{x}}_{j,u[i]}$ and $\hat{\mathbf{x}}_{j,v[i]}$) and

decode the desired signals, let us summarize the resulting rate constraints. By combining the rate constraints in (173)-(176), and extending them to the remaining message splits i , we get

$$R_{u,1} \leq C(2P) - C\left(3h^{2(\kappa-1)}\sqrt{h^2P}\right) - \frac{1}{2}\log(3), \quad (177)$$

$$R_{v,1} \leq C\left(2h^{2(\kappa-1)}\sqrt{h^2P}\right) - C(3h^{-2}P) - \frac{1}{2}\log(3), \quad (178)$$

$$R_{u,j} \leq C\left(3h^{-2(j-1)}P\right) - C\left(3h^{2(\kappa-j)}\sqrt{h^2P}\right) - \frac{1}{2}\log(3), \quad (179)$$

$$R_{v,i} \leq C\left(3h^{2(\kappa-i)}\sqrt{h^2P}\right) - C(3h^{-2i}P) - \frac{1}{2}\log(3), \quad (180)$$

with $j = 2, \dots, \kappa + 1$ and $i = 2, \dots, \kappa$. At the end of this stage, the received signal at receiver 1 becomes as shown in Figure 15(b). Next, the receivers perform interference cancellation.

2) *Interference cancellation:* In the second decoding stage, the receiver perform interference cancellation. At this point, receiver 1 knows the signals

$$\begin{aligned} &\mathbf{x}_{2,u[1]} + \mathbf{x}_{3,u[1]}, \quad \mathbf{x}_{2,v[1]} + \mathbf{x}_{3,v[1]}, \\ &\mathbf{x}_{1,u[i]} + h\mathbf{x}_{2,u[i+1]} + h\mathbf{x}_{3,u[i+1]}, \quad i = 1, \dots, \kappa, \\ &\mathbf{x}_{1,v[i]} + h\mathbf{x}_{2,v[i+1]} + h\mathbf{x}_{3,v[i+1]}, \quad i = 1, \dots, \kappa - 1. \end{aligned}$$

The receiver then scales these signal appropriately in order to cancels the repeated interference. For instance, receiver 1 scales $\mathbf{x}_{2,u[1]} + \mathbf{x}_{3,u[1]}$ by $\sqrt{\frac{\hat{P}_{u,1}}{\hat{P}_{u,1}}}$ to obtain $\hat{\mathbf{x}}_{2,u[1]} + \hat{\mathbf{x}}_{3,u[1]}$. After scaling, receiver 1 has

$$\begin{aligned} &\hat{\mathbf{x}}_{2,u[1]} + \hat{\mathbf{x}}_{3,u[1]}, \quad \hat{\mathbf{x}}_{2,v[1]} + \hat{\mathbf{x}}_{3,v[1]}, \\ &\hat{\mathbf{x}}_{1,u[i]} - h\hat{\mathbf{x}}_{2,u[i+1]} - h\hat{\mathbf{x}}_{3,u[i+1]}, \quad i = 1, \dots, \kappa - 1, \\ &\hat{\mathbf{x}}_{1,v[i]} - h\hat{\mathbf{x}}_{2,v[i+1]} - h\hat{\mathbf{x}}_{3,v[i+1]}, \quad i = 1, \dots, \kappa - 1, \\ &\sqrt{\frac{h^2\hat{P}_{u,\kappa+1}}{\hat{P}_{u,\kappa}}}\hat{\mathbf{x}}_{1,u[\kappa]} - h\hat{\mathbf{x}}_{2,u[\kappa+1]} - h\hat{\mathbf{x}}_{3,u[\kappa+1]}. \end{aligned}$$

After cancelling the interference from $\mathbf{y}_1$, the remaining signal is given by

$$\begin{aligned} \mathbf{y}'_1 &= \mathbf{x}_{1,v[\kappa]} + \mathbf{x}_{1,u[\kappa+1]} + 2\hat{\mathbf{x}}_{1,v[1]} + 2\hat{\mathbf{x}}_{1,u[1]} + \dots \\ &\quad + 2\hat{\mathbf{x}}_{1,v[\kappa-1]} + 2\hat{\mathbf{x}}_{1,u[\kappa-1]} + \hat{\mathbf{x}}_{1,v[\kappa]} \\ &\quad + \left(\sqrt{\frac{h^2\hat{P}_{u,\kappa+1}}{\hat{P}_{u,\kappa}}} + 1\right)\hat{\mathbf{x}}_{1,u[\kappa]} + \hat{\mathbf{x}}_{1,u[\kappa+1]} + \mathbf{z}_1. \end{aligned} \quad (181)$$

At this point, $\mathbf{y}'_1$ consists of desired signals only as shown in Figure 15(c), and the decoding of the desired signals can be started.

3) *Desired signal decoding:* In this stage, the receivers decode their desired signals. First, receiver 1 decodes $\mathbf{v}_{1,\kappa}$ and $\mathbf{u}_{1,\kappa+1}$ successively in this order while treating the remaining signals as noise. This decoding is possible if (182) and (183) at the top of next page are satisfied. Receiver 1 now decodes $\mathbf{v}_{1,i}$ and $\mathbf{u}_{1,i}$ (from $\hat{\mathbf{x}}_{1,v[i]}$ and $\hat{\mathbf{x}}_{1,u[i]}$, respectively) from $i = 1$ till $i = \kappa - 1$, successively, while treating the remaining signals as noise. After each decoding, the contribution of the decoded signal is subtracted from the received signal. The decoding of

these signals is possible if (184) and (185) at the top of next page are satisfied for $i = 1, \dots, \kappa - 1$. At this point, receiver 1 has

$$\mathbf{y}''_1 = \hat{\mathbf{x}}_{1,v[\kappa]} + \left(\sqrt{\frac{h^2\hat{P}_{u,\kappa+1}}{\hat{P}_{u,\kappa}}} + 1\right)\hat{\mathbf{x}}_{1,u[\kappa]} + \hat{\mathbf{x}}_{1,u[\kappa+1]} + \mathbf{z}_1.$$

It subtracts $\hat{\mathbf{x}}_{1,v[\kappa]}$ and $\hat{\mathbf{x}}_{1,u[\kappa+1]}$ from $\mathbf{y}''_1$ (since it has already decoded $\mathbf{v}_{1,\kappa}$ and $\mathbf{u}_{1,\kappa+1}$), and then it decodes $\mathbf{u}_{1,\kappa}$ from $\hat{\mathbf{x}}_{1,u[\kappa]}$ with the rate constraint

$$R_{u,\kappa} \leq \frac{1}{2}\log\left(1 + (\sqrt{h^2\hat{P}_{u,\kappa+1}} + \sqrt{\hat{P}_{u,\kappa}})^2\right). \quad (186)$$

Now, by substituting the power allocation parameters in (155)-(159) into the rate constraints (182)-(185), and using some manipulations we can rewrite the rate constraints as

$$R_{v,\kappa} \leq C(\sqrt{h^2P}) - C(4h^{-2\kappa}P) \quad (187)$$

$$R_{u,\kappa+1} \leq C(h^{-2\kappa}P) - C(4h^{-2}\sqrt{h^2P}) \quad (188)$$

$$R_{v,i} \leq C(2h^{-2i}\sqrt{h^2P}) - C(4h^{-2(\kappa+i)}P) \quad (189)$$

$$R_{u,i} \leq C(2h^{-2(\kappa+i)}P) - C(4h^{-2(i+1)}\sqrt{h^2P}), \quad (190)$$

$$R_{u,\kappa} \leq C(h^{-2\kappa}\sqrt{h^2P}), \quad (191)$$

for $i = 1, \dots, \kappa - 1$.

E. Achievable rate

By comparing (177)-(180) and (187)-(191), we can conclude that the following rates are achievable

$$R_{u,1} \leq C(2h^{-2(\kappa+1)}P) - C(4h^{-4}\sqrt{h^2P}) - \frac{1}{2}\log(3),$$

$$R_{v,1} \leq C(2h^{-2}\sqrt{h^2P}) - C(4h^{-2(\kappa+1)}P) - \frac{1}{2}\log(3),$$

$$R_{u,i} \leq C(2h^{-2(\kappa+i)}P) - C(4h^{-2(i+1)}\sqrt{h^2P}) - \frac{1}{2}\log(3),$$

$$R_{v,i} \leq C(2h^{-2i}\sqrt{h^2P}) - C(4h^{-2(\kappa+i)}P) - \frac{1}{2}\log(3),$$

with $i = 2, \dots, \kappa - 1$, and

$$R_{u,\kappa} \leq C(h^{-2\kappa}\sqrt{h^2P}) - \frac{1}{2}\log(6),$$

$$R_{v,\kappa} \leq C(\sqrt{h^2P}) - C(4h^{-2\kappa}P) - \frac{1}{2}\log\left(\frac{3}{2}\right), \quad (192)$$

$$R_{u,\kappa+1} \leq C(h^{-2\kappa}P) - C(4h^{-2}\sqrt{h^2P}) - \frac{1}{2}\log\left(\frac{3}{2}\right). \quad (193)$$

In order to get the achievable rate per user, we have to add the above inequalities as in (147). Note that the log terms above share a similar structure. Namely, the positive log term in one inequality is similar to the negative log term in another. For instance, the positive log term in (193) given by $\frac{1}{2}\log(1 + h^{-2\kappa}P)$ is approximately equal to the negative log term in (192) given by $\frac{1}{2}\log(1 + 4h^{-2\kappa}P)$ (within a constant),

$$C(h^{-2\kappa}P) - C(4h^{-2\kappa}P) > -1. \quad (194)$$

$$R_{v,\kappa} \leq C \left(\frac{P_{v,\kappa}}{1 + P_{u,\kappa+1} + 4 \sum_{i=1}^{\kappa-1} (\hat{P}_{v,i} + \hat{P}_{u,i}) + \hat{P}_{v,\kappa} + (\sqrt{h^2 \hat{P}_{u,\kappa+1}} + \sqrt{\hat{P}_{u,\kappa}})^2 + \hat{P}_{u,\kappa+1}} \right) \quad (182)$$

$$R_{u,\kappa+1} \leq C \left(\frac{P_{u,\kappa+1}}{1 + 4 \sum_{i=1}^{\kappa-1} (\hat{P}_{v,i} + \hat{P}_{u,i}) + \hat{P}_{v,\kappa} + (\sqrt{h^2 \hat{P}_{u,\kappa+1}} + \sqrt{\hat{P}_{u,\kappa}})^2 + \hat{P}_{u,\kappa+1}} \right). \quad (183)$$

$$R_{v,i} \leq C \left(\frac{4\hat{P}_{v,i}}{1 + 4\hat{P}_{u,i} + 4 \sum_{\ell=i+1}^{\kappa-1} (\hat{P}_{v,\ell} + \hat{P}_{u,\ell}) + \hat{P}_{v,\kappa} + (\sqrt{h^2 \hat{P}_{u,\kappa+1}} + \sqrt{\hat{P}_{u,\kappa}})^2 + \hat{P}_{u,\kappa+1}} \right), \quad (184)$$

$$R_{u,i} \leq C \left(\frac{4\hat{P}_{u,i}}{1 + 4 \sum_{\ell=i+1}^{\kappa-1} (\hat{P}_{v,\ell} + \hat{P}_{u,\ell}) + \hat{P}_{v,\kappa} + (\sqrt{h^2 \hat{P}_{u,\kappa+1}} + \sqrt{\hat{P}_{u,\kappa}})^2 + \hat{P}_{u,\kappa+1}} \right), \quad (185)$$

Thus, by adding (193) and (192), we conclude that the sum

$$R_{v,\kappa} + R_{u,\kappa+1} \leq C \left(\sqrt{h^2 P} \right) - C \left(4h^{-2} \sqrt{h^2 P} \right) - \log(3)$$

is achievable. By proceeding similarly, we can show that the rate

$$R_j \leq C(\sqrt{h^2 P}) - \frac{2\kappa+1}{2} \log(6), \quad j = 1, 2, 3,$$

is achievable. Notice that we loose $\frac{1}{2} \log(6)$ bits for each message split. Since $\log(1 + \sqrt{a}) > \frac{1}{2} \log(1 + a)$, and since $h^2 P > P$, we get

$$C(\sqrt{h^2 P}) > \frac{1}{2} C(h^2 P) > \frac{1}{2} C(P + h^2 P) - \frac{1}{4}. \quad (195)$$

Therefore, the following rate is achievable

$$R_j \leq \frac{1}{2} C(P + h^2 P) - \frac{1}{4} - \frac{2\kappa+1}{2} \log(6), \quad j = 1, 2, 3.$$

This proves the desired achievable rate per user as given in Theorem 6.

APPENDIX E

RATE MANIPULATIONS FOR SECTION V-B

Here, we present the rate manipulations required in Section V-B to obtain the achievable rates of our scheme for approaching the greedy-max corner points. We start with the SI1 regime.

1) *SI1*: In the SI1 regime, after substituting the power allocation parameters

$$P_a = \frac{h^2}{1+h^2}(P-h^2) \quad \text{and} \quad P_d = h^2 \quad (196)$$

in the rate constraints (41), (44), (46), (49), (52), and (54), and comparing them, we obtain

$$R_{11} \leq C \left(\frac{h^4 P - h^6}{1 + 2h^2 + h^4 + 2h^6 + 2h^2 P} - \frac{1}{2} \right) = R_{11}^*,$$

$$R_{14} \leq C \left(\frac{h^2 P - h^4}{1 + 2h^2 + 3h^4 + 2h^6} - \frac{1}{2} \right) = R_{14}^*, \quad (197)$$

$$R_{16} \leq C \left(\frac{h^4}{1+h^2} - \frac{1}{2} \right) = R_{16}^*. \quad (198)$$

Let us now lower bound R_{11}^* as follows

$$R_{11}^* = \frac{1}{2} \log \left(\frac{1}{2} + \frac{h^4 P - h^6}{1 + 2h^2 + h^4 + 2h^6 + 2h^2 P} \right) \quad (199)$$

$$= \frac{1}{2} \log \left(\frac{1 + 2h^2 + h^4 + 2h^2 P + 2h^4 P}{2(1 + 2h^2 + h^4 + 2h^6 + 2h^2 P)} \right) \quad (200)$$

$$\stackrel{(a)}{>} \frac{1}{2} \log \left(\frac{1 + 2h^2 + h^4 + P + 2h^2 P + h^4 P}{2(1 + 2h^2 + h^4 + 2h^6 + 2h^2 P)} \right) \quad (201)$$

$$= \frac{1}{2} \log \left(\frac{(1 + h^2)^2(1 + P)}{2(1 + 2h^2 + h^4 + 2h^6 + 2h^2 P)} \right) \quad (202)$$

where (a) follows since $h^4 P > P$. We also write R_{14}^* and R_{16}^* as

$$R_{14}^* = \frac{1}{2} \log \left(\frac{1 + 2h^2 + h^4 + 2h^6 + 2h^2 P}{2(1 + h^2 + 2h^4)(1 + h^2)} \right) \quad (203)$$

$$R_{16}^* = \frac{1}{2} \log \left(\frac{1 + h^2 + 2h^4}{2(1 + h^2)} \right). \quad (204)$$

We now add the three rates to obtain the achievable rate $R_1 \leq R_1^* = R_{11}^* + R_{14}^* + R_{16}^*$. This achievable rate R_1^* is hence lower bounded by

$$R_1^* = R_{11}^* + R_{14}^* + R_{16}^* \quad (205)$$

$$> \frac{1}{2} \log \left(\frac{(1 + h^2)^2(1 + P)}{2(1 + 2h^2 + h^4 + 2h^6 + 2h^2 P)} \cdot \frac{1 + 2h^2 + h^4 + 2h^6 + 2h^2 P}{2(1 + h^2 + 2h^4)(1 + h^2)} \cdot \frac{1 + h^2 + 2h^4}{2(1 + h^2)} \right) \quad (206)$$

$$= \frac{1}{2} \log \left(\frac{1 + P}{8} \right) \quad (207)$$

$$= C(P) - \frac{3}{2}. \quad (208)$$

Thus, a rate R_1 given by

$$R_1 \leq C(P) - \frac{3}{2} \quad (209)$$

is achievable.

Now consider the rate constraint for the second user, i.e., R_{21} . By comparing (38) and (49) for the case where $P_b = P_c = 0$, we can immediately see that the latter constraint is more binding. Furthermore, by plugging the value of P_d from (196) in (54) and (57), we can directly see that the former is more binding. Therefore, the remaining three constraints are ((49), (52), and (54))

$$R_{21} \leq C \left(\frac{h^2 P_a}{1 + 2P_a + (1 + 2h^2)P_d} - \frac{1}{2} \right) = R_{21}^{[1]} \quad (210)$$

$$R_{21} \leq C \left(\frac{P_a}{1 + (1 + 2h^2)P_d} - \frac{1}{2} \right) = R_{21}^{[2]} \quad (211)$$

$$R_{21} \leq C \left(\frac{h^2 P_d}{1 + P_d} - \frac{1}{2} \right) = R_{21}^{[3]}, \quad (212)$$

where we assumed P sufficiently large so that $R_{21}^{[i]} > 0$, $i = 1, 2, 3$. By substituting the values of P_a and P_d from (196) in $R_{21}^{[1]}$, we get

$$R_{21}^{[1]} = C \left(\frac{h^4 P - h^6}{1 + 2h^2 + h^4 + 2h^2 P + 2h^6} - \frac{1}{2} \right) \quad (213)$$

$$> C \left(\frac{h^4 P - h^6}{h^2 + 2h^4 + 3h^6 + 2h^2 P} - \frac{1}{2} \right), \quad (214)$$

where the inequality follows since $h^2 > 1$. Similarly, we get

$$R_{21}^{[2]} = C \left(\frac{h^4 P - h^6}{h^2 + 2h^4 + 3h^6 + 2h^8} - \frac{1}{2} \right) \quad (215)$$

$$\stackrel{(b)}{>} C \left(\frac{h^4 P - h^6}{h^2 + 2h^4 + 3h^6 + 2h^2 P} - \frac{1}{2} \right) \quad (216)$$

$$R_{21}^{[3]} = C \left(\frac{h^4}{1 + h^2} - \frac{1}{2} \right) \quad (217)$$

$$= C \left(\frac{h^4(P - h^2)}{(1 + h^2)(P - h^2)} - \frac{1}{2} \right) \quad (218)$$

$$\stackrel{(c)}{>} C \left(\frac{h^4(P - h^2)}{2h^2 P} - \frac{1}{2} \right) \quad (219)$$

$$> C \left(\frac{h^4 P - h^6}{h^2 + 2h^4 + 3h^6 + 2h^2 P} - \frac{1}{2} \right), \quad (220)$$

where (b) follows since in this regime, $h^2 < P \Rightarrow h^8 < h^2 P$, and (c) follows since $1 < h^2 < P$. Hence, the rate

$$R_2 = R_{21} \leq C \left(\frac{h^4 P - h^6}{h^2 + 2h^4 + 3h^6 + 2h^2 P} - \frac{1}{2} \right) = R_2^*$$

is achievable. Now we lower bound R_2^* as follows

$$\begin{aligned} R_2^* &= \frac{1}{2} \log \left(\frac{1 + 2h^2 + h^4 + 2P + 2h^2 P}{1 + 2h^2 + 3h^4 + 2P} \right) - \frac{1}{2} \\ &= \frac{1}{2} \log \left(\frac{\frac{1}{2} + h^2 + \frac{h^4}{2} + P}{1 + 2h^2 + 3h^4 + 2P} + \frac{h^2 P}{1 + 2h^2 + 3h^4 + 2P} \right) \\ &\stackrel{(d)}{>} \frac{1}{2} \log \left(\frac{\frac{1}{2} + h^2 + \frac{h^4}{2} + P}{2 + 4h^2 + 2h^4 + 4P} + \frac{h^2 P}{1 + 2h^2 + 3h^4 + 2P} \right) \\ &= \frac{1}{2} \log \left(\frac{1}{2} + \frac{2h^2 P}{1 + 2h^2 + 3h^4 + 2P} \right) - \frac{1}{2} \\ &\stackrel{(e)}{>} \frac{1}{2} \log \left(\frac{2}{7} + \frac{2h^2 P}{7 + 7P} \right) - \frac{1}{2} \end{aligned}$$

$$= C \left(\frac{h^2 P}{1 + P} \right) - \frac{1}{2} \log(7),$$

where (d) follows by increasing the denominator of the first fraction by $1 + 2h^2 - h^4 + 2P$ which is positive since $P > h^6 > h^4$ in this regime, and (e) follows since $1 + 2h^2 + 3h^4 + 2P < 1 + 7P < 7 + 7P$ which is true since $P > h^6 > h^4 > h^2$. Therefore, the rate

$$R_2 \leq C \left(\frac{h^2 P}{1 + P} \right) - \frac{1}{2} \log(7), \quad (221)$$

is achievable. By symmetry, a similar rate constraint holds for R_3 .

2) *SI2*: In the SI2 regime, we have

$$P_a = P - h^4, \quad P_b = h^4 - \frac{P}{h^2}, \quad P_c = 0, \quad \text{and} \quad P_d = \frac{P}{h^4}. \quad (222)$$

Now, we substitute these powers in (41)-(46), and (49)-(54). By substituting in (41) and (49), we get

$$R_{11} \leq C \left(\frac{h^2 P - h^6}{h^2 + h^6} \right) = R_{11}^{[1]} \quad (223)$$

$$R_{11} \leq C \left(\frac{h^2 P - h^6}{1 + h^2 + h^4 + 2h^6 - \frac{P}{h^2}} - \frac{1}{2} \right) = R_{11}^{[2]}. \quad (224)$$

Note that $R_{11}^{[1]}$ satisfies

$$R_{11}^{[1]} = C \left(\frac{h^2 P - h^6}{h^2 + h^6} \right) \quad (225)$$

$$\stackrel{(a)}{>} C \left(\frac{h^2 P - h^6}{1 + h^2 + h^4 + 2h^6 - \frac{P}{h^2}} \right) \quad (226)$$

$$> R_{11}^{[2]}, \quad (227)$$

where (a) follows by increasing the denominator by $1 + h^4 + h^6 - \frac{P}{h^2}$ which is positive since $h^6 > P$ in the SI2 regime. Therefore, the rate

$$R_{11} \leq C \left(\frac{h^2 P - h^6}{1 + h^2 + h^4 + 2h^6 - \frac{P}{h^2}} - \frac{1}{2} \right) = R_{11}^* \quad (228)$$

is achievable. Similarly, we can show that the rates

$$R_{12} \leq C \left(\frac{h^6 - P}{1 + h^2 + h^4 + 2P - \frac{P}{h^2}} - \frac{1}{2} \right) = R_{12}^* \quad (229)$$

$$R_{14} \leq C \left(\frac{P - h^4}{1 + h^2 + 3h^4 - \frac{P}{h^2}} - \frac{1}{2} \right) = R_{14}^* \quad (230)$$

$$R_{15} \leq C \left(\frac{h^4 - \frac{P}{h^2}}{1 + h^2 + 2\frac{P}{h^2}} - \frac{2}{3} \right) = R_{15}^* \quad (231)$$

$$R_{16} \leq C \left(\frac{\frac{P}{h^2}}{1 + h^2} - \frac{1}{2} \right) = R_{16}^*, \quad (232)$$

are achievable. By adding R_{11}^* to R_{16}^* in (228)-(232), we get the achievable rate R_1^* given by

$$R_1^* = R_{11}^* + R_{12}^* + R_{14}^* + R_{15}^* + R_{16}^* \quad (233)$$

$$= \frac{1}{2} \log \left(\frac{1 + h^2 + h^4 + 2h^2 P - \frac{P}{h^2}}{2^4(3)(1 + h^2)} \right) \quad (234)$$

$$= \frac{1}{2} \log \left(\frac{1 + h^2 + h^4 + 2h^2P - \frac{P}{h^2}}{1 + h^2} \right) - 2 - \frac{1}{2} \log(3)$$

$$= C \left(\frac{h^4 + 2h^2P - \frac{P}{h^2}}{1 + h^2} \right) - 2 - \frac{1}{2} \log(3) \quad (235)$$

$$\stackrel{(b)}{>} C \left(\frac{2h^2P}{1 + h^2} \right) - 2 - \frac{1}{2} \log(3) \quad (236)$$

$$\stackrel{(c)}{>} C \left(\frac{P + h^2P}{1 + h^2} \right) - 2 - \frac{1}{2} \log(3) \quad (237)$$

$$= C(P) - 2 - \frac{1}{2} \log(3), \quad (238)$$

where (b) follows since $h^2 > \sqrt[3]{P} \Rightarrow h^4 > \frac{P}{h^2}$, and (c) follows since $h^2 > 1$. This implies that the rate

$$R_1 \leq C(P) - 2 - \frac{1}{2} \log(3) \quad (239)$$

is achievable.

Similarly, by plugging the powers in (222) into (38) (49), (52), (54), and (57), we can show that a rate R_{21} which satisfies

$$R_{21} \leq C \left(\frac{h^2P - h^6}{1 + h^2 + h^4 + 2h^6 - \frac{P}{h^2}} - \frac{1}{2} \right) = R_{21}^{[1]} \quad (240)$$

$$R_{21} \leq C \left(\frac{h^2P - h^6}{h^4 + 3h^6 - P} - \frac{1}{2} \right) = R_{21}^{[2]} \quad (241)$$

$$R_{21} \leq C \left(\frac{h^2P}{h^2(1 + h^2)} - \frac{1}{2} \right) = R_{21}^{[3]} \quad (242)$$

is achievable. The three quantities $R_{21}^{[1]}$, $R_{21}^{[2]}$, and $R_{21}^{[3]}$ can be lower bounded by

$$R_{21}^{[1]}, R_{21}^{[2]}, R_{21}^{[3]} > C \left(\frac{h^2P}{1 + h^2 + h^4 + 3h^6} - \frac{1}{2} \right). \quad (243)$$

Therefore, the rate

$$R_{21} \leq C \left(\frac{h^2P - h^6}{1 + h^2 + h^4 + 3h^6} - \frac{1}{2} \right) = R_{21}^* \quad (244)$$

is achievable. Similarly, by plugging the powers into (39), (50), (53), and (56), we can show that a rate R_{22} satisfying

$$R_{22} \leq C \left(\frac{h^6 - P}{1 + h^2 + h^4 + 2P - \frac{P}{h^2}} - \frac{1}{2} \right) = R_{22}^{[1]} \quad (245)$$

$$R_{22} \leq C \left(\frac{h^6 - P}{h^2 + h^4 + 2P} - \frac{2}{3} \right) = R_{22}^{[2]} \quad (246)$$

$$R_{22} \leq C \left(\frac{h^6 - P}{h^4 + P} - \frac{1}{2} \right) = R_{22}^{[3]} \quad (247)$$

is achievable. Again, $R_{22}^{[1]}$, $R_{22}^{[2]}$, and $R_{22}^{[3]}$ can be lower bounded by

$$R_{22}^{[1]}, R_{22}^{[2]}, R_{22}^{[3]} > C \left(\frac{h^6 - P}{1 + h^2 + h^4 + 2P} - \frac{2}{3} \right) \quad (248)$$

leading to the achievable rate

$$R_{22} \leq C \left(\frac{h^6 - P}{1 + h^2 + h^4 + 2P} - \frac{2}{3} \right) = R_{22}^* \quad (249)$$

are achievable. By adding R_{21}^* and R_{22}^* ((244) and (249)), we get

$$R_2^* = R_{21}^* + R_{22}^* \quad (250)$$

$$= \frac{1}{2} \log \left(\frac{1 + h^2 + h^4 + h^6 + 2h^2P}{2(1 + h^2 + h^4 + 3h^6)} \cdot \frac{1 + h^2 + h^4 + 3h^6 - P}{3(1 + h^2 + h^4 + 2P)} \right) \quad (251)$$

$$\stackrel{(d)}{>} \frac{1}{2} \log \left(\frac{1 + h^2 + h^4 + h^6 + 2h^2P}{2(1 + h^2 + h^4 + 3h^6)} \cdot \frac{1 + h^2 + h^4 + 2h^6}{3(1 + h^2 + h^4 + 2P)} \right) \quad (252)$$

$$> \frac{1}{2} \log \left(\frac{1 + h^2 + h^4 + h^6 + 2h^2P}{9(1 + h^2 + h^4 + 2P)} \right) \quad (253)$$

$$\stackrel{(d)}{>} \frac{1}{2} \log \left(\frac{\frac{3}{2}(1 + P + h^2P)}{9(4 + 4P)} \right) \quad (254)$$

$$= \frac{1}{2} \log \left(1 + \frac{h^2P}{1 + P} \right) - \frac{3}{2} - \frac{1}{2} \log(3) \quad (255)$$

where (d) follows since $h^6 > P \geq h^4$ in this regime. Therefore, the rate

$$R_2 \leq \frac{1}{2} \log \left(1 + \frac{h^2P}{1 + P} \right) - \frac{3}{2} - \frac{1}{2} \log(3) \quad (256)$$

is achievable. A similar rate is achievable for user 3 by symmetry.

3) *SI3*: In this regime, we have

$$P_a = P_d = 0, \quad P_b = P - h^2, \quad \text{and} \quad P_c = h^2 - \frac{P}{h^2} - 1. \quad (257)$$

By substituting these powers into (42) and (50), we can obtain

$$R_{12} \leq C \left(\frac{h^2P - h^4}{h^4 + h^2} \right) = R_{12}^{[1]} \quad (258)$$

$$R_{12} \leq C \left(\frac{h^2P - h^4}{P + 2h^4 + 1} - \frac{1}{2} \right) = R_{12}^{[2]}. \quad (259)$$

Note that the quantity $R_{12}^{[1]}$ can be lower bounded as follows

$$R_{12}^{[1]} = C \left(\frac{h^2P - h^4}{h^4 + h^2} \right) \quad (260)$$

$$\stackrel{(a)}{>} C \left(\frac{h^2P - h^4}{P + 2h^4 + 1} - \frac{1}{2} \right) = R_{12}^{[2]}, \quad (261)$$

where (a) follows by adding $P - h^2 + h^4 + 1$, which is positive since $h^2 < P$ in this regime, to the denominator. Thus, the rate R_{12} given by

$$R_{12} \leq C \left(\frac{h^2P - h^4}{1 + P + 2h^4} - \frac{1}{2} \right) \quad (262)$$

is achievable. Similarly, by plugging the powers (257) in (43), (51), (45), and (53), we get the achievable rates

$$R_{13} \leq C \left(\frac{h^4 - P + h^2}{1 - 2h^2 + 3P} - \frac{1}{2} \right) \quad (263)$$

$$R_{15} \leq C \left(\frac{P - h^2}{1 + h^2} - \frac{2}{3} \right). \quad (264)$$

By adding the right-hand sides of (262)-(264), we can show that R_1^* is achievable where

$$R_1^* = \frac{1}{2} \log \left(\frac{1+P+2h^2P}{2(1+2h^4+P)} \cdot \frac{1+2h^4+P}{2(1-2h^2+3P)} \cdot \frac{1-2h^2+3P}{3(1+h^2)} \right) \quad (265)$$

$$= \frac{1}{2} \log \left(\frac{1+P+2h^2P}{1+h^2} \right) - \frac{1}{2} \log(12) \quad (266)$$

$$> \frac{1}{2} \log \left(\frac{1+h^2+P+h^2P}{1+h^2} \right) - \frac{1}{2} \log(12) \quad (267)$$

$$= \frac{1}{2} \log(1+P) - \frac{1}{2} \log(12). \quad (268)$$

Thus,

$$R_1 \leq \frac{1}{2} \log(1+P) - \frac{1}{2} \log(12) \quad (269)$$

is achievable. On the other hand, by substituting the power allocation parameters (257) into the rate constraints for transmitter 2 (39), (40), (50), (51), (53), (55), and (56), and using similar manipulations, we can show that the following rates are achievable

$$R_{22} \leq C \left(\frac{h^2P - h^4}{1+2h^4+P} - \frac{2}{3} \right) \quad (270)$$

$$R_{23} \leq C \left(\frac{h^2 + h^4 - P}{1-2h^2+3P} - \frac{1}{2} \right). \quad (271)$$

By adding the right-hand sides of (270)-(271), we can show that the rate R_2^* is achievable, where

$$R_2^* = \frac{1}{2} \log \left(\frac{1-h^4+P+3h^2P}{3(1+2h^4+P)} \cdot \frac{1+2h^4+P}{2(1-2h^2+3P)} \right) \quad (272)$$

$$= \frac{1}{2} \log \left(\frac{1-h^4+P+3h^2P}{1-2h^2+3P} \right) - \frac{1}{2} \log(6) \quad (273)$$

$$> \frac{1}{2} \log \left(\frac{1+P+2h^2P}{1+3P} \right) - \frac{1}{2} \log(6) \quad (274)$$

$$> \frac{1}{2} \log \left(\frac{1+\frac{3}{2}P+\frac{3}{2}h^2P}{1+3P} \right) - \frac{1}{2} \log(6) \quad (275)$$

$$= \frac{1}{2} \log \left(\frac{1+\frac{3}{2}P}{1+3P} + \frac{\frac{3}{2}h^2P}{1+3P} \right) - \frac{1}{2} \log(6) \quad (276)$$

$$> \frac{1}{2} \log \left(\frac{\frac{1}{2}+\frac{3}{2}P}{1+3P} + \frac{\frac{3}{2}h^2P}{3+3P} \right) - \frac{1}{2} \log(6) \quad (277)$$

Therefore, the rate

$$R_2 \leq \frac{1}{2} \log \left(1 + \frac{h^2P}{1+P} \right) - \frac{1}{2} \log(12), \quad (278)$$

is achievable. A similar rate can be achieved by user 3.

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