Source-receiver two-way wave extrapolation for prestack exploding-reflector mode and migration

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SUMMARY

While most of the modern seismic imaging methods perform imaging by separating input data into parts (shot gathers), we develop a formulation that is able to incorporate all available data at once while numerically propagating the recorded multidimensional wavefield backward in time. While computationally extensive, this approach has the potential of generating accurate images, free of artifacts associated with conventional approaches. We derive novel high-order partial differential equations in source-receiver-time domain. The fourth order nature of the extrapolation in time has four solutions two of which correspond to the ingoing and outgoing P-waves and reduces to the zero-offset exploding-reflector solutions when the source coincides with the receiver. Using asymptotic approximations, we develop an approach to extrapolating the full prestack wavefield forward or backward in time.

INTRODUCTION

Wave-equation depth migration methods are commonly divided into two types: one-way for wave extrapolation in depth and two-way for wave extrapolation in time or reverse-time migration (Biondi, 2006; Etgen et al., 2009). Conventionally, both methods are applied on individual shot gathers. With the one-way approach, it is also possible to combine all data (multiple shot gathers) into one wave-extrapolation procedure with the survey-sinking or DSR (double-square-root) formulation of the wave equation (Claerbout, 1985; Popovici, 1996; de Hoop et al., 2003). With modern 3-D data, DSR migration appears feasible only for single-azimuth data under the common-azimuth approximation (Biondi and Palacharla, 1996). However, it is important to preserve it in the arsenal of theoretically possible imaging methods, because one day its computational cost may become affordable for a more general case. Another limitation of the DSR formulation is the one-way nature of wave extrapolation, which limits the imaging accuracy at large structural dips. Extended coordinate systems may help (Sava and Fomel, 2005a; Shragge, 2008) but do not solve the issue completely.

In this paper, we extend the survey-sinking approach to extrapolation in time rather than depth. The possibility of such extension was first indicated by Duchkov and de Hoop (2009). We develop a constructive theory for two-way shot-receiver extrapolation in the form of a mixed-domain space-wavenumber operator, which leads

to a second-order in time and fourth-order in space partial differential equation. In practice, a two-way source-receiver extrapolation operator can be implemented with either finite-difference or spectral operators (Fomel et al., 2010; Song and Fomel, 2010).

Application of two-way extrapolators to modeling and migration follows the *exploding reflector* concept (Loewenthal et al., 1976; Claerbout, 1985). In the modeling mode, the source-receiver wavefield is initialized with signal at zero time and zero offset and corresponds to the reflectivity strength. The reflection wavefield propagates forward in time and from zero to non-zero offsets and is recorded at the surface. In the migration mode, we start with the full recorded wavefield at the surface and propagate it backward in time. The image is extracted then at zero time from the zero offset slice or, alternatively, image gathers are formed from non-zero time slices or non-zero offsets (Fomel, 2004; Sava and Fomel, 2005b, 2006; Sava and Vasconcelos, 2010). Unlike one-way survey sinking extrapolation, the two-way wave extrapolation does not impose any limits on the extrapolation direction.

THEORY

A seismic survey $P(t, \mathbf{s}, \mathbf{r}, z)$ is a function of time t and source and receiver locations $\mathbf{s}$ and $\mathbf{r}$ at the surface at depth z . Our goal is to extrapolate the four-dimensional (six-dimensional in 3-D) wavefield $P(t, \mathbf{s}, \mathbf{r}, z)$ in time. We denote the total number of dimensions as $n + 1$.

Let $\mathbf{x}$ represent the space coordinates $\mathbf{x} = \{\mathbf{s}, \mathbf{r}, z\}$ and consider a mixed-domain space-wavenumber operator for time stepping

$$P(t + \Delta t, \mathbf{x}) = \int \hat{P}(t, \mathbf{k}) e^{i\phi(\mathbf{x}, \mathbf{k}, \Delta t)} d\mathbf{k}, \quad (1)$$

where $\hat{P}(t, \mathbf{k})$ is the n -dimensional spatial Fourier transform of $P(t, \mathbf{x})$:

$$\hat{P}(t, \mathbf{k}) = \frac{1}{(2\pi)^n} \int P(t, \mathbf{x}) e^{-i\mathbf{k} \cdot \mathbf{x}} d\mathbf{x}. \quad (2)$$

In the geometrical (high-frequency) approximation, the function $\phi(\mathbf{x}, \mathbf{k}, t)$ appearing in equation (1) should satisfy the appropriate eikonal equation, which is, in the case of prestack data, the double-square-root (DSR) equation

$$\begin{aligned} \frac{\partial \phi}{\partial z} &= \sqrt{\frac{1}{v^2(s, z)} \left(\frac{\partial \phi}{\partial t} \right)^2 - |\nabla_s \phi|^2} \\ &+ \sqrt{\frac{1}{v^2(r, z)} \left(\frac{\partial \phi}{\partial t} \right)^2 - |\nabla_r \phi|^2}, \end{aligned} \quad (3)$$

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where $v(x, z)$ is the velocity of the medium. The DSR eikonal equation was first studied in the Russian literature by Belonosova and Alekseev (1967) and got applied, in the form of a pseudo-differential dispersion relation (Claerbout, 1985), to prestack migration (Yilmaz, 1979; Popovici, 1996; de Hoop et al., 2003).

In the case of VTI (vertical transverse isotropy) or TTI (tilted transverse isotropy), the DSR equation can be effectively extended with the help of the acoustic approximation of Alkhalifah (1998). For simplicity, we limit the following discussion to the isotropic case.

Equation (3) admits a form free from square roots:

$$v_-^4 \phi_t^4 + 2v_s^2 v_r^2 [v_-^2 (\phi_s^2 - \phi_r^2) - v_+^2 \phi_z^2] \phi_t^2 + v_s^4 v_r^4 [(\phi_s^2 + \phi_r^2 + \phi_z^2)^2 - 4\phi_s^2 \phi_r^2] = 0 \quad (4)$$

where we use the shortened notation $v_\xi = v(\xi, z)$, $\phi_\xi = |\nabla_\xi \phi|$, $v_+^2 = v_s^2 + v_r^2$, $v_-^2 = v_s^2 - v_r^2$.

If we consider equation (4) as an evolution in time, the initial condition for it is

$$\phi(\mathbf{x}, \mathbf{k}, 0) = \mathbf{k} \cdot \mathbf{x}, \quad (5)$$

which simply turns equation (1) into an inverse Fourier transform from wavenumber to space coordinates.

For small steps Δt , it is appropriate to replace ϕ with its Taylor series approximation

$$\phi(\mathbf{x}, \mathbf{k}, t) \approx \mathbf{k} \cdot \mathbf{x} + \phi_0(\mathbf{x}, \mathbf{k}) t. \quad (6)$$

Substituting equation (6) into (4), expanding the latter into a Taylor series, and leaving only the terms linear in t , we arrive at the following equation for ϕ_0 :

$$v_-^4 \phi_0^4 + 2v_s^2 v_r^2 [v_-^2 (k_s^2 - k_r^2) - v_+^2 k_z^2] \phi_0^2 + v_s^4 v_r^4 [(k_s^2 + k_r^2 + k_z^2)^2 - 4k_r^2 k_s^2] = 0, \quad (7)$$

whose solution is the square root of

$$\phi_0^2 = \frac{v_r^2 v_s^2 [(k_s^2 + k_r^2 + k_z^2)^2 - 4k_r^2 k_s^2]}{v_+^2 k_z^2 - v_-^2 (k_s^2 - k_r^2) + 2D}, \quad (8)$$

where

$$D = \sqrt{k_z^2 [v_r^2 v_s^2 (k_s^2 + k_r^2 + k_z^2) - v_r^4 k_r^2 - v_s^4 k_s^2]}. \quad (9)$$

In the $V(z)$ case, $v_r = v_s = v$, and equation (8) reduces to

$$\phi_0^2 = \frac{v^2 [(k_s + k_r)^2 + k_z^2] [(k_s - k_r)^2 + k_z^2]}{4k_z^2}. \quad (10)$$

In the zero-offset case, $v_r = v_s = v$, $k_s = k_r = k_x$ and the solution reduces to

$$\phi_0 = \pm \frac{v}{2} \sqrt{k_x^2 + k_z^2}, \quad (11)$$

which is appropriate for post-stack exploding-reflector reverse-time migration (Baysal et al., 1983).

Our final approximation for the wave extrapolation operator, valid for small Δt , is

$$P(t + \Delta t, \mathbf{x}) \approx \int \hat{P}(t, \mathbf{k}) e^{\pm i \phi_0(\mathbf{x}, \mathbf{k}) \Delta t + i \mathbf{k} \cdot \mathbf{x}} d\mathbf{k} \quad (12)$$

or

$$P(t + \Delta t, \mathbf{x}) + P(t - \Delta t, \mathbf{x}) \approx 2 \int \hat{P}(t, \mathbf{k}) \cos[\phi_0(\mathbf{x}, \mathbf{k}), \Delta t] e^{i \mathbf{k} \cdot \mathbf{x}} d\mathbf{k}. \quad (13)$$

Equation (13) is too expensive to implement directly. To implement it numerically, one can further approximate it with finite-difference or Fourier-domain operators. We explore one avenue for a constructive approximation in the next section.

EXTRAPOLATION ANALYSIS

The main concern with equation 8 is the behavior of ϕ_0^2 and specifically whether it is negative resulting in an imaginary ϕ_0 or contains singularities, thus, not representing a wave. Considering that we have complex conjugate solutions given by the sign of ϕ_0 one of the solutions corresponds to an ingoing wave and the other to an outgoing wave, and an imaginary ϕ_0 will insure that one of the solutions grows exponentially. Thus, our objective here is to locate the regions where ϕ_0 is imaginary or singular and try to avoid them in any application of the equation.

We start by analyzing the dependency of the ϕ_0 on the difference between v_s and v_r as equation 7 point to a clear singularity when their equal. Considering we have two sets of solutions given by equation 8 distinguished by the sign in front of D in the denominator. We assume that a_1 corresponds to ϕ_0 when the sign in front of D is positive and a_2 corresponds to ϕ_0 when the sign in front of D is negative. Figure 1 shows values of a_1 (the

Figure 1: Plots of a_1 (solid curve), and a_2 (dashed curve), from equation (8), as a function of v_r with $v_s = 2$ km/s, $k_r = -40 \text{ km}^{-1}$, $k_s = 40 \text{ km}^{-1}$ and $k_z = 40 \text{ km}^{-1}$. Areas in which the curve is missing correspond to imaginary values of a_1 and a_2 and thus unstable solutions. For comparison, the right side plot displays the phase corresponding to an equivalent medium zero-offset solution.

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solid curve), and a_2 (the dashed curve) for a range of v_r around a constant $v_s=2$ km/s for test parameters, $k_r = -40\text{km}^{-1}$, $k_s = 40\text{km}^{-1}$ and $k_z = 40\text{km}^{-1}$. We can obviously notice that this example corresponds to a 45 degree source and receiver rays. The value of a_1 near $v_s = v_r$ is stable as this is perturbation of the zero offset solution, while a_2 has a singularity at that point, and thus, tends to infinity when $v_s \approx v_r$. This implies that the solution corresponding to a_1 represents an infinitely fast traveling wave that will cause problems in any finite difference implementation. In addition, both a_1 and a_2 are imaginary when v_r is either less than 1.25 km/s or larger than 3.25 km/s. For the zero offset case, the equation 7 is second-order the corresponding solution is shown on the right side of Figure 1.

Our analysis shows that the fourth-order time extrapolation implemented in the space domain will most likely be unstable regardless of medium parameters. Thus, we need to derive a second-order space-domain extrapolation using the stable solution. This can be achieved by Fourier type solutions, asymptotic expansions around the singularity, or as we see next extrapolation of approximated solutions for small angle.

THE SECOND ORDER FORM

The singularity in equation (8) is a source of instability of the source-receiver extended wave equation, specifically when the source and receiver velocities are equal, which is always the case for zero-offset data and can be encountered in many situations otherwise. To circumvent this problem we must use approximations that allow us to extrapolate only the stable solutions. This can be done by expanding square root D in the denominator of equation (8) for only the stable solution. Recognizing that large k_z values correspond to shallow dips and small offsets, we asymptotically expand equation (8) of the stable solution (a_2) in terms of k_z , stopping at the first term to obtain:

$$\phi_0^2 \approx \frac{v_r^2 v_s^2 [(k_s^2 + k_r^2 + k_z^2)^2 - 4k_s^2 k_r^2]}{-(v_s + v_r)^2 k_z^2}. \quad (14)$$

which corresponds, by back-tracing our derivation, to the following second-order-in-time fourth-order-in-space partial differential equation

$$(v_r + v_s)^2 \frac{\partial^2}{\partial t^2} \nabla_z^2 P = -v_r^2 v_s^2 (\nabla^4 P - 4 \nabla_r^2 \nabla_s^2 P),$$

where $\nabla_\xi = \partial/\partial\xi$, and $\nabla^2 = \nabla_s^2 + \nabla_r^2 + \nabla_z^2$. Equation (15) is readily available for a variety of finite-difference or finite-element numerical schemes.

Other approximations are possible, but they will yield more complex PDEs. One possibility is approximation

$$\phi_0^2 \approx \frac{v_r^3 v_s^3 [(k_s^2 + k_r^2 + k_z^2)^2 - 4k_r^2 k_s^2]}{(v_r + v_s)^2 [(k_s^2 + k_r^2 + k_z^2) v_r v_s - k_r^2 v_r^2 - k_s^2 v_s^2]},$$

which corresponds to the following PDE:

$$(v_r + v_s)^2 \frac{\partial^2}{\partial t^2} (v_r v_s \nabla^2 P - v_r^2 \nabla_r^2 P - v_s^2 \nabla_s^2 P) = v_r^3 v_s^3 (\nabla^4 P - 4 \nabla_r^2 \nabla_s^2 P), \quad (15)$$

The best feature of equation (15) that it is stable for $v_s = v_r$ as Figure 2 (dashed curve) demonstrates that approximate ϕ_0 is real and finite for all velocities and it provides a good approximation to the exact solution (solid curve) around $v_s = v_r$.

Figure 2: The phase ϕ_0 for the exact stable solution (solid curve), same as in Figure 1, and that for the approximation of the stable solution equation (14). For this test source, receiver and depth wavenumbers are set to equal $k_s = 10\text{km}^{-1}$, $k_r = -10\text{km}^{-1}$ and $k_z = 10\text{km}^{-1}$.

Examining the approximation 14) over a wide range of angles and offsets represented by k_s and k_r , Figure 3 shows the percent error of the phase ϕ_0 for the case where $k_z = 10\text{km}^{-1}$ is relatively small as compared to k_s and k_r . As expected, the errors in the middle for zero dip and offset is zero and increases with increasing dip and offset. However, for the case of small difference between $v_s = 2\text{km/s}$ and $v_r = 2.05\text{km/s}$ the errors are generally small. In fact, equation 14) yields the exact form for the case in which $v_s = v_r$, and errors increase as we depart from this condition.

A fast way to test equation (15) is to Fourier transform in z , which yields a conventional second-order PDE in time that can be solved using finite difference methods. The initial condition is a pulse in the middle of the $s = r$ (zero-offset) plane at time equal zero. Figure 4 shows a snap shot of the wave field as a function of the source and receiver coordinates for a single k_z given by 50km^{-1} (left) and 100km^{-1} (right). To get the full wavefield as a function of depth we inverse Fourier transform the slices from k_z to z . To obtain the synthetic wavefield at the surface ($z = 0$) we simply sum up the wavefields for each time step. As expected, the shape has an overall pyramid shape associated with the DSR formula (Fowler, 1997). Of course, this type of implementation in the

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Figure 3: A contour map of the percent error of ϕ_0 using equation (14) as a function of the source k_s and receiver k_r wavenumbers for $k_z = 10 \text{ km}^{-1}$, $v_s = 2 \text{ km/s}$, and $v_r = 2.05 \text{ km/s}$ (a) and $v_r = 2.2 \text{ km/s}$ (b).

depth wavenumber domain assumes no vertical velocity variation. However, we use it here for demonstration purposes. Also, since this approach is efficient compared to a 3-D finite difference needed for equation (15), we can entertain the possibility of combining this approach with a phase-shift-plus-interpolation type methodology to handle vertical inhomogeneity.

Figure 4: A snap shot of the wavefield at 1 seconds for using a finite difference approximation of the depth wavenumber version of equation (15) for $k_z = 50 \text{ km}^{-1}$ (left) and $k_z = 100 \text{ km}^{-1}$ (right). The initial wavefield contains a band limited impulse at source and receiver location of 0.8 km and depth 0.5 km (center of the model).

MODELING EXAMPLE

One of the main applications of the new formulation is prestack modeling using the exploding reflector assumption. thus, we extrapolate the wavefield in time as function of source and receiver locations. For simplicity we consider a homogeneous medium with velocity equal to 2000 m/s in the French model shown in Figure 5.

Using equation (12) to extrapolate the wavefield in Figure 5 forward in time we obtain the Prestack wavefield at depth, $z=0$, shown in Figure 6. Clearly, all the elements of prestack data including diffractions are apparent.

CONCLUSIONS

We have developed a partial differential equation that represents the wavefield evolution as a function of time in the prestack domain. To guarantee stability, the equation is approximated with a second-order form in which we extrapolate the desired ingoing and outgoing zero-offset equivalent solutions. An asymptotic expansion of the stable solution yields a second order form that is generally accurate for small dips.

Figure 5: A section containing the zero offset reflectivity wavefield at time equal zero in a homogeneous model with velocity equal to 2000 m/s.

Figure 6: A cube plot of the resulting prestack data at the surface using the proposed extrapolation method and the initial wavefield at time equal zero shown in Figure 5. The lines crossing each section of the cube represent the location of the corresponding slices.

EDITED REFERENCES

Note: This reference list is a copy-edited version of the reference list submitted by the author. Reference lists for the 2010 SEG Technical Program Expanded Abstracts have been copy edited so that references provided with the online metadata for each paper will achieve a high degree of linking to cited sources that appear on the Web.

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