A linearized dispersion relation for orthorhombic pseudo-acoustic modeling

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SUMMARY

Wavefield extrapolation in acoustic orthorhombic anisotropic media suffers from wave-mode coupling and stability limitations in the parameter range. We introduce a linearized form of the dispersion relation for acoustic orthorhombic media to model acoustic wavefields. We apply the lowrank approximation approach to handle the corresponding space-wavenumber mixed-domain operator. Numerical experiments show that the proposed wavefield extrapolator is accurate and practically free of dispersions. Further, there is no coupling of qSv and qP waves, because we use the analytical dispersion relation. No constraints on Thomsen's parameters are required for stability. The linearized expression may provide useful application for parameter estimation in orthorhombic media.

INTRODUCTION

Nowadays, a growing number of seismic modeling and imaging techniques are being developed to handle wave propagation in transversely isotropic media (TI). Such anisotropic phenomena are typical in sedimentary rocks, in which the process of lithification usually produces identifiable layering. Tectonic movement of the crust may rotate the rocks and tilt the natural vertical preference of the symmetry axis (VTI), causing a tilted TI (TTI) anisotropy. In addition, tectonic stresses may also fracture rocks, inducing another TI with a symmetry axis parallel to the stress direction and usually normal to the sedimentation-based TI. The combination of these effects can be readily represented by an orthorhombic model with three mutually orthogonal planes of mirror symmetry; each plane functions as an independent TI model. Realization of the importance of orthorhombic models mainly comes from observation of seismic velocity azimuthal variations in flat-layered rocks, which may indicate useful fracture properties of reservoirs (Tsvankin and Grechka, 2011).

Wavefields in anisotropic media are well described by the anisotropic elastic-wave equation. However, in practice, we often have little information about shear waves and even less enthusiasm to deal with vector wavefields, especially if our objective is imaging and attaining the structure of the subsurface. Alkhalifah (2000) derived an acoustic scalar wave equation for VTI media by simply setting the S-wave velocity to zero, which provided accurate kinematics for the conventional elastic wavefield. Later on, Alkhalifah (2003) followed the same approach and introduced an acoustic wave equation of the sixth order in vertical orthorhombic media. Fowler and King (2011) presented coupled systems of partial differential equations to control pseudo-acoustic wave propagation in orthorhombic media by extending their previous work in TI media (Fowler et al., 2010). Zhang and Zhang

(2011) extended self-adjoint differential operators in TTI media (Zhang et al., 2011) and the stable TTI acoustic system (Duveneck and Bakker, 2011) to vertical and tilted orthorhombic media. Song and Alkhalifah (2012a,b) provided a dispersion relation for orthorhombic anisotropic media (Alkhalifah, 2003) and introduce a mixed-domain acoustic wave extrapolator for time marching in orthorhombic media.

In this abstract, we present a linearized solution of the dispersion relation for orthorhombic anisotropic media. The expression is derived from solving a quartic power dispersion relation using perturbation theory. Lowrank approximation (Fomel et al., 2010, 2012) is adopted to handle the resulting mixed-domain operator. We demonstrate using synthetic examples that the accuracy operator is kinematically accurate, compared to ray tracing results.

THEORY

Acoustic Wave Extrapolation

The acoustic wave equation is widely used in forward seismic modeling and reverse-time migration (Bednar, 2005; Etgen et al., 2009):

$$\frac{\partial^2 p}{\partial t^2} = v(\mathbf{x})^2 \nabla^2 p, \quad (1)$$

where $p(\mathbf{x}, t)$ is the seismic pressure wavefield and $v(\mathbf{x})$ is the wave propagation velocity.

Assuming the model is homogeneous $v(\mathbf{x}) \equiv v_0$, after Fourier transform in space, we get the following explicit expression in the wavenumber domain:

$$\frac{d^2 \hat{p}}{dt^2} = -v_0^2 |\mathbf{k}|^2 \hat{p}, \quad (2)$$

where

$$\hat{p}(\mathbf{k}, t) = \int_{-\infty}^{+\infty} p(\mathbf{x}, t) e^{i\mathbf{k} \cdot \mathbf{x}} d\mathbf{x}. \quad (3)$$

Equation (2) has the following analytical solution:

$$\hat{p}(\mathbf{k}, t + \Delta t) = e^{\pm i|\mathbf{k}|v_0\Delta t} \hat{p}(\mathbf{k}, t), \quad (4)$$

which leads to the following well-known expression (Etgen, 1989; Soubaras and Zhang, 2008):

$$P(\mathbf{x}, t + \Delta t) \approx \int \hat{P}(\mathbf{k}, t) e^{i[\mathbf{k} \cdot \mathbf{x} + v_0 |\mathbf{k}| \Delta t]} d\mathbf{k}, \quad (5)$$

Equation (5) provides an exact and efficient solution in the case of a constant-velocity medium with the aid of FFT. When seismic wave velocity varies in the medium, equation (5) turns into

a reasonable approximation by replacing v_0 with $v(\mathbf{x})$. However, FFT can no longer be applied directly to evaluate the inverse Fourier transform, because a space-wavenumber mixed-domain term appears in the integral operation:

$$W(\mathbf{x}, \mathbf{k}) = e^{i[v(\mathbf{x})|\mathbf{k}|\Delta t]} \quad (6)$$

As a result, a straight-forward numerical implementation of wave extrapolation in a variable velocity medium with mixed-domain matrix 6 will increase the cost from $O(N_x \log N_x)$ to $O(N_x^2)$, the original cost for the homogeneous case, in which N_x is the total size of the three-dimensional space grid. A number of numerical methods (Etgen and Brandsberg-Dahl, 2009; Liu et al., 2009; Zhang and Zhang, 2009; Du et al., 2010; Fomel et al., 2010, 2012; Song and Fomel, 2011; Song et al., 2011, 2012) were proposed to overcome this mixed-domain problem. In the case of orthorhombic acoustic modeling, we derive a linearized phase operator $\phi(\mathbf{x}, \mathbf{k})$ to replace $|\mathbf{k}|v(\mathbf{x})$ of the isotropic model.

Dispersion Relation for Orthorhombic Anisotropic Media

In transversely isotropic (TI) media, the model is fully characterized by five elastic parameters and density. In orthorhombic media, nine elastic parameters and density are needed to describe the elastic model. The stiffness tensor c_{ijkl} for an orthorhombic model can be represented, using the compressed two-index Voigt notation, as follows:

$$\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix}. \quad (7)$$

Tsvankin (2005) defines nine parameters determined from the above stiffness tensor:

$$\begin{aligned} v_v &= \sqrt{\frac{c_{33}}{\rho}} \\ v_{s1} &= \sqrt{\frac{c_{55}}{\rho}} \\ v_{s2} &= \sqrt{\frac{c_{44}}{\rho}} \\ v_{s3} &= \sqrt{\frac{c_{66}}{\rho}} \\ v_1 &= \sqrt{\frac{c_{13}(c_{13} + 2c_{55}) + c_{33}c_{55}}{\rho(c_{33} - c_{55})}} \\ v_2 &= \sqrt{\frac{c_{23}(c_{23} + 2c_{44}) + c_{33}c_{44}}{\rho(c_{33} - c_{44})}} \\ \eta_1 &= \frac{c_{11}(c_{33} - c_{55})}{2c_{13}(c_{13} + 2c_{55}) + 2c_{33}c_{55}} - \frac{1}{2} \\ \eta_2 &= \frac{c_{11}(c_{33} - c_{55})}{2c_{13}(c_{13} + 2c_{55}) + 2c_{33}c_{55}} - \frac{1}{2} \\ \delta &= \frac{(c_{12} + c_{66})^2 - (c_{11} - c_{66})^2}{2c_{11}(c_{11} - c_{66})}, \end{aligned} \quad (8)$$

where v_v is the P-wave vertical velocity, v_{s1} and v_{s2} are the S-wave vertical velocity polarized in the x_1 and x_2 directions, v_{s3} is the S-wave horizontal velocity polarized in the x_2 direction, but propagating in the x_1 direction, v_1 and v_2 are the NMO P-wave velocity for horizontal reflectors in the $[x_1, x_3]$ and $[x_2, x_3]$

planes, and η_1 , η_2 , and δ are the anisotropic parameters in the $[x_1, x_3]$, $[x_2, x_3]$, and $[x_1, x_2]$ planes.

A general form for the Christoffel equation in 3D anisotropic media is given by the following (Chapman, 2004):

$$\Gamma_{ik}(x_s, \mathbf{p}) = a_{ijkl}(x_s)p_j p_l - \delta_{ik}, \quad (9)$$

with $p_j = \frac{\partial \tau}{\partial x_j}$ and $a_{ijkl} = \frac{c_{ijkl}}{\rho}$, where p_j are the components of the phase vector $\mathbf{p}$, τ is the travel-time along the ray, ρ is the density, $x_s, s = 1, 2, 3$ are Cartesian coordinates for position along the ray, and δ_{ik} is the Kronecker delta function.

Alkhalifah (1998) pointed out that setting S-wave velocity to zero does not compromise the accuracy in traveltimes computations for TI media. This conclusion holds for orthorhombic media as well. Alkhalifah (2003) showed that the kinematics of wave propagation is well described by the acoustic approximation. Meanwhile, the dynamics of wave propagation also benefits from this approximation.

In orthorhombic media, Christoffel equation (9) reduces to the following form if v_{s1} , v_{s2} , and v_{s3} are set to zero:

$$\begin{bmatrix} p_1^2 v_1^2 \xi_1 - 1 & \gamma p_1 p_2 v_1^2 \xi_1 & p_1 p_3 v_1 v_v \\ \gamma p_1 p_2 v_1^2 \xi_1 & p_2^2 v_2^2 \xi_2 - 1 & p_2 p_3 v_2 v_v \\ p_1 p_3 v_1 v_v & p_2 p_3 v_2 v_v & p_3^2 v_v^2 - 1 \end{bmatrix}, \quad (10)$$

where $\xi_1 = 1 + 2\eta_1$ and $\xi_2 = 1 + 2\eta_2$.

We take the determinant of matrix (10) and set the determinant to zero. After replacing p_1 with $\frac{k_x}{\phi}$, p_2 with $\frac{k_y}{\phi}$, and p_3 with $\frac{k_z}{\phi}$, we can obtain a sixth-order polynomial equation in ϕ . Song and Alkhalifah (2012a,b) provide one root that corresponds to outgoing P-waves in acoustic media. Let $v_1 = v$, $\eta_1 = \eta$, $\Delta v = v_2 - v_1$, $\Delta \eta = \eta_2 - \eta_1$, $\gamma = \sqrt{1 + 2\delta}$, and $\Delta \gamma = \gamma - 1$. Then Δv , $\Delta \eta$ and $\Delta \gamma$ are considered small and independent, and thus we consider the following trial solution:

$$\phi = \phi_0 + \phi_v \Delta v + \phi_\eta \Delta \eta + \phi_\gamma \Delta \gamma. \quad (11)$$

Inserting the trial solution, into the cubic (in ϕ^2) dispersion relation,

$$\begin{aligned} & -\phi^6 + 2v^3 v_v^2 (v + \Delta v) k_x^2 k_y^2 k_z^2 (1 + \Delta \gamma) (1 + 2\eta) \\ & + \phi^4 (2v^2 \eta k_x^2 + v^2 k_x^2 + 2k_y^2 (v + \Delta v)^2 (\Delta \eta + \eta) + k_y^2 (v + \Delta v)^2 + v_v^2 k_z^2) \\ & + \phi^2 (v^4 k_x^2 k_y^2 (2(\Delta \gamma + 1)\eta + \Delta \gamma + 1)^2 - 2v^2 v_v^2 \eta k_x^2 k_z^2 \\ & - v^2 (2\eta + 1) k_x^2 k_y^2 (v + \Delta v)^2 (2(\Delta \eta + \eta) + 1) \\ & - 2v_v^2 k_y^2 k_z^2 (v + \Delta v)^2 (\Delta \eta + \eta)) = 0, \end{aligned} \quad (12)$$

we solve for the coefficients of the trial solution given by

$$\begin{aligned} \phi_0 &= \frac{1}{\sqrt{2}} \sqrt{k_z^2 v_v^2 + q + \sqrt{k_z^4 v_v^4 - 2(k_x^2 + k_y^2) k_z^2 v^2 v_v^2 (2\eta - 1) + q^2}}, \\ \phi_v &= \frac{k_y^2 v (k_x^2 v^2 (1 + 2\eta) - \phi_0^2) (-2k_z^2 v_v^2 \eta + (1 + 2\eta) \phi_0^2)}{u}, \\ \phi_\eta &= \frac{k_y^2 v^2 (-k_x^2 v^2 (1 + 2\eta) \phi_0^2 + \phi_0^4 - k_z^2 v_v^2 (-2k_x^2 v^2 \eta + \phi_0^2))}{u}, \end{aligned}$$

$$\phi_\gamma = \frac{k_x^2 k_y^2 v^4 (1+2\eta) (-2k_z^2 v^2 \eta + (1+2\eta) \phi_0^2)}{u},$$

$$q = (k_x^2 + k_y^2) v^2 (1+2\eta),$$

$$u = -2q \phi_0^3 + 3\phi_0^5 + 2k_z^2 v^2 \phi_0 ((k_x^2 + k_y^2) v^2 \eta - \phi_0^2).$$

This solution reduces to the isotropic P -wave solution when we set $v_v = v$, $\Delta v = 0$, $\eta = 0$, $\Delta\eta = 0$, and $\Delta\gamma = 0$, in which ϕ in expression (11) is then given by $|\mathbf{k}|v$, which is the same dispersion relation in isotropic media as that shown in equation (6).

We numerically solve equation 12 and compare the desired solution (which we consider as the exact solution within the limits of our numerical accuracy) with our perturbation based solution. Figure 1 shows the phase operator, ϕ , errors as a function of horizontal wavenumbers for (left) a moderate orthorhombic anisotropy, and (right) a stronger orthorhombic anisotropy. In both cases, the vertical wavenumber $k_z = 1 \text{ km}^{-1}$ and errors are highest along the y -axis wavenumber, as the perturbation is with respect to a VTI medium corresponding to the x -axis, where errors are zero. Nevertheless the errors, even for the stronger orthorhombic ($\Delta\eta = 0.2$) anisotropy, are small.

Figure 1: Errors in the approximate perturbation-based phase operator, ϕ , as a function of the horizontal wavenumber for (left) an orthorhombic medium with $\eta_2 = 0.15$, $v_1 = 2.05 \text{ km/s}$, and $v_2 = 2.1 \text{ km/s}$, and (right) an orthorhombic medium with $\eta_2 = 0.3$, $v_1 = 2.1 \text{ km/s}$, and $v_2 = 2.2 \text{ km/s}$. In both cases, $\eta_1 = 0.1$, $v_v = 2.0 \text{ km/s}$, $\gamma = 1.1$, and $k_z = 1 \text{ km}^{-1}$.

Tilted Orthorhombic Anisotropy

Tectonic movement of the crust may rotate the rocks and tilt the natural vertical preference of the symmetry axis, causing a tilted anisotropy. In the case of tilted orthorhombic media, k_x , k_y , and k_z need to be replaced by $\hat{k}_x$, $\hat{k}_y$, and $\hat{k}_z$, which are wavenumbers evaluated in a rotated coordinate system aligned with the symmetry axis:

$$\begin{aligned} \hat{k}_1 &= k_1 \cos \phi + k_2 \sin \phi \\ \hat{k}_2 &= -k_1 \sin \phi \cos \theta + k_2 \cos \phi \cos \theta + k_3 \sin \theta \\ \hat{k}_3 &= k_1 \sin \phi \sin \theta - k_2 \cos \phi \sin \theta + k_3 \cos \theta, \end{aligned} \quad (13)$$

where θ is the tilt angle measured with respect to vertical and ϕ is the angle between the projection of the symmetry axis in the horizontal plane and the original X -coordinate. The symmetry axis has the direction of $\{-\sin \theta \sin \phi, -\sin \theta \cos \phi, \cos \theta\}$.

NUMERICAL EXAMPLES

We use the mixed-domain operator of ϕ in equation (11) as the dispersion relation for orthorhombic media. Considering inhomogeneous media, we choose lowrank approximation (Fomel et al., 2010) to implement the mixed-domain operator. With lowrank approximation, cost can be reduced to $O(NN_x \log N_x)$, where N_x is the model size and N is a small number.

Figure 2(a)–2(c) shows wavefield snapshots (depth, inline, and crossline) in a vertical orthorhombic medium with constant parameters: $v_v = 2 \text{ km/s}$, $v_1 = 2.1 \text{ km/s}$, $v_2 = 2.05 \text{ km/s}$, $\eta_1 = 0.3$, $\eta_2 = 0.1$, and $\gamma = 1$. The time-step size is 1 ms. Because $\gamma = 1$, the depth slice is elliptical, whereas the inline and crossline display different diamond shapes, indicating different VTI properties. In Figures 2(a)–2(c), dashed lines are calculated using ray tracing. Note that the dashed lines match the wavefront from the lowrank method very well.

Our next example is wavefield snapshots in a 45° tilted orthorhombic model ($\theta = \phi = 45^\circ$) with smoothly varying velocity— v_1 : 1500–3088 m/s, v_2 : 1500–3686 m/s, v_v : 1500–3474 m/s, $\eta_1 = 0.3$, $\eta_2 = 0.1$, and $\gamma = 1.03$. The time-step is 4 ms. The Courant number for Courant-Friedrichs-Lewy (CFL) condition is around 0.65. Figure 3(a)–3(c) shows corresponding wavefield snapshots in depth, inline, and crossline slices through the central source location. The inline section (Figure 3(b)) displays the strongest anisotropic property, because η_1 is as large as 0.3. Note that the snapshots are free of dispersion and that there is no coupling of qSV and qP waves in the middle. Lowrank parameters were $M = 8$ and $N = 7$. Therefore, the cost is 7 FFTs at each time step.

CONCLUSIONS

We derive and adopt a dispersion relation for acoustic orthorhombic media so as to model seismic wavefields in such media. To handle the space-wavenumber mixed-domain operator, we apply lowrank approximation to reduce computational cost. Numerical experiments show that the proposed wavefield extrapolator is accurate. There is no coupling of qSV and qP in the wavefield snapshots because we use the exact dispersion relation. In addition, our approach yields practically dispersion-free wavefields, and it is also free of instability issues with Thomsen's parameters.

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Depth Slice at 1s

(a)

X Slice at 1s

(b)

Y Slice at 1s

(c)

Figure 2: Three slices of the wavefield snapshot at 1 second in a vertical orthorhombic medium: (a) Depth Slice; (b) Inline Slice; (c) Crossline Slice. Also plotted are yellow curves

Depth Slice

(a)

Inline Slice

(b)

Crossline Slice

(c)

Figure 3: Wavefield snapshots in an orthorhombic medium with 45° tilted symmetry axis ($\theta = \phi = 45^\circ$) and variable velocity: (a) Depth Slice; (b) Inline Slice; (c) Crossline Slice.

EDITED REFERENCES

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