Effective anisotropy through traveltimes and amplitude matching

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SUMMARY

Introducing anisotropy to seismic wave propagation reveals more realistic physics of our Earth's subsurface as compared to the isotropic assumption. However wavefield modeling, the engine of seismic inverse problems, in anisotropic media still suffers from computational burdens, in particular with complex anisotropy such as transversely isotropic (TI) and Orthorhombic anisotropy. We develop effective isotropic velocity and density models to package the effects of anisotropy such that the wave propagation behavior using these effective models approximate those of the original anisotropic model. We build these effective models through the high frequency asymptotic approximation based on the eikonal and transport equations. We match the geometrical behavior of the wavefields, given by traveltimes, from the anisotropic and isotropic eikonal equations. This matching yields the effective isotropic velocity that approximates the kinematics of the anisotropic wavefield. Equivalently, we calculate the effective densities by equating the anisotropic and isotropic transport equations. The effective velocities and densities are then fed into the isotropic acoustic variable density wave equation to obtain cheaper anisotropic wavefields. We justify our approach by testing it on an elliptical anisotropic model. The numerical results demonstrate a good matching of both traveltimes and amplitude between anisotropic and effective isotropic wavefields.

INTRODUCTION

Seismic anisotropy has justified its indispensable position in seismic migration and processing given the fact that incorporating anisotropy outputs better interpretable images (Tsvankin, 2001). However for more general nonlinear seismic inverse problems like migration velocity analysis (MVA) and full waveform inversion (FWI), many are still contending with the isotropic assumption of the Earth. This is partly due to two factors. One is by using the anisotropic parameters, we significantly increase the degrees of freedom in the inverse problems; the other barrier that hampers the utility of anisotropy is the computational cost of modeling in an anisotropic world, especially in the case of inverse problems where modeling is the building block for the inverse problem.

Starting from the acoustic approximation of anisotropy introduced by Alkhalifah (2000), numerous pseudo-acoustic or qP wave equations have been derived to account for anisotropy (Du et al., 2008; Zhan et al., 2011; Zhang et al., 2008). Despite the variety, such wave equations all suffer from the complexity in formulations and the high computational cost compared to isotropic wavefield extrapolation. To alleviate this cost challenge, Alkhalifah et al. (2013) describe an approach to embed anisotropy in the inhomogeneity of an effective velocity. The approach they take only matches the first arrival kinematics of the anisotropic and isotropic wavefields, which are governed

by their corresponding eikonal equations in the high frequency asymptotic sense. They also claim that the wavefields at later arrivals match well in practice. However, in their approach, they ignore the dynamic part of the wavefields, which tend to be crucial in amplitude related imaging techniques, and especially in inversion. We extend the effective anisotropy travel-time fitting to dynamic matching, thereby accounting for amplitude matching of the two wavefields.

The essence of this traveltimes and amplitude matching idea is solving anisotropic eikonal and transport equations to get effective velocities and densities. We, therefore, need efficient and accurate eikonal and transport solvers for general anisotropic media. Eikonal equations are often formulated into static Hamilton-Jacobi (HJ) equations then are solved for weak solutions known as viscosity solutions (Crandall and Lions, 1983). These solutions happen to be first arrivals of our wavefields. Such efficient numerical solvers for these boundary value nonlinear systems include Dijkstra-like methods e.g. fast marching method (Sethian, 1996) and fast sweeping method (Zhao, 2005). Another category of eikonal solvers is through solving "time"-dependent HJ equations using the level-set idea (Osher, 1993). Paraxial formulations (Qian and Symes, 2001) and fixed-point iterations (Zhang et al., 2006) all fall into this category. Nevertheless the convergence speed of these "time"-dependent HJ solvers are restricted by the CFL conditions, which are usually slow.

The transport equations from first-order high frequency asymptotic series are often linear and can also be solved through HJ formulations. Because second-order derivatives of travel-time appear in the transport equations, we need at least third-order accurate eikonal solutions to achieve a first order accurate amplitude (Qian and Symes, 2002). We, however, in this abstract provide another way for amplitude matching rather than solving the transport equations twice (once for anisotropic and the other for isotropic media). The traveltimes matching is done through solving the eikonal equations with a third-order WENO (Weighted Essentially Non-Oscillatory) scheme fast sweeping method. The effective velocity together with the effective density are then used to extrapolate isotropic wavefields, which approximate the corresponding anisotropic wavefields both kinematically and dynamically.

THEORY

For simplicity, we focus on acoustic vertically transversely isotropic (VTI) media (Alkhalifah, 2000) to illustrate this concept. Extensions to other anisotropic models are possible and can be treated in a similar manner.

VTI eikonal and transport equations

The full VTI *P*-wave phase velocity written in Thomsen pa-

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rameters (Tsvankin, 1996) has the form of

$$\frac{V^2(\theta)}{V_{P0}^2} = 1 + \varepsilon \sin^2 \theta - \frac{f}{2} + \frac{f}{2} \sqrt{1 + \frac{4 \sin^2 \theta}{f} (2\delta \cos^2 \theta - \varepsilon \cos 2\theta) + \frac{4\varepsilon^2 \sin^4 \theta}{f^2}}, \quad (1)$$

where

$$f \equiv 1 - V_{S0}^2/V_{P0}^2 = 1 - c_{44}/c_{33} \quad (2)$$

is the only term containing the S-wave vertical velocity. Alkhalifah (1998) proposed an acoustic approximation by setting S-wave vertical velocity V_{S0} to zero and reformulating the phase velocity using parameters V_{nmo} and η . The 2D VTI eikonal equation is recovered by substituting $\sin \theta/V_{P0} = \partial \tau / \partial x$ and $\cos \theta/V_{P0} = \partial \tau / \partial z$ into phase velocity formula, thus has the form of

$$V_{nmo}^2 (1 + 2\eta) \left(\frac{\partial \tau}{\partial x} \right)^2 + V_{P0}^2 \left(\frac{\partial \tau}{\partial z} \right)^2 \left(1 - 2\eta V_{nmo}^2 \left(\frac{\partial \tau}{\partial x} \right)^2 \right) = 1. \quad (3)$$

Equation (3) degenerates to the isotropic eikonal equation:

$$|\nabla \tau|^2 = 1/V^2, \quad (4)$$

if we set $\eta = 0$, $V_{nmo} = V_{P0} = V$, where V is the isotropic velocity.

Using the dispersion relation $\mathbf{k} = \omega \mathbf{p} = \omega \nabla \tau$ and taking inverse Fourier transform in wavenumbers for equation (3) yields the 2D acoustic VTI wave equation

$$\omega^2 V_{P0}^2 \frac{\partial^2 U}{\partial z^2} + 2\eta V_{nmo}^2 V_{P0}^2 \frac{\partial^4 U}{\partial x^2 \partial z^2} + \omega^2 (1 + 2\eta) V_{P0}^2 \frac{\partial^2 U}{\partial x^2} + \omega^4 U = 0. \quad (5)$$

The above frequency domain wave equation (5) has a trial solution in the high frequency asymptotic sense given by the Debye series (Bleistein et al., 2001)

$$U(\mathbf{x}, \omega) \sim e^{i\omega \tau(\mathbf{x})} \sum_{n=0}^{\infty} \frac{A_n(\mathbf{x})}{(i\omega)^n}. \quad (6)$$

Substitute a zeroth-order term of the series into equation (5) and collect coefficients of ω^3 then equate them to zero, we will arrive at the constant density VTI transport equation

$$\begin{aligned} & \left((1 + 2\eta) V_{nmo}^2 - 2\eta V_{nmo}^2 V_{P0}^2 \left(\frac{\partial \tau}{\partial z} \right)^2 \right) \times \left(2 \frac{\partial A}{\partial x} \frac{\partial \tau}{\partial x} + A \frac{\partial^2 \tau}{\partial x^2} \right) \\ & + V_{P0}^2 \left(2 \frac{\partial A}{\partial z} \frac{\partial \tau}{\partial z} + A \frac{\partial^2 \tau}{\partial z^2} \right) \\ & - 2\eta V_{nmo}^2 V_{P0}^2 \left(A \frac{\partial^2 \tau}{\partial z^2} + 2 \frac{\partial A}{\partial z} \frac{\partial \tau}{\partial z} \right) \times \left(\frac{\partial \tau}{\partial x} \right)^2 \\ & - 8\eta V_{nmo}^2 V_{P0}^2 A \frac{\partial \tau}{\partial z} \times \left(\frac{\partial \tau}{\partial x} \frac{\partial^2 \tau}{\partial x \partial z} \right) = 0. \end{aligned} \quad (7)$$

Setting $\eta = 0$, $V_{nmo} = V_{P0} = V$, equation (7) degenerate to constant density isotropic transport equation:

$$\nabla \cdot (A^2 \nabla \tau) = 0. \quad (8)$$

A variable density version of the isotropic transport equation can be similarly derived and is given by

$$\nabla \cdot \left(\frac{A^2}{\rho} \nabla \tau \right) = 0. \quad (9)$$

By recognizing the consistent form of the isotropic transport equations (8) and (9), we now give the conservation form of equation (7), which is crucial in our development of effective density without evaluating the amplitude. If we ignore the spatial derivative of V_{nmo} , V_{P0} and η , we can arrive at the conservation form of the VTI transport equation:

$$\nabla \cdot A^2 \begin{pmatrix} \left((1 + 2\eta) V_{nmo}^2 - 2\eta V_{nmo}^2 V_{P0}^2 \left(\frac{\partial \tau}{\partial z} \right)^2 \right) \frac{\partial \tau}{\partial x} \\ \left((1 + 2\eta) V_{nmo}^2 - 2\eta V_{nmo}^2 V_{P0}^2 \left(\frac{\partial \tau}{\partial z} \right)^2 \right) \frac{\partial \tau}{\partial y} \\ \left(V_{P0}^2 - 2\eta V_{nmo}^2 V_{P0}^2 \left(\left(\frac{\partial \tau}{\partial x} \right)^2 + \left(\frac{\partial \tau}{\partial y} \right)^2 \right) \right) \frac{\partial \tau}{\partial z} \end{pmatrix} = 0. \quad (10)$$

Equation (10) differs from equation (7) in discarding a weighted divergence term of anisotropic parameters. This is acceptable in the high frequency asymptotic sense since we have already assumed the parameters to be slowly varying.

Effective velocity and density

The effective isotropic velocity evaluation consists of following steps:

1. Solve the anisotropic eikonal equation (3) for traveltime $T(\mathbf{x})$.
2. Solve the isotropic eikonal equation (4) for effective velocity $V_{\text{eff}}(\mathbf{x})$ knowing traveltime $T(\mathbf{x})$.

Naturally, we can do the same trick to compute the effective density with following steps:

1. Solve the anisotropic transport equation (7) for anisotropic amplitude field $A(\mathbf{x})$ knowing traveltime field $T(\mathbf{x})$, which is obtained from the anisotropic eikonal solver.
2. Solve the variable density isotropic transport equation (9) for effective density $\rho_{\text{eff}}(\mathbf{x})$ knowing the amplitude $A(\mathbf{x})$ and traveltime $T(\mathbf{x})$.

However this approach requires a highly accurate anisotropic eikonal solution to ensure an acceptable accurate amplitude. Furthermore, the initialization of boundary values for density could be an issue if we use either fast marching or fast sweeping methods to solve equation (9). We therefore evaluate the effective density directly without ever computing the amplitude of the field. We compare equations (9) and (10), the amplitude and traveltime should be same and all equal to the corresponding anisotropic eikonal and transport solutions. We then directly equate the flux functions inside the divergence, i.e.

$$\frac{\nabla \tau}{\rho} = \begin{pmatrix} \left((1 + 2\eta) V_{nmo}^2 - 2\eta V_{nmo}^2 V_{P0}^2 \left(\frac{\partial \tau}{\partial z} \right)^2 \right) \frac{\partial \tau}{\partial x} \\ \left((1 + 2\eta) V_{nmo}^2 - 2\eta V_{nmo}^2 V_{P0}^2 \left(\frac{\partial \tau}{\partial z} \right)^2 \right) \frac{\partial \tau}{\partial y} \\ \left(V_{P0}^2 - 2\eta V_{nmo}^2 V_{P0}^2 \left(\left(\frac{\partial \tau}{\partial x} \right)^2 + \left(\frac{\partial \tau}{\partial y} \right)^2 \right) \right) \frac{\partial \tau}{\partial z} \end{pmatrix}. \quad (11)$$

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This is by no means the only solution to matching the two transport equations (9) and (10). However it can serve as one possible solution if we want the system to be solved. To make it more convenient and allow the units of density to normalize, we multiply both sides in equation (9) by the effective velocity V_{eff}^2 , apply high frequency asymptotic approximation to get

$$\nabla \cdot \left(\frac{A^2}{\rho} V_{\text{eff}}^2 \nabla \tau \right) = 0, \quad (12)$$

equate the flux function with that in equation (10) again, and then drop the gradient of traveltime, we will then have

$$V_{\text{eff}}^2 \begin{pmatrix} 1/\rho_x \\ 1/\rho_y \\ 1/\rho_z \end{pmatrix} = \begin{pmatrix} (1+2\eta)V_{\text{nmo}}^2 - 2\eta V_{\text{nmo}}^2 V_{P0}^2 \left(\frac{\partial \tau}{\partial z}\right)^2 \\ (1+2\eta)V_{\text{nmo}}^2 - 2\eta V_{\text{nmo}}^2 V_{P0}^2 \left(\frac{\partial \tau}{\partial z}\right)^2 \\ V_{P0}^2 - 2\eta V_{\text{nmo}}^2 V_{P0}^2 \left(\left(\frac{\partial \tau}{\partial x}\right)^2 + \left(\frac{\partial \tau}{\partial y}\right)^2\right) \end{pmatrix}. \quad (13)$$

We now propose to treat the effective density as an anisotropic parameter, and match equation (13) term by term to get a effective density field $\rho(x, y, z) = (\rho_x, \rho_y, \rho_z)$. This will not increase the computational cost of isotropic wavefield extrapolation too much since the density term $\frac{1}{\rho} \nabla \rho \cdot \nabla U$ in the wave equation is evaluated axis by axis. Through this approach, we avoid evaluating the amplitude while obtaining an effective density to match the dynamic part of the original anisotropic wavefield with that of effective isotropic wavefield.

With the effective isotropic velocity and effective anisotropic density at hand, we easily feed them into the variable density acoustic wave equation

$$\frac{1}{v^2} \frac{\partial^2 U}{\partial t^2} - \nabla^2 U + \begin{pmatrix} \partial_x \rho / \rho_x \\ \partial_y \rho / \rho_y \\ \partial_z \rho / \rho_z \end{pmatrix} \cdot \nabla U = 0, \quad (14)$$

where the density is no longer a scalar field, but a vector.

However other solutions for equation (11) are possible, for example, we can solve for an isotropic scalar density field through optimization applied to the system based on some norm. We could also solve through variational formulations by integrating the two transport equations (9) and (10).

NUMERICAL EXAMPLES

We now demonstrate the use of our effective isotropic velocity and effective anisotropic density approach with a VTI Marmousi example. For simplicity, we ignore η parameter in this example to make the model elliptically anisotropic, however, we claim the same procedure could apply to VTI case.

Figure 1 shows the vertical P -wave velocity (V_z) and δ distribution of VTI Marmousi model. Our source location is placed at lateral position 4000 m and depth position 500 m. We choose this shot position because anisotropy is relatively stronger in this region. We then use a third-order WENO scheme Godunov type Hamiltonian fast sweeping method to compute the traveltime. In our implementation, the first-order scheme is executed for several iterations to ensure a good traveltime initialization then we switch to the WENO scheme. Boundaries are

numerically filled with large values as well as large variations so that the Godunov Hamiltonian can automatically decide the upwind direction, the convergence criterion is implemented in the ℓ_1 norm. The traveltime field is then shown in Figure 2(a). A wavefiled snapshot from anisotropic wave equation modeling at $t = 0.39$ s overlaid with first arrival traveltime is shown in Figure 2(b).

An effective isotropic velocity field is obtained by solving equation (4) and shown in Figure 3. From the figure, we see that the largest difference between this effective isotropic velocity and the original vertical velocity is located in the upper-left part of the model where the source is located (an amplitude singularity) with large traveltime gradient.

Using the effective isotropic velocity, we then compute the anisotropic density field using equation (13). The resulting horizontal and vertical component of the effective density field is shown in Figures 4 and 5. This effective density field is also source dependent and has obvious source signature influence at and around the source.

We then do forward modeling with the effective models and compare the wavefields with that using the original anisotropic model. Figures 6(a) and 6(b) show the effective modeled wavefield snapshots without using and using effective density, respectively. Both of them show a good match with the anisotropic eikonal solver which confirms the kinematic match of effective modeled wavefield with the original anisotropic wavefield. To check the dynamic match we slice the wavefields at lateral position 4000 m, and show the slices in Figures 7(a) and 7(b). The amplitudes are normalized. It is clear from the figures that adding an effective density correction gives better matching in amplitudes between the effective modeled wavefield and the original anisotropic modeled wavefield, especially near the first arrival areas, which are about 1000 m in depth. This, thereby, justifies our effective density approach.

CONCLUSION

We develop an effective anisotropy scheme to embed anisotropy into the inhomogeneity of the velocity and density models in the high frequency asymptotic sense to reduce the cost of extrapolating anisotropic wavefields. This is done by matching both the kinematic and dynamic parts of the approximated wavefields to original ones, specifically matching the anisotropic eikonal and transport equations to the isotropic eikonal and transport equations. We also provide one possibility for evaluating the effective density efficiently without solving the transport equation. Our numerical example demonstrates the feasibility of our approach.

ACKNOWLEDGMENTS

We wish to thank KAUST for financial support. We wish to thank members of Seismic Wave Analysis Group at KAUST for useful discussions. We also would like to thank the Madagascar community for providing the computational environment (Fomel et al., 2013) used here.

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Figure 1: VTI Marmousi V_z and δ models.

Figure 2: (a) Traveltime field; (b) Wavefield snapshot at $t = 0.39$ s overlaid with anisotropic eikonal solution.

Figure 3: Effective isotropic velocity field.

Figure 4: Horizontal component of effective density field.

Figure 5: Vertical component of effective density field.

Figure 6: Wavefield snapshots at $t = 0.39$ s overlaid with anisotropic eikonal solution. (a) Isotropic modeled wavefield only using effective velocity; (b) Isotropic modeled wavefield using effective velocity and effective density.

Figure 7: (a) Traces of wavefield snapshots at $t = 0.39$ s from Figure 6(a); (b) Traces of wavefield snapshots at $t = 0.39$ s from Figure 6(b). Black lines are reference trace from original anisotropic wavefield snapshot. Red lines in (a) and (b) are traces from effective modeled wavefield snapshots without using and using effective density, respectively.

EDITED REFERENCES

Note: This reference list is a copy-edited version of the reference list submitted by the author. Reference lists for the 2014 SEG Technical Program Expanded Abstracts have been copy edited so that references provided with the online metadata for each paper will achieve a high degree of linking to cited sources that appear on the Web.

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