Complexity Reduction of Multiphase Flows in Heterogeneous Porous Media

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Summary

In this paper, we apply mode decomposition and interpolatory projection methods to speed up simulations of two-phase flows in heterogeneous porous media. We propose intrusive and nonintrusive model-reduction approaches that enable a significant reduction in the size of the subsurface flow problem while capturing the behavior of the fully resolved solutions. In one approach, we use the dynamic mode decomposition. This approach does not require any modification of the reservoir simulation code but rather post-processes a set of global snapshots to identify the dynamically relevant structures associated with the flow behavior. In the second approach, we project the governing equations of the velocity and the pressure fields on the subspace spanned by their proper-orthogonal-decomposition modes. Furthermore, we use the discrete empirical interpolation method to approximate the mobility-related term in the global-system assembly and then reduce the online computational cost and make it independent of the fine grid. To show the effectiveness and usefulness of the aforementioned approaches, we consider the SPE-10 benchmark permeability field, and present a numerical example in two-phase flow. One can efficiently use the proposed model-reduction methods in the context of uncertainty quantification and production optimization.

Introduction

High-fidelity reservoir-simulation models are shown to yield better predictions in optimization problems and in planning for new reservoir developments. However, the computational time of such large-scale models becomes the bottleneck of fast turnarounds in the decision-making process and the assimilation of real-time data into reservoir models (namely, the closed-loop reservoir management) to improve the accuracy of such models (Gildin and Lopez 2011). Even in the case of unconventional reservoirs, numerical reservoir simulation was used with several modifications with the current models, particularly in the nature of the flow. To this end, natural fractures and pore-space networks were incorporated in the simulation process to account for flow in the fractures and matrix (Moridis et al. 2010; Yan et al. 2013; Alfi et al. 2014).

Despite the great advances in reservoir-modeling tools and the advent of high-performance computing, high-fidelity physics-based numerical simulation still remains a challenging step in understanding the physics of the reservoir because of the large-scale nature of the discretization of the underlying partial-differential equations. Computationally intensive simulations, such as in the case of history matching (Afra et al. 2014), optimization, and uncertainty quantification (Doren et al. 2006), become impractical to be performed in a timely manner if real-time data need to be assimilated into the model (Jansen et al. 2009; Gildin and Lopez 2011).

The importance of obtaining a simpler model that can represent the physics of the full system is paramount to speed up the work flows that require several (from dozens to thousands) calls of the forward model. This is usually the case in history matching and the optimization process. A variety of complexity-reduction

techniques were proposed to ease this problem and to reduce the computational cost in the optimization under the uncertainty paradigm (Heijn et al. 2004; Gildin 2010). In general, one can classify them in three broader areas if one is dealing with the forward simulations (production optimization) or the inverse problem (parameter estimation) (Oliver et al. 2008):

- Reduction of the cost of forward simulations: surrogate and reduced-order models, multiscale approach, upscaling
- Reduction of the input space dimension: sparsity-based parameterizations, direct cosine transformation, polynomial chaos
- Reduction of the number of forward simulations: design of experiments, Markov-chain-Monte Carlo sampling

In this paper, we are interested in designing model-reduction techniques for forward problems. To this end, reduced-order modeling techniques are shown to be a viable way of mitigating the large-scale nature of the simulation models, providing surrogates (or proxy models) to alleviate and, very often, to accelerate the computations taking place in subsurface applications. The options range from nonintrusive methods (i.e., that do not depend on modifications of a reservoir-simulation code) to more-intrusive and -sophisticated methods that depend on several modifications of legacy code or the development of new simulator codes. In the case of nonintrusive methods, data-driven model reduction has been the choice in material-balance-type modeling such as the capacitance/resistance models (Yousef et al. 2006) and flow-network models (Lerlertpakdee et al. 2014), and in the use of artificial intelligence, such as neural networks and fuzzy-logic techniques (Mohaghegh et al. 2012). For the intrusive schemes, reduced-order modeling by projection has been used in the systems/controls-like framework, such as the balanced truncation (Heijn et al. 2004), proper orthogonal decompositions (PODs) (Volkwein and Hinze 2005), and the trajectory-piecewise linear (TPWL) techniques (Cardoso and Durlofsky 2010), bilinear Krylov subspace methods (Ghasemi et al. 2014), and quadratic bilinear model order reduction (Gildin and Ghasemi 2014).

For nonlinear systems, one of the most common methodologies used is POD and its variants because of its computational simplicity, ease of implementation, and good approximation. This method was coupled with the TPWL model reduction in Cardoso and Durlofsky (2010). In this approach, one performs several linearizations of the nonlinear systems about several known states so that one can use POD or other linear model-reduction techniques to derive good approximations. In reservoir simulation and optimization, these methodologies have shown to yield reduction in the size of the models of several orders of magnitude, but only a modest reduction in the computational time. One can attribute this mainly to the fact that in the nonlinear case, one still needs to deal with the fact that the projection is not performed in a direct manner, because one needs to compute full-size Jacobians in the linearization process. Also, as in TPWL, several state trajectories need to be saved, and one can achieve instabilities in this process. To avoid this problem Chaturantabut and Sorensen (2010, 2011), Gildin et al. (2013), and Ghasemi et al. (2015) developed a variant of the POD methodology—the discrete empirical interpolation method (DEIM), in which the nonlinear terms are approximated by some form of interpolation, and therefore, one can achieve great reductions in computational time. A different approach was taken in Efendiev et al. (2012). Here, the authors use a combination of multiscale methods (Aarnes et al. 2006; Efendiev and Hou

2009) used in the solution of partial-differential equations together with reduced-order modeling.

In this work, we follow two approaches to reduce the computational complexity associated with the simulation of two-phase flows in heterogeneous porous media. The first approach is based on non-intrusive methods that use dynamic mode decomposition. This approach does not require any modification of the reservoir-simulation code but rather post-processes a set of global snapshots to identify the dynamically relevant structures associated with the flow behavior. In the second approach, we design the POD-DEIM scheme for two-phase flow and transport. To have a mass-conservative velocity-field basis, we use POD within a mixed multiscale finite-element coupling. The POD basis functions for both pressure and the velocity fields are constructed and coupled within mixed finite-element framework. To assemble the global reduced-order model, the mobility needs to be estimated. We use DEIM to approximate the mobility-related term in the global-system assembly. Further, the reduced-order system for the flow equation is solved. In this paper, the saturation equation is solved on the fine grid. The POD basis for the saturation is computed on the basis of an offline simulation, whereas in the online stage, we use a sequential approach in which the flow equation is solved at a reduced cost, whereas the saturation equation is solved on the fine grid. We present numerical results with the SPE-10 benchmark permeability field. POD modes are assembled with an offline well schedule, and these modes are used for a different well schedule. Our numerical results show that POD coupled with DEIM can reduce the dimension and computational cost of simulating a reservoir model successfully.

Model Description and Problem Formulation

We consider two-phase flows in a reservoir domain (denoted by Ω) under the assumption that the displacement is dominated by viscous effects (i.e., we neglect the effects of gravity, compressibility, and capillary pressure). The two phases are referred to as water and oil. We write Darcy's law for each phase as follows,

$$u_l = -\frac{k_{rl}(s)}{\mu_l} K \nabla p, \quad (1)$$

where u_l is the phase velocity, K is the permeability tensor, k_{rl} is the relative permeability to phase l ($l = o, w$), s is the water saturation (volume fraction), and p is pressure. Throughout the paper, we will assume that the permeability tensor is isotropic and a single set of relative permeability curves is used. Combining Darcy's law with a statement of conservation of mass allows us to express the governing equations in terms of the so-called pressure and saturation equations:

$$-\nabla \cdot [K \lambda(s) \nabla p] = q_w + q_o \quad (2)$$

and

$$\phi \frac{\partial s}{\partial t} + \nabla \cdot [f(s) u] = q_w / \rho_w, \quad (3)$$

where ϕ is the porosity, λ is the total mobility, $f(s)$ is the flux function, and u is the total velocity, which are respectively given by

$$\lambda(s) = \lambda_w(s) + \lambda_o(s) = \frac{k_{rw}(s)}{\mu_w} + \frac{k_{ro}(s)}{\mu_o}, \quad (4)$$

$$f(s) = \frac{\lambda_w(s)}{\lambda(s)} = \frac{k_{rw}(s)}{k_{rw}(s) + \frac{\mu_w}{\mu_o} k_{ro}(s)}, \quad (5)$$

and

$$u = u_w + u_o = -\lambda(s) K \nabla p. \quad (6)$$

Moreover, q_w and q_o are volumetric source terms for water and oil, respectively, and μ_l is the fluid (oil or water) viscosity ($l = o, w$).

The previous descriptions are referred to as the fine model of the two-phase-flow problem.

In this paper, we use mixed multiscale finite-element methods (MsFEMs) to use conservative velocity-field space. As such, we rewrite the two-phase-flow equation as

$$(\lambda K)^{-1} u + \nabla p = 0 \text{ in } \Omega \quad (7)$$

$$\nabla \cdot u = h \text{ in } \Omega \quad (8)$$

$$(\lambda K) \nabla p \cdot n = g \text{ on } \partial \Omega. \quad (9)$$

For simplicity of the presentation, we assume Neumann boundary conditions.

Let $V_h \subset H(\text{div}, \Omega)$ and $Q_h \subset L^2(\Omega)/R$ be finite dimensional spaces and $V_h^0 = V_h \cap H_0(\text{div}, \Omega)$, where $H_0(\text{div}, \Omega)$ is $H(\text{div}, \Omega)$ with homogeneous boundary conditions. The numerical approximation of Eq. 7 on the fine grid is to find $\{u, p\} \in V_h \times Q_h$ such that $u \cdot n = g$ on $\partial \Omega$ and

$$\begin{aligned} [(\lambda K)^{-1} u, v_h] + (\nabla \cdot v_h, p) &= 0 \quad \forall v_h \in V_h^0, \\ (\nabla \cdot u, q_h) &= (h, q_h) \quad \forall q_h \in Q_h, \end{aligned} \quad (10)$$

where $(\cdot, \cdot)$ is the usual L^2 inner product.

In a mixed finite-element-method (FEM) discretization, we replace the pressure and velocity solutions by finite dimensional subspaces that typically consist of low-order piecewise polynomials. In a mixed MsFEM, one attempts to design the approximation space for velocity in such a way that it embodies the impact of subgrid variations in k . Furthermore, multiscale basis functions are used for the velocity field, whereas the pressure field is approximated with piecewise constant functions, denoted by Φ_i . In particular, one constructs multiscale basis functions for the velocity field for each edge (face in three dimensions) of every block. Throughout, we denote by e_i^k an edge or a face i of the block k (i.e., a subset of ∂k that has positive length in 2D space and positive area in 3D space).

With a permeability field K , the corresponding multiscale basis function for e_i^k is defined as follows. First, we compute an auxiliary function w_i^k by solving the following equations in K :

$$-\nabla \cdot (K \nabla w_i^k) = \frac{1}{|k|} \text{ in } k \quad (11)$$

$$K \nabla w_i^k \cdot n^k = \begin{cases} g_i^k & \text{on } e_i^k, \\ 0 & \text{else,} \end{cases} \quad (12)$$

where the choice of g_i^k is discussed later. To each auxiliary function w_i^k , we associate the multiscale basis function $\Psi_i^k = K \nabla w_i^k$. When basis functions are computed for each edge, we define the corresponding finite-dimensional velocity-approximation space for the mixed MsFEM by

$$V_h(K) = \bigoplus_k \{\Psi_i^k\}, \quad (13)$$

$$V_h^0(K) = V_h(K) \cap H_0(\text{div}, \Omega). \quad (14)$$

Here, we have emphasized that the velocity-approximation space depends on K , because it appears in the equations that determine the multiscale basis functions. Hence, for each permeability field K , we associate a velocity-approximation space $V_h(K)$ that is spanned by the set of basis functions consisting of one multiscale basis function for each edge of every gridblock in the coarse grid. In this paper, we use an approximation space that is defined for multiple K ; that is, we use

$$V_h = \bigoplus_i V_h(K_i), \quad (15)$$

where K_i are permeability fields.

The choice of boundary conditions g_i^k in Eq. 12 can greatly influence the accuracy of a mixed MsFEM. In Chen and Hou (2003), the authors use piecewise constant coarse-scale fluxes on the boundary of the coarse elements; that is, $g_i^k = 1/|e_i^k|$. To obtain accurate approximations of the fluxes, the velocity should contain the fine-scale features similar to the solution of the original problem on the boundary. Piecewise constant boundary conditions can introduce a strong mismatch along gridblock boundaries between the mixed MsFEM solution and the solution of the original problem. To alleviate this problem, an oversampling method is introduced in Chen and Hou (2003) (see also Hou and Wu 1997). The main idea of the oversampling methods is to use larger regions for local-problem computations (i.e., the local problems are solved in larger regions, and then the basis functions are computed with only the interior information). These approaches tend to be conservative only on the coarse grid. For our purposes, we would like approaches that are conservative also on the fine grid.

One can write the numerical approximation of Eq. 7 in a matrix form as

$$\begin{pmatrix} A & -C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} 0 \\ g \end{pmatrix}, \quad (16)$$

where $A = (a_{ij}) = \int (\lambda K)^{-1} \Psi_i \Psi_j$ and $C = (c_{ij}) = \int \nabla \cdot (\Psi_i) \Phi_j$.

Our solution approach is sequential, meaning that the preceding pressure equation (Eq. 16) is solved on a coarse grid with precomputed multiscale basis. Then, the saturation equation (Eq. 3) is solved on the basis of the finite volume spatial discretization and the implicit time discretization. Our primary focus in this work is to present a reduced-order model for the pressure equation, which is a computationally expensive part of the entire simulation.

Model Order Reduction

In the subsequent analysis, we propose different approaches to reduce the computational complexity associated with the simulation of two-phase flows. This can be helpful to perform an optimization analysis or to conduct uncertainty-quantification studies that require solving the forward problem many times. Throughout the paper, we refer to the fully resolved solution as the solution obtained from mixed finite-element method (FEM). To reduce the computational complexity associated with nonlinear dynamical systems, two common mode-decomposition methods were successfully used: dynamic mode decomposition (DMD) and proper orthogonal decomposition (POD). Both of them are based on processing information from a sequence of snapshots (or instantaneous solutions) to identify a low-dimensional set of basis functions. These functions are then used to derive a low-dimensional dynamical system that is typically obtained by Galerkin projection (Wang et al. 2011). Later, to reduce the computational time of these methods even more, we integrate the discrete empirical interpolation method (DEIM).

Nonintrusive Method. This technique is intended to accurately extract the coherent and dynamically relevant structures rather than selecting the dominant modes that capture most of the flow's energy. DMD enables the computation, from simulation and empirical data, of the eigenvalues and eigenvectors of a linear model that best represents the underlying dynamics, even if those dynamics are produced by a nonlinear process. One important feature of this method is its ability to extract dynamic information from flow fields without depending on the availability of a model, but rather being based on a sequence of snapshots (Schmid et al. 2011). As such, we propose the application of this mode-decomposition technique to speed up significantly the numerical simulation of the two-phase flow in a reservoir while preserving a good accuracy.

DMD Description. DMD was recently proposed by Schmid (2010). The DMD-based approach involves the following steps (Tu et al. 2013; Ghommam et al. 2013):

1. Collect and store the snapshots, instantaneous solutions of the saturation separated by a constant timestep Δt , in two consecutive snapshot sequences as follows:

$$\begin{aligned} S_1^{n_s-1} &= \begin{pmatrix} | & | & \cdots & | \\ s_1 & s_2 & \cdots & s_{n_s-1} \\ | & | & & | \end{pmatrix} \text{ and} \\ S_2^{n_s} &= \begin{pmatrix} | & | & \cdots & | \\ s_2 & s_3 & \cdots & s_{n_s} \\ | & | & & | \end{pmatrix}. \quad (17) \end{aligned}$$

Here, $s_i \in \mathbb{R}^{n_p}$ is the vector collecting the saturation values at the gridpoints at the instant $t_i = i \Delta t$, where n_p denotes the number of gridpoints and n_s is the number of snapshots. In our present study, we assume that $n_p \gg n_s$. The objective is to evaluate only the first few instantaneous saturation fields to compute the dynamically relevant structures that enable the forecast of the evolution of the saturation with good accuracy while gaining significantly in the computational cost. Assuming that

$$S_2^{n_s} = A S_1^{n_s-1}. \quad (18)$$

2. For improved convergence and algorithmic robustness, we perform the economy-sized singular-value decomposition (SVD) of the first snapshot matrix $S_1^{n_s-1}$:

$$S_1^{n_s-1} = U \Sigma V^T, \quad (19)$$

where U and V matrices are unitary, U contains the proper orthogonal modes of $S_1^{n_s-1}$ and V^T is the transpose of right singular matrix, Σ is a square diagonal matrix containing the nonzero singular values of $S_1^{n_s-1}$.

3. Calculate the matrix $\tilde{A}$ as

$$\tilde{A} \triangleq U^T S_2^{n_s} V \Sigma^{-1}. \quad (20)$$

4. Compute the eigenvalues and eigenvectors of the matrix $\tilde{A}$; that is, solve the following eigenvalue problem,

$$\tilde{A} W = W \Lambda, \quad (21)$$

where Λ and W are matrices containing the eigenvalues and eigenvectors of matrix $\tilde{A}$, respectively.

5. Compute the dynamic modes $\Phi = \{\phi_j\}_{j=1}^r$, where $r = \min(n_p, n_s)$ as

$$\phi_j = U W_{(:,j)}. \quad (22)$$

The corresponding dynamic mode spectrum is defined as, $\lambda_j = \log(\Lambda_{jj})/\Delta t$, where Λ_{jj} is the j th diagonal element and Δt is the sampling time.

6. Approximate the saturation as

$$s(x, t^k) \approx \tilde{s}(x, t^k) = \sum_{j=1}^r \lambda_j^k \bar{\phi}_j, \quad k \in \{1, \dots, n_s\}, \quad (23)$$

where $\bar{\phi}_j$ are the scaled DMD modes such that the sum of the modes equals the first data vector (Tu et al. 2013).

One can follow the same steps to approximate the pressure field as well. Unlike the POD-based reduced-order model (Akhtar et al. 2009; Wang et al. 2011; Ghommam et al. 2013) that requires the Galerkin projection of the original governing equations, the DMD-based approximation, as given by Eq. 23, does not require a priori knowledge of the governing equations, but rather only involves the analysis of a set of snapshots.

Intrusive Method. Even though nonintrusive methods are relatively easy to implement, the results are highly dependent on the training inputs (set of snapshots). Also, they usually have poor prediction. On the contrary, the intrusive methods require access to the source code, but they are more reliable in approximating a system. POD is one of these approaches that are considered in this work. However, because there are nonlinear-function evaluations in the governing equations, the performance of the POD will be hindered to completely reduce the dimension of the system. Thus,

	Mixed-Pressure Eq.	Saturation Eq.	Total Simulation Time	Speedup
Fine scale	0.89 T_{fs}	0.11 T_{fs}	1.0 T_{fs}	1
POD-DEIM	0.05 T_{fs}	0.11 T_{fs}	0.16 T_{fs}	6.25

Table 1—Relative computational time of different models.

we apply DEIM to approximate these nonlinear functions. DEIM is based on the approximation of a linear or nonlinear function by means of an interpolatory projection of few selected global snapshots of the function. The idea is to represent a function over the full domain while using only information in some locations. This is motivated by many important applications in which the saturation data could not be available everywhere and can be measured only in some locations.

DEIM Description. We briefly review the DEIM as presented in Chaturantabut and Sorensen (2010). Let $\mathbf{f}(\tau) \in \mathbb{R}^n$ denote a linear or nonlinear function, where τ refers to time or any control parameter. We assume an approximation of the function $\mathbf{f}$ obtained by projecting it into a subspace spanned by the basis functions $\mathbf{U}_f = (\mathbf{u}_{f_1}, \dots, \mathbf{u}_{f_m}) \in \mathbb{R}^{n \times m}$ as

$$\mathbf{f}(\tau) \approx \mathbf{U}_f \mathbf{c}(\tau). \quad (24)$$

To compute the coefficient vector $\mathbf{c}$, we consider a matrix

$$\mathbf{P} = (\mathbf{e}_{\varphi_1}, \dots, \mathbf{e}_{\varphi_m}) \in \mathbb{R}^{n \times m},$$

where $\mathbf{e}_{\varphi_i} = (0, \dots, 0, 1, 0, \dots, 0)^T \in \mathbb{R}^n$ is the φ_i^{th} column of the identity matrix $\mathbf{I}_n \in \mathbb{R}^{n \times n}$ for $i = 1, \dots, m$. By multiplying Eq. 24 by $\mathbf{P}^T$ and assuming that the matrix $\mathbf{P}^T \mathbf{U}_f$ is nonsingular, we obtain

$$\mathbf{f}(\tau) \approx \mathbf{U}_f \mathbf{c}(\tau) = \mathbf{U}_f (\mathbf{P}^T \mathbf{U}_f)^{-1} \mathbf{P}^T \mathbf{f}(\tau). \quad (25)$$

To summarize, the approximation of the nonlinear function $\mathbf{f}(\tau)$, as given by Eq. 25, requires first the computation of the projection basis $\mathbf{U}_f = (\mathbf{u}_{f_1}, \dots, \mathbf{u}_{f_m})$, and next, the identification of the indices $\{\varphi_1, \dots, \varphi_m\}$.

To determine the projection basis $\mathbf{U}_f$, we assemble the function evaluations in an $n \times n_s$ matrix $[\mathbf{f}(\tau_1), \dots, \mathbf{f}(\tau_{n_s})]$ and use the SVD to compute the POD modes. These modes are used as the projection basis in the approximation given by Eq. 24. As for the interpolation indices $\{\varphi_1, \dots, \varphi_m\}$, they are selected with the DEIM algorithm (Chaturantabut and Sorensen 2010), as given next.

Algorithm 1 DEIM procedure (Chaturantabut and Sorensen 2010):

- 1: **procedure** DEIM [$\mathbf{U}_f = (\mathbf{u}_{f_1}, \dots, \mathbf{u}_{f_m}) \in \mathbb{R}^{n \times m}$]
- 2: $(|\rho|, \varphi_1) = \max\{|\mathbf{u}_{f_1}|\}$
- 3: $\mathbf{U}_f = (\mathbf{u}_{f_1})$, $\mathbf{P} = (\mathbf{e}_{\varphi_1})$
- 4: **for** $\ell = 2, \dots, m$ **do**
- 5: $(\mathbf{P}^T \mathbf{U}_f) \mathbf{c} = \mathbf{P}^T \mathbf{u}_{f_\ell}$
- 6: $\mathbf{r} = \mathbf{u}_{f_\ell} - \mathbf{U}_f \mathbf{c}$
- 7: $(|\rho|, \varphi_\ell) = \max\{|\mathbf{r}|\}$
- 8: $\mathbf{U}_f \leftarrow (\mathbf{U}_f \mathbf{u}_{f_\ell})$, $\mathbf{P} \leftarrow (\mathbf{P} \mathbf{e}_{\varphi_\ell})$
- 9: **end for**
- 10: **return** $\vec{\varphi} = (\varphi_1, \dots, \varphi_m)^T$
- 11: **end procedure**,

The approximation of the function $\mathbf{f}(\tau)$ is written then as

$$\begin{aligned} \mathbf{f}(\tau) &\approx \tilde{\mathbf{f}}(\tau) = \mathbf{U}_f (\mathbf{P}^T \mathbf{U}_f)^{-1} \mathbf{P}^T \mathbf{f}(\tau) \\ &= \mathbf{U}_f (\mathbf{P}^T \mathbf{U}_f)^{-1} \begin{pmatrix} \mathbf{f}_{\varphi_1}(\tau) \\ \mathbf{f}_{\varphi_2}(\tau) \\ \vdots \\ \mathbf{f}_{\varphi_m}(\tau) \end{pmatrix}. \quad (26) \end{aligned}$$

In Eq. 26, the term $\mathbf{U}_f (\mathbf{P}^T \mathbf{U}_f)^{-1} \in \mathbb{R}^{n \times m}$ is computed once and stored, and the remaining term is obtained by computing only the values of the functions at m points with the indices $\varphi_1, \dots, \varphi_m$ (identified from Algorithm 1). Here comes the computational saving: such operation requires only the evaluation of the function at m selected points rather than performing a full evaluation over the whole system domain.

Coupled POD-DEIM Approach. POD constitutes a powerful mode-decomposition technique for extracting the modes that capture the highest amount of energy in a linear or nonlinear dynamical process (Wang et al. 2011). Because the main computational cost of the mixed multiscale FEM is associated with solving the linear system of Eq. 16 as it is shown in Table 1, we discuss POD-DEIM for this equation. The application of this method in the saturation equation should be investigated in future work. This method involves finding the basis for states (i.e., velocity and pressure), and the nonlinear functions (i.e., mobility).

First, we collect and store instantaneous coarse-scale solutions of the velocity and pressure in snapshot matrices as follows:

$$\begin{aligned} \mathbf{U}_1^{n_s} &= \begin{pmatrix} | & | & & | \\ \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_{n_s} \\ | & | & & | \end{pmatrix} \\ \text{and } \mathbf{P}_1^{n_s} &= \begin{pmatrix} | & | & & | \\ \mathbf{p}_1 & \mathbf{p}_2 & \dots & \mathbf{p}_{n_s} \\ | & | & & | \end{pmatrix}. \end{aligned}$$

We perform the SVD on the snapshot matrices to compute the velocity and pressure POD modes denoted by Φ^u and Φ^p , respectively. We assume an approximation of the velocity and pressure as an expansion in terms of the POD modes; that is, we let

$$\begin{aligned} \mathbf{u}(x, t) &\approx \sum_{i=1}^{N_u} \alpha_i^u(t) \phi_i^u(x) \text{ and} \\ \mathbf{p}(x, t) &\approx \sum_{i=1}^{N_p} \alpha_i^p(t) \phi_i^p(x) \quad (27) \end{aligned}$$

or in a matrix form

$$\mathbf{u} \approx \Phi^u \boldsymbol{\alpha}^u \text{ and } \mathbf{p} \approx \Phi^p \boldsymbol{\alpha}^p, \quad (28)$$

where N^u and N^p are the number of velocity and pressure POD modes, respectively.

We apply DEIM to approximate the nonlinear mobility function $\lambda(s)$ to reduce the computational complexity associated with assembling the matrix $\mathbf{A}$ of Eq. 16. Note that the cost of assembling the resulting coarse-grid system is substantially reduced and is independent of the fine-grid system size.

We use the solution expansion given by Eq. 28 and project Eq. 16 into the space formed by the POD modes to obtain the following linear system

$$\begin{pmatrix} (\Phi^u)^T \mathbf{A} \{[\tilde{\lambda}(s)]\} \Phi^u & -(\Phi^u)^T \mathbf{C}^T \Phi^p \\ (\Phi^p)^T \mathbf{C} \Phi^u & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\alpha}^u \\ \boldsymbol{\alpha}^p \end{pmatrix} = \begin{bmatrix} 0 \\ (\Phi^p)^T \mathbf{g} \end{bmatrix}. \quad (29)$$

Thus, the original system with large number of unknowns (number of edges + number of gridblocks) is transformed into a reduced order dynamical system with fewer unknowns (number of velocity basis + number of pressure basis). Moreover, the cost of the assembly of the coarse-grid system is reduced and only

Fig. 1—SPE-10 model with an inverted-five-spot pattern.

depends on the number of selected DEIM points and coarse-grid system size (and independent of the fine-grid mesh size).

We elaborate on the importance of choosing a suitably accurate enriched multiscale space and the selection of POD modes and DEIM points when using the coupled approach. The maximum error between the fully resolved and approximate solutions may be bounded by using the triangle inequality. In particular, we may write

$$\max_{t \in [0, \text{PVI}]} \|s(x, t) - \tilde{s}(x, t)\|_2 \leq \max_{t \in [0, \text{PVI}]} \|s(x, t) - s^{ms}(x, t)\|_2 + \max_{t \in [0, \text{PVI}]} \|s^{ms}(x, t) - \tilde{s}(x, t)\|_2, \quad \dots \quad (30)$$

where the first term on the right-hand side of the inequality is the error between the fully resolved solution and the multiscale solution, and the second term on the right-hand side is the error between the multiscale solution and the approximate solution. Guaranteeing a suitable bound will involve choosing the dimension of the coarse space and picking properly the number of POD modes and DEIM interpolation points such that both error terms are small.

Next, we consider a time-varying schedule for the water injections at the four corner of the reservoirs (see Fig. 1a). We follow the approach described previously to derive a reduced-order model obtained by projecting the pressure equation into the space spanned by the POD modes and using DEIM to approximate the nonlinear mobility function.

Case Study

In the present study, we adopt the SPE-10 comparative model (Christie and Blunt 2001) (ten upper layers) for the permeability field and relative permeability, as shown in Figs. 1a and 1b, respectively. The water/oil-viscosity ratio is set to 0.1, ($\mu_w/\mu_o = 0.1$).

This constitutes a highly heterogeneous field with high contrast. We solve the model on a 3D domain with a regular Cartesian grid of $60 \times 220 \times 10$ gridblocks; each gridblock is $20 \times 10 \times 2$ ft.

We consider two-phase flows (oil and water) in a reservoir domain under the assumption that the displacement is dominated by viscous effects (i.e., we neglect the effects of gravity, compressibility, and capillary pressure).

All the wells have a single perforation at the bottom layer. The oil in the reservoir is produced from a vertical producer in the center of the reservoir, controlled by constant bottomhole pressure, and driven by four water injectors at the corners with a constant injection rate, with schedule shown later in Fig. 6 by solid line. All the simulations are performed for 1 pore volume (PV) of injection (PVI).

Nonintrusive Results. We use the instantaneous solutions to compute the DMD modes, as detailed in the Nonintrusive Method subsection. For this example, we run the case study for 1 PVI. The simulation was run for 300 days, and the saturation is saved every $\Delta t = 1$ day; thus, we have 300 snapshots. Note that here we applied uniform-timestep DMD, but one can refer to Tu et al. (2013) for nonuniform time sampling DMD.

Two sets of DMD bases are constructed by use of first 100 and 150 snapshots, respectively. They are used for predicting the solution for the rest of the simulation. Fig. 2 depicts the final saturation obtained from the fully resolved solution and that obtained from the DMD-based approximation while varying the number of modes. We observe good agreement between the two sets of data. The final relative error in final saturation is equal to 9.93 and 4.59% when considering 100 and 150 modes, respectively. Note that one can reduce the error by increasing the number of basis in this problem. We note that the dimension of the full 3D domain is

Fig. 2—Saturation obtained from (a and b) the DMD-based approximation and (c) the mixed FEM.

Fig. 3—Saturation obtained from (a) the DEIM-based approximation and (b) the mixed FEM. The first 20 selected interpolation points obtained by the DEIM algorithm are denoted by green squares.

Fig. 4—Variations of the relative error with the number of DEIM points.

132,000. Thus, the flow-problem size was reduced by three orders of magnitude. These observations show the capability of DMD modes to capture the main characteristics of the saturation evolution and to forecast the saturation with good accuracy while gaining in terms of computational cost.

Intrusive Results. We follow the DEIM-based approach to approximate the saturation with information from the fully resolved solution at a few points in the domain. The basis and interpolation points were found with the DEIM algorithm, as explained in the Intrusive Methods subsection. These bases were used to reconstruct the saturation in a fine scale at the final timestep. For example, with 20 bases, the saturation was evaluated at 20 points, as shown in **Fig. 3**, and the final saturation was reconstructed by using the DEIM basis, as described in Eq. 25, and compared with fine-scale result in this figure. The relative error in final saturation is equal to 2.62%. In **Fig. 4**, we show the variations of the relative

error in saturation between the fully resolved and approximate solution with increasing the number of DEIM interpolation points. As expected, the error decreases as the number of DEIM points is increased. We note that the dimension of the full 3D domain is 132,000. These results demonstrate the ability of DEIM to reconstruct the saturation field over the full domain while using information at only few points.

Clearly, the use of DEIM enables a significant reduction in the computational cost (solution evaluation at significantly fewer nodal points) while constructing the saturation field over the full domain with good accuracy.

We follow the projection-based approach described previously to speed up the simulation of two-phase flows. Here, we integrated the POD-DEIM approach in the pressure equation, as explained in the previous section. In **Fig. 5**, we plot the saturation obtained from the mixed finite-element method (FEM), the mixed multiscale FEM, and coupled POD-DEIM approaches. We observe good agreement between the three sets of data. The results shown are obtained at PVI/6. Furthermore, two pressure and two velocity POD modes and 20 DEIM interpolation points are used for the coupled approach. The relative error in final saturation between the approximate and multiscale solutions is equal to 5.3% and that between the approximate and fully resolved solutions is equal to 10.34%. The relative error in final pressure is approximately 2% for both cases. Furthermore, the relative computational time is compared for the fine-scale and reduced models in Table 1. T_{fs} is the simulation time to run the full-scale model. Clearly, the computational time of the mixed pressure solver is reduced more than one order of magnitude in this problem. One should note that the successful application of model reduction on the saturation equation is a task that one should consider in future modeling efforts. These observations indicate the suitability of the projection-based coupled approach for model reduction of two-phase flows in highly heterogeneous porous media.

To investigate the suitability of the coupled POD-DEIM approach to model two-phase flow while varying the schedule of

Fig. 5—Saturation obtained from (a) the coupled POD-DEIM approach, (b) the mixed multiscale FEM, and (c) the mixed FEM.

Fig. 6—Original and 10%-perturbed schedule of injection rates.

water injection, we perturb the injection rates 10%, as shown in Fig. 6.

This analysis is motivated by many applications in which one needs to solve many forward problems corresponding to different injection schedules—for instance, when the injection rates may be subject to uncertainty or when conducting optimization studies to maximize the oil recovery. In Fig. 7, we plot the temporal variations of the relative error between the fully resolved and approximate solutions, as defined in the following:

$$E(t^k) = \frac{\|s(x, t^k) - \tilde{s}(x, t^k)\|_2}{\|s(x, t^k)\|_2}, \quad t^k \in (0, \text{PVI}), \quad \dots \quad (31)$$

where t^k is the k th timestep and s and $\tilde{s}$ are the saturation results from the fine-scale model and the reduced model, respectively.

We note that the pressure and velocity POD modes along with the DEIM interpolation points are generated when considering the unperturbed schedule. We observe error values reaching a maximum value of 10%. One can reduce this error, as discussed in Cardoso and Durlofsky (2010), by excluding the gridblocks containing wells from reduced subspace and solve the pressure and saturation equations in original space. This shows the robustness of the projection-based approach, combining multiscale FEM, POD, and DEIM, to derive a reliable reduced-order model while moderately varying forcing inputs (injection rates).

Conclusions

In this paper, we developed reduced-order complexity-reduction techniques for simulations of two-phase-flow systems. In one approach, we use nonintrusive methods that used the dynamic-mode decomposition. This approach does not require any modification of the reservoir-simulation code but rather post-processes a set of global snapshots to identify the dynamically relevant structures associated with the flow behavior.

In a second approach, we followed an intrusive method that is based on projecting the governing equations of the velocity and the pressure fields on the subspace spanned by their proper orthogonal decomposition (POD) modes. Without appropriate nonlinear-interpolation tools, one needs to evaluate the residual on the fine grid, which can be expensive. As such, we combined the discrete empirical-interpolation method (DEIM) with POD to reduce the online computational cost and make it independent of the fine grid. We considered constant and time-varying water-injection rates and presented the corresponding numerical results while using the coupled POD-DEIM approach. We showed that, with a fraction of the fine-grid computational cost, we can compute the forward response for different well schedules. One can efficiently use the proposed model-reduction methods when performing uncertainty quantification or optimization studies.

Fig. 7—Temporal variation of the relative error between the fully resolved and approximate saturation with perturbed-injection schedule.

Nomenclature

K = permeability, ft²
 N = number of basis
 p = pressure, psi
 q = volumetric flow rate, B/D
 s = water saturation
 t = time, days
 u = phase velocity, ft/D
 μ = viscosity, cp

Subscripts

n_s = number of snapshots
 n_p = number of gridpoints
 o = oil
 ro = relative to oil
 rw = relative to water
 w = water

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