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MIXED FINITE ELEMENT-BASED FULLY CONSERVATIVE METHODS FOR SIMULATING WORMHOLE PROPAGATION*

JISHENG KOU[†], SHUYU SUN[‡], AND YUANQING WU[§]

Abstract. Wormhole propagation during reactive dissolution of carbonates plays a very important role in the product enhancement of oil and gas reservoir. Because of high velocity and nonuniform porosity, the Darcy–Forchheimer model is applicable for this problem instead of conventional Darcy framework. We develop a mixed finite element scheme for numerical simulation of this problem, in which mixed finite element methods are used not only for the Darcy–Forchheimer flow equations but also for the solute transport equation by introducing an auxiliary flux variable to guarantee full mass conservation. In theoretical analysis aspects, based on the cut-off operator of solute concentration, we construct an analytical function to control and handle the change of porosity with time; we treat the auxiliary flux variable as a function of velocity and establish its properties; we employ the coupled analysis approach to deal with the fully coupling relation of multivariables. From this, the stability analysis and *a priori* error estimates for velocity, pressure, concentration and porosity are established in different norms. Numerical results are also given to verify theoretical analysis and effectiveness of the proposed scheme.

Key words. Mixed finite element methods; Wormhole; Error estimate; Darcy–Forchheimer model.

AMS subject classifications. 65M60; 65M12; 65M15

1. Introduction. Matrix acidization technique plays an important role in enhancing oil production rate when muds and fines deposit at the perforated well pore pipe [4, 5, 7, 9, 15, 20, 26, 29, 30]. In the technique, acid is injected into matrix to dissolve the rocks and deposits around the well bore and a channel with high porosities is formed. Such channel is called wormhole. Oil and gas components in the reservoir can be pushed to the surface easily through the channel. In the procedure, the solute transport equation can be expressed as below [20]

$$\frac{\partial(\phi c_f)}{\partial t} + \nabla \cdot (\mathbf{u} c_f) - \nabla \cdot (\phi \mathbf{D}_e \nabla c_f) = k_c a_v (c_s - c_f) + f_P c_f + f_I c_I, \quad (1.1)$$

where c_f is the cup-mixing concentration of the acid in the fluid phase, ϕ is the porosity, t is the time, $\mathbf{u}$ is the velocity, $\mathbf{D}_e$ is the effective dispersion tensor, k_c is the local mass-transfer coefficient, a_v is the interfacial area available for reaction per unit volume of the medium. The functions f_P and f_I are production and injection rates respectively, and c_I is the injected concentration. The variable c_s is the concentration of the acid at the fluid-solid interface, and the relationship between c_f and c_s is shown as

$$c_s = \frac{c_f}{1 + k_s/k_c}, \quad (1.2)$$

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where k_s is the surface reaction rate constant. The first three terms represent the accumulation, convection and dispersion of the acid respectively, and the fourth term represents the transfer of acid species from the fluid phase to the fluid-solid interface [20].

The advancement of chemical reaction front does not propagate uniformly along the injection direction. In fact, it advances in certain direction more than the other directions. As a result, a wormhole pattern is established. On the other hand, the heterogeneity of porosity and permeability in subsurface formations also promotes the chemical reaction front to be non-uniform. Suppose the permeability (denoted by K) is isotropic, then the relationship between the permeability and the porosity is established by the Carman-Kozeny correlation [17]

$$\frac{K}{K_0} = \frac{\phi}{\phi_0} \left(\frac{\phi(1 - \phi_0)}{\phi_0(1 - \phi)} \right)^2, \quad (1.3)$$

where K_0 and ϕ_0 are the initial permeability and porosity respectively. As a result of reaction, the evolution of porosity in the domain is expressed as

$$\frac{\partial \phi}{\partial t} = \frac{\alpha k_c a_v (c_f - c_s)}{\rho_s}, \quad (1.4)$$

where α is the dissolving power of the acid and ρ_s is the density of the solid phase. Interfacial area can be calculated from porosity and permeability as

$$\frac{a_v}{a_0} = \frac{\phi}{\phi_0} \sqrt{\frac{K_0 \phi}{K \phi_0}}, \quad (1.5)$$

where a_0 is the initial interfacial area.

Numerical simulation based on Darcy framework is applicable for the scenarios where the change of porosity is small, but it can be quite incorrect in cases where there is large change of porosity (and consequently the permeability). Apparently, there are situations in which the porosity becomes large (but less than one) and its spatial distribution also becomes very heterogeneous, and as a result, the fluid velocity is high in the high-porosity area. Therefore, the Darcy–Forchheimer model that accounts for higher velocity and nonuniform porosity is suggested [24]. In this work, Darcy–Forchheimer model is used to describe matrix acidization for the purpose of retrieving more reasonable results. The flow model formulated by the continuity equation and Darcy–Forchheimer equation can be expressed as

$$\gamma \frac{\partial p}{\partial t} + \frac{\partial \phi}{\partial t} + \nabla \cdot \mathbf{u} = f, \quad (1.6)$$

$$\frac{\rho}{\phi} \frac{\partial \mathbf{u}}{\partial t} + \frac{\mu}{K} \mathbf{u} + \frac{\rho F}{\sqrt{K}} |\mathbf{u}| \mathbf{u} = -\nabla p + \rho \mathbf{g}, \quad (1.7)$$

where $f = f_P + f_I$, p is the pressure, $F = \frac{1.75}{\sqrt{150\phi^3}}$ is the Forchheimer number, μ is the fluid viscosity, ρ is the mass density and $\mathbf{g}$ is the gravity vector. In the continuity equation (1.6), γ is a pseudo-compressibility parameter [10] that accounts for slight change of the density of the fluid phase in the dissolution process. As stated in [14], in some conditions of $\gamma = 0$, the linear system may be singular and cannot be solved

by the linear solver, and as a result, one suggested solution is to set γ to be a very small positive number to ensure an invertible coefficient matrix. The second term in Eq. (1.6) accounts for the effect of local volume change during dissolution on the flow field. In eq. (1.7), $|\mathbf{u}|$ represents the 2-norm of $\mathbf{u}$, and the term $-\frac{\mu}{K}\mathbf{u} - \nabla p + \rho\mathbf{g}$ is the Darcy term. For very high velocities in porous media, inertial effects can also become significant, so the inertial term $\frac{\rho F}{\sqrt{K}}|\mathbf{u}|\mathbf{u}$, which is also called the Forchheimer term [24], is included in (1.7).

In summary, the mathematical model of wormhole propagation in porous media is formulated by a nonlinear system of equations including the mass conservation equation (Eq. (1.6)), the Darcy-Forchheimer equation (Eq. (1.7)), the solute reaction transport equation (Eq. (1.1)) and a number of semi-empirical equations that bridge the scale differences (Eqs. (1.2), (1.3), (1.4) and (1.5)). This model is mostly based on [20] except for the Darcy-Forchheimer equation.

Because of its importance in enhancing production rate, matrix acidization in carbonate reservoirs has been extensively studied in [4, 5, 7, 9, 15, 20, 26, 29, 30], but the numerical methods are restricted to finite difference and finite volume methods. In [31], the combination of the finite element method and the finite difference method has been used to simulate the wormhole generation and propagation in carbonate rocks. In this paper, we use mixed finite element methods [2, 3, 12, 13] to simulate wormhole propagation. The advantages of mixed finite element methods can be listed as below: firstly, they are capable to achieve very accurate and stable approximations for both pressure variables and flux variables across grid-cell interfaces; secondly, they also have the local mass conservation property. Therefore, mixed finite element methods are preferable instrument for wormhole simulation.

Mixed finite element methods have been developed for Darcy-Forchheimer framework in [8, 11, 16, 18, 21, 25]. In our proposed mixed finite element methods for wormhole simulation, Darcy-Forchheimer equation is similar to that introduced in [18]. However, our methods take into account the change of porosity and permeability in (1.6) and (1.7), which is a distinct physical property of wormhole. A numerical scheme has been proposed in [19] to solve incompressible miscible displacement problem with the Darcy-Forchheimer model. In their scheme, a mixed finite element method is used to handle the Darcy-Forchheimer equation and a standard finite element method is used to approximate concentration. Compared to the problem considered in [19], the wormhole problem is more complicated because of the change of porosity and permeability. Moreover, different from [19], our proposed scheme uses mixed finite element method for the solute reaction transport equation instead of standard finite element method. Actually, in our proposed scheme, in order to guarantee full mass conservation of fluid, mixed finite element methods are used not only for the Darcy-Forchheimer equation (1.6) and mass conservation equation (1.7), but also for the solute reaction transport equation by introducing an auxiliary flux variable to reform (1.1). Thus, there are at least three issues in theoretical analysis of our proposed methods for wormhole simulation: the first one is to control and handle the change of porosity with time, the second one is the complication resulted from an auxiliary flux variable and the third one is the fully coupling relation of multivariables. To resolve the first issue, based on the cut-off operator of solute concentration, we construct an analytical function to bound the porosity; for the second issue, we define an auxiliary function of velocity and establish its properties; for the third issue, we employ the coupled analysis approach to achieve the estimates of multivariables.

The rest of this paper is organized as follows. In Section 2, we present the mixed

weak formulation and its mixed finite element approximation of wormhole problem. In Section 3, *a priori* bounds of discrete solutions are proved. In Section 4, the *a priori* error estimates of mixed finite element approximations are analyzed. In Section 5, we introduce an upwind quadrature rule and describe a time discretization scheme. In Section 6, we carry out numerical experiments using the lowest order Raviart–Thomas finite element. Finally, we make conclusions in Section 7.

2. Mixed finite element methods for wormhole problem. Suppose that $\Omega \subset \mathbf{R}^d (d = 2, 3)$ is a polygonal and bounded domain. The traditional notations are used for the Lebesgue spaces $L^p(\Omega)$ and Sobolev spaces $W^{k,p}(\Omega)$ with $1 \leq p \leq \infty$ over the domain Ω . In particular, $W^{k,2}(\Omega)$ is denoted by $H^k(\Omega)$. The corresponding norms are denoted by $\|\cdot\|_{L^p(\Omega)}$, $\|\cdot\|_{W^{k,p}(\Omega)}$ and $\|\cdot\|_{H^k(\Omega)}$ respectively. For simplicity, we also denote $\|\cdot\|_{L^p(\Omega)}$ by $\|\cdot\|_{L^p}$. The inner product of $L^p(\Omega)$ or $(L^p(\Omega))^d$ is denoted by $(\cdot, \cdot)$, and the inner product of the functions in the boundary Γ is denoted by $(\cdot, \cdot)_\Gamma$. We also use $|\cdot|$ to represent the 2-norm of a vector in $\mathbf{R}^d$ and a matrix in $\mathbf{R}^{d \times d}$. The norms for a vector-value function $\mathbf{v} \in (L^p(\Omega))^d$ and a matrix-value function $\mathbf{A} \in (L^p(\Omega))^{d \times d}$ are defined as

$$\|\mathbf{v}\|_{(L^p(\Omega))^d} := \|\|\mathbf{v}\|\|_{L^p(\Omega)}, \quad \|\mathbf{A}\|_{(L^p(\Omega))^{d \times d}} := \|\|\mathbf{A}\|\|_{L^p(\Omega)}.$$

For notation simplicity, the norms $\|\cdot\|_{L^2(\Omega)}$, $\|\cdot\|_{(L^2(\Omega))^d}$ and $\|\cdot\|_{(L^2(\Omega))^{d \times d}}$ are also written as $\|\cdot\|$ for scalar-value, vector-value, and matrix-value functions respectively. Let T be the final simulation time. For a given normed space X and $1 \leq p \leq \infty$, we define

$$L^p(0, T; X) := \left\{ \varphi : \varphi(t) \in X, \quad \|\varphi\|_X \in L^p(0, T) \right\},$$

which is a normed linear space equipped with the norm given as

$$\|\varphi\|_{L^p(0, T; X)} := \|(\|\varphi\|_X)\|_{L^p(0, T)}.$$

We now describe the initial-boundary conditions of wormhole problem. The initial pressure p_0 , initial velocity $\mathbf{u}_0$ and initial solute concentration c_0 are provided. Assume $0 \leq c_0 \leq 1$. In addition, we also need to specify the initial porosity ϕ_0 , initial permeability K_0 and initial interfacial area a_0 . We consider the following boundary conditions for wormhole problem:

$$\mathbf{u} \cdot \mathbf{n} = 0, \quad \mathbf{x} \in \partial\Omega, \quad t \in (0, T],$$

$$(\mathbf{u}c_f - \phi \mathbf{D}_e \nabla c_f) \cdot \mathbf{n} = 0, \quad \mathbf{x} \in \partial\Omega, \quad t \in (0, T].$$

Here, $\mathbf{n}$ denotes the unit outward normal vector to $\partial\Omega$.

For the physical quantities, we assume that $0 < \phi_{0*} \leq \phi_0 \leq \phi_0^* < 1$, $0 < K_{0*} \leq K_0 \leq K_0^*$ and $0 \leq a_{0*} \leq a_0 \leq a_0^*$. We assume that $f_P, f_I, c_I \in L^\infty(0, T; L^\infty(\Omega))$ and $f_P \leq 0, f_I \geq 0, 0 \leq c_I \leq 1$. In addition, we assume that ρ, μ, k_c and k_s are all given positive constants.

Define two functional spaces as

$$\mathbf{V} = \{\mathbf{v} \in (L^3(\Omega))^d : \nabla \cdot \mathbf{v} \in L^2(\Omega)\},$$

and

$$\mathbf{V}_0 = \{\mathbf{v} \in \mathbf{V} : \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}.$$

Furthermore, we define another functional spaces as

$$H(\text{div}; \Omega) = \{\mathbf{v} \in (L^2(\Omega))^d : \nabla \cdot \mathbf{v} \in L^2(\Omega)\},$$

and

$$H_0(\text{div}; \Omega) = \{\mathbf{v} \in H(\text{div}; \Omega) : \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}.$$

Using the relation (1.2), the transport equation (1.1) becomes

$$\frac{\partial(\phi c_f)}{\partial t} + \nabla \cdot (\mathbf{u} c_f) - \nabla \cdot (\phi \mathbf{D}_e \nabla c_f) = \frac{k_c k_s}{k_c + k_s} a_v c_f + f_P c_f + f_I c_I. \quad (2.1)$$

Here, the effective dispersion tensor $\mathbf{D}_e$ is defined as

$$\mathbf{D}_e(\mathbf{u}) = d_m \mathbf{I} + |\mathbf{u}| \{\alpha_l \mathbf{E}(\mathbf{u}) + \alpha_t (\mathbf{I} - \mathbf{E}(\mathbf{u}))\}, \quad (2.2)$$

where

$$(\mathbf{E}(\mathbf{u}))_{ij} = \frac{u_i u_j}{|\mathbf{u}|^2}, \quad 1 \leq i, j \leq d. \quad (2.3)$$

In (2.2), d_m is the molecular diffusivity that is assumed to be strictly positive; α_l and α_t are the longitudinal and the transverse dispersivities respectively, and they are also assumed to be positive.

In order to apply mixed finite element methods to (2.1), we define an auxiliary flux variable $\mathbf{q}$ as

$$\mathbf{q} = \mathbf{u} c_f - \phi \mathbf{D}_e \nabla c_f. \quad (2.4)$$

Then (2.1) is rewritten as

$$\frac{\partial(\phi c_f)}{\partial t} + \nabla \cdot \mathbf{q} = \frac{k_c k_s}{k_c + k_s} a_v c_f + f_P c_f + f_I c_I. \quad (2.5)$$

Furthermore, we define an auxiliary function of velocity as

$$\mathbf{M}_e(\mathbf{u}) = (\mathbf{D}_e(\mathbf{u}))^{-1} \mathbf{u}. \quad (2.6)$$

Thus, the mixed weak form describing wormhole problem is formulated as below

$$\gamma \left(\frac{\partial p}{\partial t}, \varphi \right) + \left(\frac{\partial \phi}{\partial t}, \varphi \right) + (\nabla \cdot \mathbf{u}, \varphi) = (f, \varphi), \quad \varphi \in L^2(\Omega), \quad (2.7)$$

$$\begin{aligned} & \left(\frac{\rho}{\phi} \frac{\partial \mathbf{u}}{\partial t}, \mathbf{v} \right) + \left(\frac{\mu}{K(\phi)} \mathbf{u}, \mathbf{v} \right) + \left(\frac{\rho F(\phi)}{\sqrt{K(\phi)}} |\mathbf{u}|, \mathbf{v} \right) \\ & = (p, \nabla \cdot \mathbf{v}) + (\rho \mathbf{g}, \mathbf{v}), \quad \mathbf{v} \in \mathbf{V}_0, \end{aligned} \quad (2.8)$$

$$\begin{aligned} & \left(\frac{\partial(\phi c_f)}{\partial t}, \psi \right) + (\nabla \cdot \mathbf{q}, \psi) + \frac{k_c k_s}{k_c + k_s} (a_v(\phi) c_f, \psi) \\ & - (f_P c_f, \psi) = (f_I c_I, \psi), \quad \psi \in L^2(\Omega), \end{aligned} \quad (2.9)$$

$$((\phi \mathbf{D}_e(\mathbf{u}))^{-1} \mathbf{q}, \mathbf{w}) = (\phi^{-1} \mathbf{M}_e(\mathbf{u}) c_f, \mathbf{w}) + (c_f, \nabla \cdot \mathbf{w}), \quad \mathbf{w} \in H_0(\text{div}; \Omega). \quad (2.10)$$

Denote $R_\phi = \frac{\alpha a_0 k_c k_s}{\rho_s(k_c + k_s)}$. Note that because of the boundedness of a_0 , R_ϕ is also bounded as $0 \leq R_{\phi*} \leq R_\phi \leq R_\phi^*$. Using the relations (1.2), (1.3) and (1.5), the evolution of porosity is reduced to

$$\frac{\partial \phi}{\partial t} = \frac{R_\phi (1 - \phi)}{1 - \phi_0} c_f. \quad (2.11)$$

We use a partition $\mathcal{E}_h$ of the domain Ω , which is composed of triangles or quadrilaterals if $d = 2$, or tetrahedra, prisms or hexahedra if $d = 3$. Denote by h_E the diameter of $E \in \mathcal{E}_h$ and let $h = \max_{E \in \mathcal{E}_h} h_E$ such that $h/h_E \leq C$. We assume that $\mathcal{E}_h$ is non-degenerate; that is, each element is convex, and that there exists $\rho > 0$ such that each of the sub-triangles (for $d = 2$) or sub-tetrahedra (for $d = 3$) of element E contains a ball of radius ρh_E in its interior. Denote by Γ_h the set of element boundaries. We also make an assumption that no element crosses the boundaries of $\partial\Omega$. Let the approximating subspaces $\mathcal{X} \subset \mathbf{V}$, $\mathcal{X}_0 \subset \mathbf{V}_0$, $\mathcal{Y} \subset H(\text{div}; \Omega)$, $\mathcal{Y}_0 \subset H_0(\text{div}; \Omega)$ and $\mathcal{W} \subset L^2(\Omega)$ be the r -th order ($r \geq 0$) Raviart-Thomas space (RT_r) on the partition $\mathcal{E}_h$.

Define the cut-off operator $\mathcal{M}$ for velocity as

$$\mathcal{M}(\mathbf{u}) := \begin{cases} \mathbf{u}, & \text{if } |\mathbf{u}| \leq M, \\ M\mathbf{u}/|\mathbf{u}|, & \text{if } |\mathbf{u}| > M, \end{cases} \quad (2.12)$$

where M is a given large positive constant.

Denote $J = (0, T]$, where $T > 0$. The mixed finite element formulation is stated as below: find $p_h(\cdot, t), c_h(\cdot, t) \in \mathcal{W}$ and $\mathbf{u}_h(\cdot, t) \in \mathcal{X}_0, \mathbf{q}_h(\cdot, t) \in \mathcal{Y}_0$ ($t \in J$) such that

$$\gamma \left(\frac{\partial p_h}{\partial t}, \varphi \right) + \left(\frac{\partial \phi_h}{\partial t}, \varphi \right) + (\nabla \cdot \mathbf{u}_h, \varphi) = (f, \varphi), \quad \varphi \in \mathcal{W}, \quad (2.13)$$

$$\begin{aligned} \left(\frac{\rho}{\phi_h} \frac{\partial \mathbf{u}_h}{\partial t}, \mathbf{v} \right) + \left(\frac{\mu}{K(\phi_h)} \mathbf{u}_h, \mathbf{v} \right) + \left(\frac{\rho F(\phi_h)}{\sqrt{K(\phi_h)}} |\mathbf{u}_h|, \mathbf{v} \right) \\ = (p_h, \nabla \cdot \mathbf{v}) + (\rho \mathbf{g}, \mathbf{v}), \quad \mathbf{v} \in \mathcal{X}_0, \end{aligned} \quad (2.14)$$

$$\begin{aligned} \left(\frac{\partial(\phi_h c_h)}{\partial t}, \psi \right) + (\nabla \cdot \mathbf{q}_h, \psi) + \frac{k_c k_s}{k_c + k_s} (a_v(\phi_h) c_h, \psi) \\ - (f_{PC_h}, \psi) = (f_{IC_I}, \psi), \quad \psi \in \mathcal{W}, \end{aligned} \quad (2.15)$$

$$((\phi_h \mathbf{D}_e(\mathbf{u}_h^M))^{-1} \mathbf{q}_h, \mathbf{w}) = (\phi_h^{-1} \mathbf{M}_e(\mathbf{u}_h) c_h, \mathbf{w}) + (c_h, \nabla \cdot \mathbf{w}), \quad \mathbf{w} \in \mathcal{Y}_0, \quad (2.16)$$

where $\mathbf{u}_h^M = \mathcal{M}(\mathbf{u}_h)$. The discrete porosity is computed as

$$\frac{\partial \phi_h}{\partial t} = \frac{R_\phi (1 - \phi_h)}{1 - \phi_0} \bar{c}_h, \quad (2.17)$$

where $\phi_h(\cdot, 0) = \phi(\cdot, 0) = \phi_0$ and $\bar{c}_h = \max(0, \min(c_h, 1))$. The discrete porosity is point-wise defined, and it will be shown that $\phi_h \in L^\infty(J \times \Omega)$ and $\frac{\partial \phi_h}{\partial t} \in L^\infty(J \times \Omega)$.

3. Stability analysis. In this section, we will prove the *a priori* bounds for discrete solutions $\phi_h, \mathbf{u}_h, p_h, \mathbf{q}_h$ and c_h . In what follows, we use C to indicate a generic positive constant independent of mesh size, but the values of C are probably different in different occurrences.

For the purpose of theoretical analysis, we first analyze the properties of $\mathbf{D}_e$ and $\mathbf{M}_e$.

LEMMA 3.1. *Let $\mathbf{D}_e(\mathbf{u})$ be defined as in (2.2) and $d_m > 0$. Suppose that α_l and α_t are non-negative functions of domain $\mathbf{x} \in \Omega$ and are uniformly bounded as $\alpha_l(\mathbf{x}) \leq \alpha_l^*$ and $\alpha_t(\mathbf{x}) \leq \alpha_t^*$. Then for $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$,*

$$|\mathbf{D}_e(\mathbf{u})^{-1}\mathbf{v}| \leq (d_m + \min(\alpha_l, \alpha_t)|\mathbf{u}|)^{-1}|\mathbf{v}| \leq d_m^{-1}|\mathbf{v}|, \quad (3.1)$$

$$|\mathbf{D}_e(\mathbf{u})\mathbf{v}| \leq (d_m + \max(\alpha_l^*, \alpha_t^*)|\mathbf{u}|)|\mathbf{v}|, \quad (3.2)$$

$$|\mathbf{D}_e(\mathbf{u})^{-1} - \mathbf{D}_e(\mathbf{v})^{-1}| \leq d_m^{-2} (7\alpha_t^* + 6\alpha_l^*) d^{3/2} |\mathbf{u} - \mathbf{v}|, \quad (3.3)$$

where d is the dimension of domain Ω .

Proof. The inequalities (3.1) and (3.2) are deduced from uniform positive definiteness of $\mathbf{D}(\mathbf{u})$ proven in [28]. Since

$$\begin{aligned} |\mathbf{D}_e(\mathbf{u})^{-1} - \mathbf{D}_e(\mathbf{v})^{-1}| &\leq |\mathbf{D}_e(\mathbf{u})^{-1}| |\mathbf{D}_e(\mathbf{u}) - \mathbf{D}_e(\mathbf{v})| |\mathbf{D}_e(\mathbf{v})^{-1}| \\ &\leq \frac{|\mathbf{D}_e(\mathbf{u}) - \mathbf{D}_e(\mathbf{v})|}{(d_m + \min(\alpha_l, \alpha_t)|\mathbf{u}|)(d_m + \min(\alpha_l, \alpha_t)|\mathbf{v}|)}, \end{aligned}$$

the inequality (3.3) is obtained by applying the uniform Lipschitz continuity of $\mathbf{D}(\mathbf{u})$ proven in [28]. $\square$

LEMMA 3.2. *Let $\mathbf{M}_e(\mathbf{u})$ be defined as in (2.6) and $d_m > 0$. Suppose that α_l and α_t are positive functions of domain $\mathbf{x} \in \Omega$ and $0 < \alpha_{l*} \leq \alpha_l(\mathbf{x}) \leq \alpha_l^*$ and $0 < \alpha_{t*} \leq \alpha_t(\mathbf{x}) \leq \alpha_t^*$. Then for $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$,*

$$|\mathbf{M}_e(\mathbf{u})| \leq (\min(\alpha_{l*}, \alpha_{t*}))^{-1}, \quad (3.4)$$

$$|\mathbf{M}_e(\mathbf{u}) - \mathbf{M}_e(\mathbf{v})| \leq L_M |\mathbf{u} - \mathbf{v}|, \quad (3.5)$$

where $L_M = d_m^{-1} + (d_m \min(\alpha_{l*}, \alpha_{t*}))^{-1} (7\alpha_t^* + 6\alpha_l^*) d^{3/2}$ and d is the dimension of domain Ω .

Proof. With the definition of $\mathbf{M}_e(\mathbf{u})$, the inequality (3.4) is a direct result of (3.1) when we take $\mathbf{v} = \mathbf{u}$. From (3.4) and Lemma 3.1, we have

$$\begin{aligned} |\mathbf{M}_e(\mathbf{u}) - \mathbf{M}_e(\mathbf{v})| &= |\mathbf{D}_e(\mathbf{u})^{-1}(\mathbf{u} - \mathbf{v})| + |(\mathbf{D}_e(\mathbf{u})^{-1} - \mathbf{D}_e(\mathbf{v})^{-1})\mathbf{v}| \\ &\leq |\mathbf{D}_e(\mathbf{u})^{-1}(\mathbf{u} - \mathbf{v})| + |\mathbf{D}_e(\mathbf{u})^{-1}| |\mathbf{D}_e(\mathbf{u}) - \mathbf{D}_e(\mathbf{v})| |\mathbf{M}_e(\mathbf{v})| \\ &\leq d_m^{-1} |\mathbf{u} - \mathbf{v}| + (d_m \min(\alpha_{l*}, \alpha_{t*}))^{-1} |\mathbf{D}_e(\mathbf{u}) - \mathbf{D}_e(\mathbf{v})|. \end{aligned}$$

Applying the uniform Lipschitz continuity of $\mathbf{D}(\mathbf{u})$ proven in [28], we obtain the inequality (3.5). $\square$

We now prove the stability results for the discrete solutions of (2.13)-(2.17).

THEOREM 3.1. *The discrete porosity ϕ_h is bounded almost everywhere, i.e., $\phi_0 \leq \phi_h \leq 1$, where $\phi_0 > 0$ is the initial porosity, and more precisely,*

$$\phi_0 \leq \phi_h \leq 1 - (1 - \phi_0)e^{-\zeta t}, \quad t \in J, \quad (3.6)$$

where $\zeta = \frac{R_\phi}{1-\phi_0}$. It also holds that

$$0 \leq \frac{\partial \phi_h}{\partial t} \leq R_\phi, \quad t \in J. \quad (3.7)$$

Proof. It is known that $\zeta > 0$. Define $(\phi_h - 1)^+ = \max(\phi_h - 1, 0)$. We rewrite (2.17) as

$$\frac{\partial(\phi_h - 1)}{\partial t} = -\zeta \bar{c}_h(\phi_h - 1). \quad (3.8)$$

Multiplying (3.8) by $(\phi_h - 1)^+$ gives

$$\frac{1}{2} \frac{\partial ((\phi_h - 1)^+)^2}{\partial t} = -\zeta \bar{c}_h ((\phi_h - 1)^+)^2 \leq 0. \quad (3.9)$$

Noticing that $(\phi_h - 1)^+ = 0$ at $t = 0$, we deduce from (3.9) that $((\phi_h - 1)^+)^2 \leq 0$ for $t > 0$. This means that $((\phi_h - 1)^+)^2 = 0$ for $t > 0$ because we have also $((\phi_h - 1)^+)^2 \geq 0$. Thus, $\phi_h \leq 1$ always holds for $t > 0$. It is observed from (2.17) that the discrete porosity is increasing with time, so $\phi_h \geq \phi_0$.

We now estimate the more precise bound of ϕ_h with time. Since $0 < \phi_0 < 1$, we can choose $t > 0$ such that $\phi_h < 1$. Noticing that

$$\frac{\partial(\phi_h - 1)}{\partial t} \leq \zeta(1 - \phi_h),$$

we deduce that

$$1 - \phi_h \geq (1 - \phi_0)e^{-\zeta t}. \quad (3.10)$$

Thus, the estimate (3.6) is obtained. The inequality (3.7) is reached by the boundedness of ϕ_h . $\square$

LEMMA 3.3. *The permeability $K(\phi_h)$ is bounded below, i.e., $K(\phi_h) \geq K_0$, where $K_0 > 0$ is the initial permeability. Moreover, for given time $T > 0$,*

$$K(\phi_h) \leq K_0 \frac{\phi_T}{\phi_0} \left(\frac{\phi_T(1 - \phi_0)}{\phi_0(1 - \phi_T)} \right)^2, \quad (3.11)$$

where $\phi_T = 1 - (1 - \phi_0)e^{-\zeta T}$ and $\zeta = \frac{R_\phi}{1-\phi_0}$. In addition, as the functions of ϕ ($\phi_0 \leq \phi < 1$), $K(\phi)^{-1}$ and $F(\phi)K(\phi)^{-1/2}$ are uniformly Lipschitz continuous.

Proof. The proof is immediately obtained by combining Theorem 3.1 and monotonicity of $K(\phi)$ and $F(\phi)$. $\square$

THEOREM 3.2. *The approximate solutions of (2.13)-(2.16) satisfy*

$$\begin{aligned} & \|p_h\|_{L^\infty(J;L^2)}^2 + \|\mathbf{u}_h\|_{L^\infty(J;L^2)}^2 + \|\mathbf{u}_h\|_{L^3(J;L^3)}^3 \\ & \leq C \left(|\rho \mathbf{g}|^2 + R_\phi^{*2} + \|p_0\|^2 + \|\mathbf{u}_0\|^2 + \|f\|_{L^2(J;L^2)}^2 \right), \end{aligned} \quad (3.12)$$

$$\|c_h\|_{L^\infty(J;L^2)}^2 + \left\| (\mathbf{D}_e(\mathbf{u}_h^M))^{-1/2} \mathbf{q}_h \right\|_{L^2(J;L^2)}^2 \leq C \left(\|c_0\|^2 + \|f_{IC_I}\|_{L^2(J;L^2)}^2 \right). \quad (3.13)$$

Proof. Take $\varphi = p_h$ in (2.13) and $\mathbf{v} = \mathbf{u}_h$ in (2.14), we obtain

$$\frac{\gamma}{2} \frac{\partial}{\partial t} \|p_h\|^2 + \left(\frac{\partial \phi_h}{\partial t}, p_h \right) + (\nabla \cdot \mathbf{u}_h, p_h) = (f, p_h), \quad (3.14)$$

$$\begin{aligned} \left(\frac{\rho}{\phi_h} \frac{\partial \mathbf{u}_h}{\partial t}, \mathbf{u}_h \right) + \left(\frac{\mu}{K(\phi_h)} \mathbf{u}_h, \mathbf{u}_h \right) + \left(\frac{\rho F(\phi_h)}{\sqrt{K(\phi_h)}} |\mathbf{u}_h| \mathbf{u}_h, \mathbf{u}_h \right) \\ = (p_h, \nabla \cdot \mathbf{u}_h) + (\rho \mathbf{g}, \mathbf{u}_h). \end{aligned} \quad (3.15)$$

Notice that

$$\left(\frac{\rho}{\phi_h} \frac{\partial \mathbf{u}_h}{\partial t}, \mathbf{u}_h \right) = \frac{1}{2} \frac{\partial}{\partial t} \left(\frac{\rho}{\phi_h} \mathbf{u}_h, \mathbf{u}_h \right) + \frac{1}{2} \left(\frac{\rho}{\phi_h^2} \frac{\partial \phi_h}{\partial t} \mathbf{u}_h, \mathbf{u}_h \right), \quad (3.16)$$

we sum the two equations (3.14) and (3.15) to get

$$\begin{aligned} \frac{\gamma}{2} \frac{\partial}{\partial t} \|p_h\|^2 + \frac{1}{2} \frac{\partial}{\partial t} \left(\frac{\rho}{\phi_h} \mathbf{u}_h, \mathbf{u}_h \right) + \left(\frac{\mu}{K(\phi_h)} \mathbf{u}_h, \mathbf{u}_h \right) + \left(\frac{\rho F(\phi_h)}{\sqrt{K(\phi_h)}} |\mathbf{u}_h| \mathbf{u}_h, \mathbf{u}_h \right) \\ = (f, p_h) + (\rho \mathbf{g}, \mathbf{u}_h) - \left(\frac{\partial \phi_h}{\partial t}, p_h \right) - \frac{1}{2} \left(\frac{\rho}{\phi_h^2} \frac{\partial \phi_h}{\partial t} \mathbf{u}_h, \mathbf{u}_h \right). \end{aligned} \quad (3.17)$$

It follows by using Theorem 3.1 and Young's inequality that

$$\begin{aligned} \frac{\gamma}{2} \frac{\partial}{\partial t} \|p_h\|^2 + \frac{1}{2} \frac{\partial}{\partial t} \left(\frac{\rho}{\phi_h} \mathbf{u}_h, \mathbf{u}_h \right) + \mu \|K(\phi_h)^{-1/2} \mathbf{u}_h\|^2 \\ + \left\| \left(\frac{\rho F(\phi_h)}{\sqrt{K(\phi_h)}} \right)^{1/3} \mathbf{u}_h \right\|_{L^3}^3 \leq \frac{1}{2} |\rho \mathbf{g}|^2 |\Omega| + \frac{1}{2} \|\mathbf{u}_h\|^2 \\ + \frac{1}{2} R_\phi^{*2} + \frac{1}{2} \|f\|^2 + \|p_h\|^2, \end{aligned} \quad (3.18)$$

where we have also used the fact $\left(\frac{\rho}{\phi_h} \frac{\partial \phi_h}{\partial t} \mathbf{u}_h, \mathbf{u}_h \right) \geq 0$. With the boundedness of ϕ_h , $F(\phi_h)$ and $K(\phi_h)$, integrating (3.18) in time from 0 to t ($t \leq T$) gives

$$\begin{aligned} \|p_h\|(t)^2 + \|\mathbf{u}_h\|^2(t) + \int_0^t \|\mathbf{u}_h\|^2 + \int_0^t \|\mathbf{u}_h\|_{L^3}^3 \leq C \|p_0\|^2 + C \|\mathbf{u}_0\|^2 \\ + C |\rho \mathbf{g}|^2 |\Omega| + C R_\phi^{*2} + C \|f\|_{L^2(J; L^2)}^2 + C \int_0^t (\|\mathbf{u}_h\|^2 + \|p_h\|^2). \end{aligned} \quad (3.19)$$

The estimate (3.12) is obtained by using Gronwall's lemma to (3.19).

We now turn to prove (3.13). Taking $\psi = c_h$ in (2.15) and $\mathbf{w} = \mathbf{q}_h$ (2.16) leads to

$$\begin{aligned} \left(\frac{\partial(\phi_h c_h)}{\partial t}, c_h \right) + (\nabla \cdot \mathbf{q}_h, c_h) + \frac{k_c k_s}{k_c + k_s} (a_v(\phi_h) c_h, c_h) \\ - (f_P c_h, c_h) = (f_I c_I, c_h), \end{aligned} \quad (3.20)$$

$$\left((\phi_h \mathbf{D}_e(\mathbf{u}_h^M))^{-1} \mathbf{q}_h, \mathbf{q}_h \right) = (\phi_h^{-1} \mathbf{M}_e(\mathbf{u}_h) c_h, \mathbf{q}_h) + (c_h, \nabla \cdot \mathbf{q}_h). \quad (3.21)$$

Since $\left(\frac{\partial \phi_h}{\partial t} c_h, c_h\right) \geq 0$, we get

$$\begin{aligned} \left(\frac{\partial(\phi_h c_h)}{\partial t}, c_h\right) &= \frac{1}{2} \frac{\partial}{\partial t} (\phi_h c_h, c_h) + \frac{1}{2} \left(\frac{\partial \phi_h}{\partial t} c_h, c_h\right) \\ &\geq \frac{1}{2} \frac{\partial}{\partial t} (\phi_h c_h, c_h). \end{aligned} \quad (3.22)$$

Noticing that $\frac{k_c k_s}{k_c + k_s} (a_v(\phi_h) c_h, c_h) \geq 0$ and $f_P \leq 0$, we sum the equations (3.20) and (3.21) and then obtain

$$\frac{1}{2} \frac{\partial}{\partial t} \|\phi_h^{1/2} c_h\|^2 + \left\| (\phi_h \mathbf{D}_e(\mathbf{u}_h^M))^{-1/2} \mathbf{q}_h \right\|^2 \leq (\phi_h^{-1} \mathbf{M}_e(\mathbf{u}_h) c_h, \mathbf{q}_h) + (f_I c_I, c_h) \quad (3.23)$$

For the first term on the right-hand side of (3.23), the boundedness of ϕ_h , $\mathbf{D}_e(\mathbf{u}_h^M)$ and $\mathbf{M}_e(\mathbf{u}_h)$ gives us

$$(\phi_h^{-1} \mathbf{M}_e(\mathbf{u}_h) c_h, \mathbf{q}_h) \leq C \|c_h\|^2 + \frac{1}{2} \left\| (\phi_h \mathbf{D}_e(\mathbf{u}_h^M))^{-1/2} \mathbf{q}_h \right\|^2. \quad (3.24)$$

With the estimate $(f_I c_I, c_h) \leq \frac{1}{2} (\|f_I c_I\|^2 + \|c_h\|^2)$, it follows by integrating (3.23) in time from 0 to t ($t \leq T$) and combining (3.24) that

$$\begin{aligned} \|c_h\|^2(t) + \int_0^t \left\| (\phi_h \mathbf{D}_e(\mathbf{u}_h^M))^{-1/2} \mathbf{q}_h \right\|^2 &\leq C \|c_0\|^2 \\ &+ C \|f_I c_I\|_{L^2(J; L^2)}^2 + C \int_0^t \|c_h\|^2, \end{aligned} \quad (3.25)$$

and thus, (3.13) is obtained by Gronwall's lemma. $\square$

4. Error estimates. In this section, we will estimate the approximate errors of mixed finite element solutions of (2.13)-(2.17). Denote by h the mesh size and $h \leq h_0$ ($h_0 > 0$). Furthermore, denote by r the order of Raviart-Thomas spaces. Let P_h be L^2 -projection operator, there exists a unique $P_h p \in \mathcal{W}$ such that

$$(p - P_h p, w) = 0, \quad \forall w \in \mathcal{W}. \quad (4.1)$$

Moreover, P_h satisfies the following approximation property

$$\|P_h p - p\| \leq C \|p\|_{H^j(\Omega)} h^j, \quad 0 \leq j \leq r + 1. \quad (4.2)$$

Let Π_h denote the usual Raviart-Thomas projection, which has the following properties [1, 3, 22, 23, 27]

$$(\nabla \cdot (\mathbf{u} - \Pi_h \mathbf{u}), w) = 0, \quad w \in \mathcal{W}, \quad (4.3)$$

$$\|\mathbf{u} - \Pi_h \mathbf{u}\|_{L^p} \leq C \|\mathbf{u}\|_{(W^{j,p}(\Omega))^d} h^j, \quad 1 \leq j \leq r + 1. \quad (4.4)$$

Assume that the exact solutions satisfy the following regularity requirements:

- (A1) $p \in L^\infty(J; H^s(\Omega))$, $\frac{\partial p}{\partial t} \in L^2(J; H^s(\Omega))$ and $p_0 \in H^s(\Omega)$;
- (A2) $\mathbf{u} \in (L^\infty(J; W^{s,p}(\Omega)))^d \cap (L^\infty(J \times \Omega))^d$, $\frac{\partial \mathbf{u}}{\partial t} \in (L^2(J; H^s(\Omega)))^d \cap (L^\infty(J \times \Omega))^d$, and $\mathbf{u}_0 \in (H^s(\Omega))^d \cap (L^\infty(\Omega))^d$;

(A3) $c_f \in L^\infty(J; H^{s+1}(\Omega)) \cap W^{2,\infty}(J \times \Omega)$, $\frac{\partial c_f}{\partial t} \in L^2(J; H^s(\Omega))$, $c_0 \in H^s(\Omega)$; where $s \geq r+1$ and $p \geq 3$.

For the cut-off operator for velocity defined in (2.12), we assume that the constant M is chosen as $M > 2\|\mathbf{u}\|_{(L^\infty(J \times \Omega))^d}$. Under this assumption, we can deduce that $|\mathbf{u}_h^M - \mathbf{u}| \leq |\mathbf{u}_h - \mathbf{u}|$ pointwise. In fact, it obviously holds if $|\mathbf{u}_h| \leq M$, so we only need to consider the case $|\mathbf{u}_h| > M$. In this case, denote $\kappa = M/|\mathbf{u}_h|$, and then $\kappa < 1$. By the definition of $\mathbf{u}^M$, we have

$$|\mathbf{u}_h^M - \mathbf{u}| = |\kappa \mathbf{u}_h - \mathbf{u}| \leq \kappa |\mathbf{u}_h - \mathbf{u}| + (1 - \kappa) |\mathbf{u}|.$$

On the other hand, the assumption implies $|\mathbf{u}_h| > M > 2|\mathbf{u}|$, and as a result

$$|\mathbf{u}_h - \mathbf{u}| \geq |\mathbf{u}_h| - |\mathbf{u}| \geq |\mathbf{u}|.$$

Consequently, the required result is obtained.

For the continuous-in-time scheme, we define finite element solution errors as

$$E_\phi = \phi - \phi_h, \quad E_p = p - p_h, \quad E_{\mathbf{u}} = \mathbf{u} - \mathbf{u}_h, \quad E_c = c_f - c_h, \quad E_{\mathbf{q}} = \mathbf{q} - \mathbf{q}_h.$$

Furthermore, define the projection errors as

$$E_p^I = P_h p - p, \quad E_{\mathbf{u}}^I = \Pi_h \mathbf{u} - \mathbf{u}, \quad E_c^I = P_h c_f - c_f, \quad E_{\mathbf{q}}^I = \Pi_h \mathbf{q} - \mathbf{q},$$

and also define the auxiliary errors as

$$E_p^A = P_h p - p_h, \quad E_{\mathbf{u}}^A = \Pi_h \mathbf{u} - \mathbf{u}_h, \quad E_c^A = P_h c_f - c_h, \quad E_{\mathbf{q}}^A = \Pi_h \mathbf{q} - \mathbf{q}_h.$$

LEMMA 4.1. *The approximate error of discrete porosity satisfies*

$$\|E_\phi\|^2(t) \leq R_\phi^* \int_0^t (\|E_\phi\|^2 + \|E_c\|^2), \quad (4.5)$$

$$\int_0^t \left\| \frac{\partial E_\phi}{\partial t} \right\|^2 \leq C \int_0^t (\|E_\phi\|^2 + \|E_c\|^2), \quad (4.6)$$

where $t \in (0, T]$.

Proof. It is observed that

$$\frac{\partial E_\phi}{\partial t} = -\frac{R_\phi}{1 - \phi_0} c_f E_\phi + \frac{R_\phi}{1 - \phi_0} (1 - \phi_h) (c_f - \bar{c}_h). \quad (4.7)$$

Multiplying (4.7) by E_ϕ , and then integrating it over Ω , we obtain that

$$\begin{aligned} \frac{1}{2} \frac{\partial}{\partial t} \|E_\phi\|^2 &= -\left(\frac{R_\phi}{1 - \phi_0} c_f E_\phi, E_\phi \right) \\ &\quad + \left(\frac{R_\phi}{1 - \phi_0} (1 - \phi_h) (c_f - \bar{c}_h), E_\phi \right) \\ &\leq \frac{R_\phi^*}{2} (\|E_\phi\|^2 + \|E_c\|^2), \end{aligned} \quad (4.8)$$

where we used the fact $|c_f - \bar{c}_h| \leq |c_f - c_h|$. The inequality (4.5) is obtained by integrating (4.8) in time from 0 to t and taking into account $\|E_\phi\|(0) = 0$. On the other hand, it follows from (4.7) that

$$\left\| \frac{\partial E_\phi}{\partial t} \right\| \leq \left\| \frac{R_\phi c_f}{1 - \phi_0} E_\phi \right\| + \left\| \frac{R_\phi}{1 - \phi_0} (1 - \phi_h) (c_f - \bar{c}_h) \right\|$$

$$\leq \left\| \frac{R_\phi c_f}{1 - \phi_0} \right\|_{L^\infty} \|E_\phi\| + R_\phi^* \|E_c\|. \quad (4.9)$$

Thus, the second estimate is obtained from (4.9). $\square$

LEMMA 4.2. *The auxiliary errors of discrete pressure and velocity satisfy*

$$\begin{aligned} & \|E_p^A\|^2(t) + \|E_u^A\|^2(t) + \int_0^t \|K(\phi_h)^{-1/2} E_u^A\|^2 \\ & + \int_0^t \left\| \left(\frac{\rho F(\phi_h)}{\sqrt{K(\phi_h)}} \right)^{1/3} E_u^A \right\|_{L^3}^3 \\ & \leq C \int_0^t \left(\|E_p^A\|^2 + \|E_u^A\|^2 + \|E_c^A\|^2 + \|E_\phi\|^2 \right) + Ch^{2r+2}, \end{aligned} \quad (4.10)$$

where $t \in (0, T]$ and the values of different C depend on T and the exact solutions, but independent of h .

Proof. Firstly, from the mixed finite element formulations of pressure and velocity, using the approximation properties, we obtain

$$\gamma \left(\frac{\partial E_p^A}{\partial t}, \varphi \right) + (\nabla \cdot E_u^A, \varphi) = \left(\frac{\partial E_\phi}{\partial t}, \varphi \right), \quad (4.11)$$

$$\begin{aligned} & \left(\frac{\rho}{\phi_h} \frac{\partial E_u^A}{\partial t}, \mathbf{v} \right) + \left(\frac{\mu}{K(\phi_h)} E_u^A, \mathbf{v} \right) + \left(\frac{\rho F(\phi_h)}{\sqrt{K(\phi_h)}} (|\mathbf{u}| \mathbf{u} - |\mathbf{u}_h| \mathbf{u}_h), \mathbf{v} \right) \\ & = (E_p^A, \nabla \cdot \mathbf{v}) + \left(\frac{\rho}{\phi_h} \frac{\partial E_u^I}{\partial t}, \mathbf{v} \right) + \left(\frac{\rho E_\phi}{\phi_h \phi} \frac{\partial \mathbf{u}}{\partial t}, \mathbf{v} \right) + \left(\frac{\mu}{K(\phi_h)} E_u^I, \mathbf{v} \right) \\ & + \left(\left(\frac{\mu}{K(\phi_h)} - \frac{\mu}{K(\phi)} \right) \mathbf{u}, \mathbf{v} \right) + \left(\left(\frac{\rho F(\phi_h)}{\sqrt{K(\phi_h)}} - \frac{\rho F(\phi)}{\sqrt{K(\phi)}} \right) |\mathbf{u}| \mathbf{u}, \mathbf{v} \right). \end{aligned} \quad (4.12)$$

Taking $\varphi = E_p^A$ in (4.11) and $\mathbf{v} = E_u^A$ in (4.12), and then summing these two equations, we get

$$\begin{aligned} & \frac{\gamma}{2} \frac{\partial}{\partial t} \|E_p^A\|^2 + \left(\frac{\rho}{\phi_h} \frac{\partial E_u^A}{\partial t}, E_u^A \right) + \mu \|K(\phi_h)^{-1/2} E_u^A\|^2 \\ & + \left(\frac{\rho F(\phi_h)}{\sqrt{K(\phi_h)}} (|\mathbf{u}| \mathbf{u} - |\mathbf{u}_h| \mathbf{u}_h), E_u^A \right) = \left(\frac{\partial E_\phi}{\partial t}, E_p^A \right) \\ & + \left(\frac{\rho E_\phi}{\phi_h \phi} \frac{\partial \mathbf{u}}{\partial t}, E_u^A \right) + \left(\frac{\rho}{\phi_h} \frac{\partial E_u^I}{\partial t}, E_u^A \right) + \left(\frac{\mu}{K(\phi_h)} E_u^I, E_u^A \right) \\ & + \left(\left(\frac{\mu}{K(\phi_h)} - \frac{\mu}{K(\phi)} \right) \mathbf{u}, E_u^A \right) + \left(\left(\frac{\rho F(\phi_h)}{\sqrt{K(\phi_h)}} - \frac{\rho F(\phi)}{\sqrt{K(\phi)}} \right) |\mathbf{u}| \mathbf{u}, E_u^A \right). \end{aligned} \quad (4.13)$$

The second term on the left-hand side of (4.13) is bounded below as

$$\left(\frac{\rho}{\phi_h} \frac{\partial E_u^A}{\partial t}, E_u^A \right) = \frac{1}{2} \left(\frac{\partial}{\partial t} \left(\frac{\rho}{\phi_h} E_u^A, E_u^A \right) + \left(\frac{\rho}{\phi_h^2} \frac{\partial \phi_h}{\partial t} E_u^A, E_u^A \right) \right)$$

$$\geq \frac{1}{2} \frac{\partial}{\partial t} \left(\frac{\rho}{\phi_h} E_{\mathbf{u}}^A, E_{\mathbf{u}}^A \right) \quad (4.14)$$

since $\frac{\rho}{\phi_h^2} \frac{\partial \phi_h}{\partial t} \geq 0$. The last term on the left-hand side of (4.13) can be rewritten as

$$\begin{aligned} \left(\frac{\rho F(\phi_h)}{\sqrt{K(\phi_h)}} (|\mathbf{u}| \mathbf{u} - |\mathbf{u}_h| \mathbf{u}_h), E_{\mathbf{u}}^A \right) &= \left(\frac{\rho F(\phi_h)}{\sqrt{K(\phi_h)}} (|\mathbf{u}| - |\mathbf{u}_h|) \mathbf{u}, E_{\mathbf{u}}^A \right) \\ &\quad + \left(\frac{\rho F(\phi_h)}{\sqrt{K(\phi_h)}} |\mathbf{u} - \mathbf{u}_h| (\mathbf{u} - \mathbf{u}_h), E_{\mathbf{u}}^A \right) \\ &\quad + \left(\frac{\rho F(\phi_h)}{\sqrt{K(\phi_h)}} (|\mathbf{u}_h| - |\mathbf{u} - \mathbf{u}_h|) (\mathbf{u} - \mathbf{u}_h), E_{\mathbf{u}}^A \right) \\ &:= T_1 + T_2 + T_3. \end{aligned} \quad (4.15)$$

We recall the elementary lemma

$$||\mathbf{w}| - |\mathbf{w} - \mathbf{v}|| \leq |\mathbf{v}|$$

for any d -dimensional vectors $\mathbf{w}$ and $\mathbf{v}$. The terms T_1 and T_3 are estimated as

$$|T_1| + |T_3| \leq 2 \left\| \frac{\rho F(\phi_0)}{\sqrt{K_0}} \right\|_{L^\infty} \|\mathbf{u}\|_{L^\infty} \|\mathbf{u} - \mathbf{u}_h\| \|E_{\mathbf{u}}^A\| \leq C (\|E_{\mathbf{u}}^A\|^2 + \|E_{\mathbf{u}}^I\|^2) \quad (4.16)$$

With Young's inequality, the term T_2 is estimated as

$$\begin{aligned} T_2 &= \left(\frac{\rho F(\phi_h)}{\sqrt{K(\phi_h)}} |E_{\mathbf{u}}^A| E_{\mathbf{u}}^A, E_{\mathbf{u}}^A \right) + \left(\frac{\rho F(\phi_h)}{\sqrt{K(\phi_h)}} (|E_{\mathbf{u}}| - |E_{\mathbf{u}}^A|) E_{\mathbf{u}}^A, E_{\mathbf{u}}^A \right) \\ &\quad - \left(\frac{\rho F(\phi_h)}{\sqrt{K(\phi_h)}} |E_{\mathbf{u}}| E_{\mathbf{u}}^I, E_{\mathbf{u}}^A \right) \\ &\geq \left\| \left(\frac{\rho F(\phi_h)}{\sqrt{K(\phi_h)}} \right)^{1/3} E_{\mathbf{u}}^A \right\|_{L^3}^3 - 2 \left\| \left(\frac{\rho F(\phi_h)}{\sqrt{K(\phi_h)}} \right)^{1/3} E_{\mathbf{u}}^I \right\|_{L^3} \left\| \left(\frac{\rho F(\phi_h)}{\sqrt{K(\phi_h)}} \right)^{1/3} E_{\mathbf{u}}^A \right\|_{L^3}^2 \\ &\quad - \left\| \left(\frac{\rho F(\phi_h)}{\sqrt{K(\phi_h)}} \right)^{1/3} E_{\mathbf{u}}^A \right\|_{L^3} \left\| \left(\frac{\rho F(\phi_h)}{\sqrt{K(\phi_h)}} \right)^{1/3} E_{\mathbf{u}}^I \right\|_{L^3}^2 \\ &\geq \frac{1}{2} \left\| \left(\frac{\rho F(\phi_h)}{\sqrt{K(\phi_h)}} \right)^{1/3} E_{\mathbf{u}}^A \right\|_{L^3}^3 - C \left\| \frac{\rho F(\phi_0)}{\sqrt{K_0}} \right\|_{L^\infty} \|E_{\mathbf{u}}^I\|_{L^3}^3. \end{aligned} \quad (4.17)$$

We now turn to estimate the terms on the right-hand side of (4.13). Using Cauchy-Schwarz inequality, we obtain the estimates

$$\left(\frac{\partial E_\phi}{\partial t}, E_p^A \right) \leq \frac{1}{2\gamma} \left\| \frac{\partial E_\phi}{\partial t} \right\|^2 + \frac{1}{2} \gamma \|E_p^A\|^2, \quad (4.18)$$

$$\left(\frac{\rho E_\phi}{\phi_h \phi} \frac{\partial \mathbf{u}}{\partial t}, E_{\mathbf{u}}^A \right) \leq \frac{1}{2} \left\| \frac{\rho}{\phi_0^2} \right\|_{L^\infty}^2 \left\| \frac{\partial \mathbf{u}}{\partial t} \right\|_{L^\infty}^2 \|E_\phi\|^2 + \frac{1}{2} \|E_{\mathbf{u}}^A\|^2. \quad (4.19)$$

It is derived by the boundedness of ϕ_h and $K(\phi_h)^{-1}$ that

$$\left(\frac{\rho}{\phi_h} \frac{\partial E_{\mathbf{u}}^I}{\partial t}, E_{\mathbf{u}}^A \right) + \left(\frac{\mu}{K(\phi_h)} E_{\mathbf{u}}^I, E_{\mathbf{u}}^A \right) \leq C \left\| \frac{\partial E_{\mathbf{u}}^I}{\partial t} \right\|^2 + C \|E_{\mathbf{u}}^I\|^2 + \|E_{\mathbf{u}}^A\|^2. \quad (4.20)$$

Using the uniform Lipschitz continuity of $K(\cdot)^{-1}$ and $\frac{F(\cdot)}{\sqrt{K(\cdot)}}$, we get

$$\begin{aligned} & \left(\left(\frac{\mu}{K(\phi_h)} - \frac{\mu}{K(\phi)} \right) \mathbf{u}, E_{\mathbf{u}}^A \right) + \left(\left(\frac{\rho F(\phi_h)}{\sqrt{K(\phi_h)}} - \frac{\rho F(\phi)}{\sqrt{K(\phi)}} \right) |\mathbf{u}|, E_{\mathbf{u}}^A \right) \\ & \leq C \|E_{\phi}\|^2 + \|E_{\mathbf{u}}^A\|^2. \end{aligned} \quad (4.21)$$

Combining the above estimates gives

$$\begin{aligned} & \frac{\gamma}{2} \frac{\partial}{\partial t} \|E_p^A\|^2 + \frac{1}{2} \frac{\partial}{\partial t} \left(\frac{\rho}{\phi_h} E_{\mathbf{u}}^A, E_{\mathbf{u}}^A \right) + \mu \|K(\phi_h)^{-1/2} E_{\mathbf{u}}^A\|^2 \\ & + \frac{1}{2} \left\| \left(\frac{\rho F(\phi_h)}{\sqrt{K(\phi_h)}} \right)^{1/3} E_{\mathbf{u}}^A \right\|_{L^3}^3 \\ & \leq C \left\| \frac{\partial E_{\mathbf{u}}^I}{\partial t} \right\|^2 + C \|E_{\mathbf{u}}^I\|^2 + C \|E_{\mathbf{u}}^I\|_{L^3}^3 + \frac{\gamma}{2} \|E_p^A\|^2 + C \|E_{\mathbf{u}}^A\|^2 \\ & + \left\| \frac{\partial E_{\phi}}{\partial t} \right\|^2 + C \|E_{\phi}\|^2. \end{aligned} \quad (4.22)$$

Integrating (4.22) in time from 0 to t ($t \in J$), and using Lemma 4.1 and approximation results, we reach the estimate (4.10). $\square$

LEMMA 4.3. *The auxiliary errors of discrete concentration and auxiliary flux variable satisfy*

$$\begin{aligned} \|E_c^A\|^2(t) + \int_0^t \left\| (\phi_h \mathbf{D}_e(\mathbf{u}_h^M))^{-1/2} E_{\mathbf{q}}^A \right\|^2 & \leq C \int_0^t \left(\|E_c^A\|^2 + \|E_{\mathbf{u}}^A\|^2 + \|E_{\phi}\|^2 \right) \\ & + Ch^{2r+2}, \end{aligned} \quad (4.23)$$

where $t \in (0, T]$ and the values of different C depend on T and the exact solutions, but independent of h .

Proof. Firstly, it is easy to know that for the transport equation,

$$\begin{aligned} & \left(\frac{\partial(\phi_h E_c)}{\partial t}, \psi \right) + (\nabla \cdot E_{\mathbf{q}}^A, \psi) + \frac{k_c k_s}{k_c + k_s} (a_v(\phi_h) E_c^A, \psi) \\ & = - \left(\frac{\partial(E_{\phi} c_f)}{\partial t}, \psi \right) + \frac{k_c k_s}{k_c + k_s} (a_v(\phi_h) E_c^I, \psi) \\ & + \frac{k_c k_s}{k_c + k_s} ((a_v(\phi_h) - a_v(\phi)) c_f, \psi) + (f_P E_c, \psi), \end{aligned} \quad (4.24)$$

$$\begin{aligned} & ((\phi_h \mathbf{D}_e(\mathbf{u}_h^M))^{-1} E_{\mathbf{q}}^A, \mathbf{w}) = (E_c^A, \nabla \cdot \mathbf{w}) + (\phi_h^{-1} \mathbf{M}_e(\mathbf{u}_h) E_c, \mathbf{w}) \\ & + ((\phi_h \mathbf{D}_e(\mathbf{u}_h^M))^{-1} E_{\mathbf{q}}^I, \mathbf{w}) + (((\phi_h \mathbf{D}_e(\mathbf{u}_h^M))^{-1} - (\phi \mathbf{D}_e(\mathbf{u}))^{-1}) \mathbf{q}, \mathbf{w}) \\ & + ((\phi^{-1} \mathbf{M}_e(\mathbf{u}) - \phi_h^{-1} \mathbf{M}_e(\mathbf{u}_h)) c_f, \mathbf{w}). \end{aligned} \quad (4.25)$$

Take $\psi = E_c^A$ in (4.24) and $\mathbf{w} = E_{\mathbf{q}}^A$ in (4.25) respectively, and then sum these two equations

$$\begin{aligned}
 & \left(\frac{\partial(\phi_h E_c^A)}{\partial t}, E_c^A \right) + \left\| (\phi_h \mathbf{D}_e(\mathbf{u}_h^M))^{-1/2} E_{\mathbf{q}}^A \right\|^2 + \frac{k_c k_s}{k_c + k_s} (a_v(\phi_h) E_c^A, E_c^A) \\
 &= \left(\frac{\partial(\phi_h E_c^I)}{\partial t}, E_c^A \right) - \left(\frac{\partial(E_\phi c_f)}{\partial t}, E_c^A \right) + \frac{k_c k_s}{k_c + k_s} (a_v(\phi_h) E_c^I, E_c^A) \\
 &+ \frac{k_c k_s}{k_c + k_s} ((a_v(\phi_h) - a_v(\phi)) c_f, E_c^A) + (\phi_h^{-1} \mathbf{M}_e(\mathbf{u}_h) E_c, E_{\mathbf{q}}^A) \\
 &+ ((\phi_h \mathbf{D}_e(\mathbf{u}_h^M))^{-1} E_{\mathbf{q}}^I, E_{\mathbf{q}}^A) + (((\phi_h \mathbf{D}_e(\mathbf{u}_h^M))^{-1} - (\phi \mathbf{D}_e(\mathbf{u}))^{-1}) \mathbf{q}, E_{\mathbf{q}}^A) \\
 &+ ((\phi_h^{-1} \mathbf{M}_e(\mathbf{u}) - \phi_h^{-1} \mathbf{M}_e(\mathbf{u}_h)) c_f, E_{\mathbf{q}}^A) + (f_P E_c, E_c^A). \tag{4.26}
 \end{aligned}$$

Because $\frac{\partial \phi_h}{\partial t} \geq 0$, the first term on the left-hand side of (4.26) is estimated as

$$\begin{aligned}
 \left(\frac{\partial(\phi_h E_c^A)}{\partial t}, E_c^A \right) &= \frac{1}{2} \left(\frac{\partial}{\partial t} (\phi_h E_c^A, E_c^A) + \left(\frac{\partial \phi_h}{\partial t} E_c^A, E_c^A \right) \right) \\
 &\geq \frac{1}{2} \frac{\partial}{\partial t} (\phi_h E_c^A, E_c^A). \tag{4.27}
 \end{aligned}$$

Since k_c, k_s and $a_v(\phi_h)$ are all nonnegative, the third term on the left-hand side of (4.26) is also nonnegative. The first term on the right-hand side of (4.26) is estimated as

$$\begin{aligned}
 \left(\frac{\partial(\phi_h E_c^I)}{\partial t}, E_c^A \right) &= \left(\frac{\partial \phi_h}{\partial t} E_c^I, E_c^A \right) + \left(\phi_h \frac{\partial E_c^I}{\partial t}, E_c^A \right) \\
 &\leq \|E_c^A\|^2 + \frac{1}{2} R_\phi^{*2} \|E_c^I\|^2 + \frac{1}{2} \left\| \frac{\partial E_c^I}{\partial t} \right\|^2, \tag{4.28}
 \end{aligned}$$

where we have used the boundedness of ϕ_h and $\frac{\partial \phi_h}{\partial t}$. The second term on the right-hand side of (4.26) is estimated as

$$\begin{aligned}
 - \left(\frac{\partial(E_\phi c_f)}{\partial t}, E_c^A \right) &= - \left(c_f \frac{\partial E_\phi}{\partial t}, E_c^A \right) - \left(E_\phi \frac{\partial c_f}{\partial t}, E_c^A \right) \\
 &\leq \|E_c^A\|^2 + \frac{1}{2} \left\| \frac{\partial c_f}{\partial t} \right\|_{L^\infty} \|E_\phi\|^2 + \frac{1}{2} \|c_f\|_{L^\infty} \left\| \frac{\partial E_\phi}{\partial t} \right\|^2. \tag{4.29}
 \end{aligned}$$

Noticing that $0 \leq a_v(\phi_h) \leq a_0$ and $|a_v(\phi_h) - a_v(\phi)| \leq \frac{a_0}{1-\phi_0} |E_\phi|$, we obtain the estimates

$$\frac{k_c k_s}{k_c + k_s} (a_v(\phi_h) E_c^I, E_c^A) \leq \frac{1}{2} \|E_c^A\|^2 + \frac{1}{2} \frac{(k_c k_s a_0^*)^2}{(k_c + k_s)^2} \|E_c^I\|^2, \tag{4.30}$$

$$\begin{aligned}
 & \frac{k_c k_s}{k_c + k_s} ((a_v(\phi_h) - a_v(\phi)) c_f, E_c^A) \leq \frac{1}{2} \|E_c^A\|^2 \\
 &+ \frac{1}{2} \frac{(k_c k_s a_0^*)^2}{(k_c + k_s)^2 (1 - \phi_{0*})^2} \|c_f\|_{L^\infty}^2 \|E_\phi\|^2. \tag{4.31}
 \end{aligned}$$

Using the boundedness of $\mathbf{M}_e(\mathbf{u}_h)$ and $\mathbf{D}_e(\mathbf{u}_h^M)$, we get

$$(\phi_h^{-1} \mathbf{M}_e(\mathbf{u}_h) E_c, E_{\mathbf{q}}^A) + ((\phi_h \mathbf{D}_e(\mathbf{u}_h^M))^{-1} E_{\mathbf{q}}^I, E_{\mathbf{q}}^A)$$

$$\leq \frac{1}{4} \left\| (\phi_h \mathbf{D}_e(\mathbf{u}_h^M))^{-1/2} E_{\mathbf{q}}^A \right\|^2 + C \|E_c\|^2 + C \|E_{\mathbf{q}}^I\|^2. \quad (4.32)$$

From the hypothesis regarding the choice of M , we have $|\mathbf{u}_h^M - \mathbf{u}| \leq |\mathbf{u}_h - \mathbf{u}|$ pointwise, so using the Lipschitz continuity of $\mathbf{M}_e(\cdot)$ and $\mathbf{D}_e(\cdot)$, we get

$$\begin{aligned} & ((\phi_h \mathbf{D}_e(\mathbf{u}_h^M))^{-1} - (\phi \mathbf{D}_e(\mathbf{u}))^{-1}) \mathbf{q}, E_{\mathbf{q}}^A) + ((\phi^{-1} \mathbf{M}_e(\mathbf{u}) - \phi_h^{-1} \mathbf{M}_e(\mathbf{u}_h)) c_f, E_{\mathbf{q}}^A) \\ & \leq \frac{1}{4} \left\| (\phi_h \mathbf{D}_e(\mathbf{u}_h^M))^{-1/2} E_{\mathbf{q}}^A \right\|^2 + C (\|c_f\|_{L^\infty}^2 + \|\mathbf{q}\|_{L^\infty}^2) (\|E_{\mathbf{u}}\|^2 + \|E_\phi\|^2). \end{aligned} \quad (4.33)$$

Since $f_P \leq 0$, the last term is estimated as

$$\begin{aligned} (f_P E_c, E_c^A) &= (f_P E_c^A, E_c^A) - (f_P E_c^I, E_c^A) \\ &\leq - (f_P E_c^I, E_c^A) \leq \frac{1}{2} (\|f_P\|_{L^\infty}^2 \|E_c^I\|^2 + \|E_c^A\|^2). \end{aligned} \quad (4.34)$$

We summarize the above estimates and get

$$\begin{aligned} & \frac{1}{2} \frac{\partial}{\partial t} (\phi_h E_c^A, E_c^A) + \frac{1}{2} \left\| (\phi_h \mathbf{D}_e(\mathbf{u}_h^M))^{-1/2} E_{\mathbf{q}}^A \right\|^2 \leq C \|E_c^A\|^2 + C \|E_c^I\|^2 \\ & + \frac{1}{2} \left\| \frac{\partial E_c^I}{\partial t} \right\|^2 + C \|E_{\mathbf{q}}^I\|^2 + C \|E_{\mathbf{u}}\|^2 + C \|E_\phi\|^2 + C \left\| \frac{\partial E_\phi}{\partial t} \right\|^2. \end{aligned} \quad (4.35)$$

Finally, the estimate (4.23) is obtained by integrating (4.35) in time from 0 to t ($t \in J$), and using Lemma 4.1 and approximation results. $\square$

Based on the above lemmas, we are now able to prove the main error estimates as below.

THEOREM 4.1. *The mixed finite element approximate solutions of (2.13)-(2.17) satisfy the following error estimates*

$$\begin{aligned} & \|E_\phi\|_{L^\infty(J; L^2)}^2 + \|E_p\|_{L^\infty(J; L^2)}^2 + \|E_{\mathbf{u}}\|_{L^\infty(J; L^2)}^2 + \|E_c\|_{L^\infty(J; L^2)}^2 \\ & + \left\| \frac{\partial E_\phi}{\partial t} \right\|_{L^2(J; L^2)}^2 + \left\| K(\phi_h)^{-1/2} E_{\mathbf{u}}^A \right\|_{L^2(J; L^2)}^2 + \left\| \left(\frac{\rho F(\phi_h)}{\sqrt{K(\phi_h)}} \right)^{1/3} E_{\mathbf{u}}^A \right\|_{L^3(J; L^3)}^3 \\ & + \left\| (\phi_h \mathbf{D}_e(\mathbf{u}_h^M))^{-1/2} E_{\mathbf{q}}^A \right\|_{L^2(J; L^2)}^2 \leq C h^{2r+2}, \end{aligned} \quad (4.36)$$

where $J = (0, T]$ and the value of C depends on T and the exact solutions, but independent of h .

Proof. It is followed from Lemmas 4.1-4.3 that

$$\begin{aligned} & \|E_\phi\|^2(t) + \|E_p^A\|^2(t) + \|E_{\mathbf{u}}^A\|^2(t) + \|E_c^A\|^2(t) + \int_0^t \left\| \frac{\partial E_\phi}{\partial t} \right\|^2 \\ & + \int_0^t \left\| K(\phi_h)^{-1/2} E_{\mathbf{u}}^A \right\|^2 + \int_0^t \left\| \left(\frac{\rho F(\phi_h)}{\sqrt{K(\phi_h)}} \right)^{1/3} E_{\mathbf{u}}^A \right\|_{L^3}^3 \\ & + \int_0^t \left\| (\phi_h \mathbf{D}_e(\mathbf{u}_h^M))^{-1/2} E_{\mathbf{q}}^A \right\|^2 \\ & \leq C \int_0^t (\|E_\phi\|^2 + \|E_p^A\|^2 + \|E_{\mathbf{u}}^A\|^2 + \|E_c^A\|^2) + C h^{2r+2}, \end{aligned} \quad (4.37)$$

where $t \in (0, T]$ and the values of different C depend on T and the exact solutions, but independent of h . Thus, the estimate (4.36) is obtained by applying Gronwall's lemma to (4.37) and then using the triangle rule. $\square$

In many practical applications, the recovery rate, denoted by Θ , is an important indicator in oil production, which is defined as

$$\Theta = 1 - \frac{\int_{\Omega} c_f(\mathbf{x}, t) d\mathbf{x}}{\int_{\Omega} c_0 d\mathbf{x}} \quad (4.38)$$

and the discrete recovery rate Θ_h as

$$\Theta_h = 1 - \frac{\int_{\Omega} c_h(\mathbf{x}, t) d\mathbf{x}}{\int_{\Omega} c_0 d\mathbf{x}}. \quad (4.39)$$

From Theorem 4.1, we immediately deduce the following approximate error of discrete recovery rate.

COROLLARY 4.1. *The error between the exact and discrete recovery rates is bounded as*

$$|\Theta_h - \Theta| \leq Ch^{r+1}, \quad (4.40)$$

where C is independent of h .

5. Upwind quadrature rule and time discretization scheme.

5.1. Upwind quadrature rule. In numerical implement of finite element methods, the quadrature rule is usually necessary for integral calculations. But the conventional quadrature rule lacks upwind mechanism to characterize the convection term. Notice that the functions in $\mathcal{W}$ may be discontinuous, so we introduce the following upwind quadrature rule for c_h as

$$(\phi_h^{-1} \mathbf{M}_e(\mathbf{u}_h) c_h, \mathbf{w})_E \approx \sum_{i=1}^{Q_d} \omega_i \phi_h^{-1}(\mathbf{x}_i) \mathbf{M}_e(\mathbf{u}_h(\mathbf{x}_i)) \cdot \mathbf{w}(\mathbf{x}_i) c_h^*(\mathbf{x}_i), \quad (5.1)$$

where E is an element, Q_d is the number of quadrature points and ω_i are the quadrature weights. If the point $\mathbf{x}_i \in e$, where $e \in \Gamma_h$ is shared by the two elements E_e^1 and E_e^2 with the unit normal vector $\mathbf{n}_e$ exterior to E_e^1 , then the upwind value of c_h^* at $\mathbf{x}_i$ is defined as

$$c_h^*(\mathbf{x}_i) = \begin{cases} c_h(\mathbf{x}_i)|_{E_e^1}, & \text{if } \mathbf{u}_h(\mathbf{x}_i) \cdot \mathbf{n}_e \geq 0, \\ c_h(\mathbf{x}_i)|_{E_e^2}, & \text{if } \mathbf{u}_h(\mathbf{x}_i) \cdot \mathbf{n}_e < 0. \end{cases} \quad (5.2)$$

In numerical tests, we employ the lowest order Raviart-Thomas space (RT_0) on a rectangular mesh, and furthermore, we apply the upwind trapezoid quadrature rule to $(\phi_h^{-1} \mathbf{M}_e(\mathbf{u}_h) c_h, \mathbf{w})$ and apply the conventional trapezoid quadrature rule to $((\phi_h \mathbf{D}_e(\mathbf{u}_h^M))^{-1} \mathbf{q}_h, \mathbf{w})$ respectively, for decoupling the system and getting explicit formula for each individual diffusive flux [2].

5.2. Time discretization scheme. We now present the time discretization of the mixed finite element forms used in numerical experiment. Let the total time interval $J = (0, T]$, where $T > 0$. Divide J into N subintervals $I_n = (t_n, t_{n+1}]$, where $t_0 = 0$ and $t_N = T$, and denote $\delta t_n = t_{n+1} - t_n$. For any scalar variable $v(t)$ or vector variable $\mathbf{u}(t)$, we denote by v^n or $\mathbf{u}^n$ its approximation at the time point t_n .

Suppose that $p_h^n, c_h^n \in \mathcal{W}$ and $\mathbf{u}_h^n \in \mathcal{X}_0, \mathbf{q}_h^n \in \mathcal{Y}_0$ ($0 \leq n \leq N-1$) are given or calculated, the semi-implicit mixed finite element formulation is stated as below: find $p_h^{n+1}, c_h^{n+1} \in \mathcal{W}$ and $\mathbf{u}_h^{n+1} \in \mathcal{X}_0, \mathbf{q}_h^{n+1} \in \mathcal{Y}_0$ such that

$$\gamma \left(\frac{p_h^{n+1} - p_h^n}{\delta t_n}, \varphi \right) + \left(\frac{\phi_h^{n+1} - \phi_h^n}{\delta t_n}, \varphi \right) + (\nabla \cdot \mathbf{u}_h^{n+1}, \varphi) = (f(\mathbf{x}, t^{n+1}), \varphi), \quad \varphi \in \mathcal{V}, \quad (5.3)$$

$$\begin{aligned} & \left(\frac{\rho}{\phi_h^{n+1}} \frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\delta t_n}, \mathbf{v} \right) + \left(\frac{\mu}{K(\phi_h^{n+1})} \mathbf{u}_h^{n+1}, \mathbf{v} \right) + \left(\frac{\rho F(\phi_h^{n+1})}{\sqrt{K(\phi_h^{n+1})}} |\mathbf{u}_h^n| \mathbf{u}_h^{n+1}, \mathbf{v} \right) \\ & = (p_h^{n+1}, \nabla \cdot \mathbf{v}) + (\rho \mathbf{g}, \mathbf{v}), \quad \mathbf{v} \in \mathcal{X}_0, \end{aligned} \quad (5.4)$$

$$\begin{aligned} & \left(\frac{\phi_h^{n+1} c_h^{n+1} - \phi_h^n c_h^n}{\delta t_n}, \psi \right) + (\nabla \cdot \mathbf{q}_h^{n+1}, \psi) + \frac{k_c k_s}{k_c + k_s} (a_v(\phi_h^{n+1}) c_h^{n+1}, \psi) \\ & - (f_P(\mathbf{x}, t^{n+1}) c_h^{n+1}, \psi) = (f_I(\mathbf{x}, t^{n+1}) c_I(\mathbf{x}, t^{n+1}), \psi), \quad \psi \in \mathcal{W}, \end{aligned} \quad (5.5)$$

$$\begin{aligned} & \left((\phi_h^{n+1} \mathbf{D}_e(\mathbf{u}_h^{n+1, M}))^{-1} \mathbf{q}_h^{n+1}, \mathbf{w} \right) = \left((\phi_h^{n+1})^{-1} \mathbf{M}_e(\mathbf{u}_h^{n+1}) c_h^{n+1}, \mathbf{w} \right) \\ & + (c_h^{n+1}, \nabla \cdot \mathbf{w}), \quad \mathbf{w} \in \mathcal{Y}_0, \end{aligned} \quad (5.6)$$

where $\mathbf{u}_h^{n+1, M} = \mathcal{M}(\mathbf{u}_h^{n+1})$.

For the calculation of discrete porosity, we use a semi-implicit scheme as

$$\frac{\phi_h^{n+1} - \phi_h^n}{\delta t_n} = \frac{R_\phi (1 - \phi_h^{n+1})}{1 - \phi_0} \bar{c}_h^n, \quad (5.7)$$

where $\bar{c}_h^n = \max(0, \min(c_h^n, 1))$. Denote

$$\theta_n = \delta t_n \frac{R_\phi}{1 - \phi_0} \bar{c}_h^n. \quad (5.8)$$

From (5.7), the porosity ϕ_h^{n+1} can be calculated as

$$\phi_h^{n+1} = \frac{\theta_n}{1 + \theta_n} + \frac{\phi_h^n}{1 + \theta_n}. \quad (5.9)$$

We note that the porosity ϕ_h^{n+1} computed by this method satisfies the maximum principle for any time step size. Indeed, suppose that $0 < \phi_h^n < 1$, then we deduce from (5.9) that $0 < \phi_h^n \leq \phi_h^{n+1} < 1$ holds for any time step size. By recurrence relation, we obtain $0 < \phi_0 \leq \phi_h^n < 1$ for $n \geq 1$.

6. Numerical experiments. In this section, we carry out two numerical examples to verify the theoretical analysis and effectiveness of the proposed scheme. In these examples, we use the lowest order Raviart-Thomas space with the upwind quadrature rule on a rectangular mesh. The sparse direct solver UMFPACK [6] is used to solve the linear system.

6.1. Numerical error analysis. In this example, we test a simple problem to verify the convergence rate of the proposed scheme. Suppose there is a 0.2×0.2 meter porous medium. The initial porosity in the porous medium is 0.2 everywhere. The initial permeability of the porous medium is 10^{-12}m^2 everywhere. A kind of acid flow is injected from the left boundary of the porous medium and the velocity of the flow is 10^{-6}m/s . A constant pressure of 10^5Pa is imposed on the right boundary of the porous medium. The up and down boundaries of the porous medium are closed. The initial pressure inside the porous medium is also 10^5Pa . Two high-porosity entries are set on the left boundary of the porous medium, one 0.05 meter above the bottom with a porosity of 0.6, and the other 0.1 meter above the bottom with a porosity of 0.5. The permeabilities of the two entries are 10^{-10}m^2 and 10^{-11}m^2 respectively. The other properties of acid flow and the porous medium can be seen in Table 6.1. The parameter γ is set to 0. The initial concentration of acid flow is zero.

There are four kinds of grid in the experiment and they have 20×20 , 40×40 , 80×80 and 160×160 cells respectively. The errors of physical parameters such as porosity, concentration, pressure, and velocity in the grids of 20×20 , 40×40 , 80×80 cells are calculated by comparing them with the corresponding values in the grid of 160×160 cells. The data of fine grid of 160×160 cells is linearly interpolated to the coarse grid. We calculated the errors at the end of 25th time step, 50th time step, 75th time step and 100th time step respectively and each time step is 100 seconds. The listed results in Tables 6.2, 6.3, 6.4 and 6.5 are the relative errors, which are calculated as

$$\|E_\phi\|_{\text{rel}} = \frac{\|E_\phi\|}{\|\phi_h\|}, \|E_p\|_{\text{rel}} = \frac{\|E_p\|}{\|p_h\|}, \|E_c\|_{\text{rel}} = \frac{\|E_c\|}{\|c_h\|}, \|E_{\mathbf{u}}\|_{\text{rel}} = \frac{\|E_{\mathbf{u}}\|}{\|\mathbf{u}_h\|}.$$

It is observed from the present numerical error results that the convergence rate of all MFE approximations is almost first order and the porosity convergence depends on the concentration. This is in agreement with the theoretical analysis results.

In Figures 6.1 and 6.2, we show the pressure contours and flow streamlines at different time steps respectively, while we also show the configurations of the porosity and concentration (plotted by $c_h \times 10^5 \text{mole/m}^2$) in Figures 6.3 and 6.4 respectively. These results are computed on the fine grid of 160×160 cells.

Table 6.1: The properties of acid flow and porous medium in error analysis

Property	Value
Viscosity of acid flow (μ)	$10^{-2} \text{Pa}\cdot\text{s}$
Density of acid flow (ρ)	10^3kg/m^3
Longitudinal dispersion coefficient	10^{-3}
Transverse dispersion coefficient	10^{-4}
Molecular diffusion coefficient	10^{-9}
Surface reaction rate constant (k_s)	10^{-3}m/s
Local mass-transfer coefficient (k_c)	10^{-4}m/s
Dissolving power of acid (α)	0.1kg/mole
Density of the rock (ρ_s)	2500kg/m^3
Initial interfacial area (a_0)	0.5m^{-1}
Injection concentration of acid flow ($c_I \times 10^5 \text{mole/m}^2$)	10^5mole/m^2

6.2. Dissolution experiment. In the dissolution experiment, a 1-meter by 0.2-meter porous medium is considered. The initial porosities in the porous medium

Table 6.2: Numerical errors of porosities on different rectangular meshes at different time steps

Mesh \ Time Steps	25	50	75	100
20 × 20	0.046493	0.06579	0.066958	0.065767
40 × 40	0.029612	0.03473	0.032795	0.032507
80 × 80	0.011837	0.012807	0.012462	0.013261

Table 6.3: Numerical errors of concentrations on different rectangular meshes at different time steps

Mesh \ Time Steps	25	50	75	100
20 × 20	0.26962	0.22998	0.21524	0.20962
40 × 40	0.15376	0.11824	0.10478	0.10284
80 × 80	0.057591	0.041013	0.036898	0.03921

comply with the continuously uniform distribution and the range is from 0.05 to 0.35. The initial permeability of the porous medium also obeys the uniform distribution and the average value is 10^{-8}m^2 . A kind of acid flow is injected from the left boundary of the porous medium and the velocity of the flow is 0.01m/s; that is, $q_I(x, y, t) = 0.01$ at the boundary $x = 0$ otherwise it equals zero. The acid flow goes out of the porous medium from the right boundary and the velocity is also 0.01m/s; that is, $q_P(x, y, t) = -0.01$ at the boundary $x = 1$ otherwise it equals zero. The up and down boundaries of the porous medium are closed. The initial concentration of acid flow is zero. The initial pressure inside the porous medium is 10^5Pa . The other properties of acid flow and the porous medium can be seen in Table 6.6. The parameter γ is set to 0.01. The simulation period is 10^5 seconds and each time step is 1000 seconds.

The pressure contours and flow streamlines at different time steps are shown in Figures 6.5 and 6.6 respectively, while we also show the configurations of the porosity and concentration (plotted by $c_h \times 10\text{mole/m}^2$) in Figures 6.7 and 6.8 respectively.

It is obviously observed from Figures 6.7 that the chemical concentration front advances in certain direction more than the other directions. As shown in 6.8, the wormholes with high porosities are formed with time because of nonuniform advancement of concentration. On the other hand, as shown in Figures 6.5 and 6.6, the heterogeneity of porosity and permeability in wormhole formations also has great effect on the velocity field, which further promotes the non-uniformity of the chemical reaction front. This example shows that the proposed method is capable of effectively simulating wormhole propagation.

7. Conclusions. We have developed a mixed finite element scheme for simulating wormhole formation during reactive dissolution of carbonates, which is frequently used to enhance oil and gas production rate. In the proposed scheme, in order to guarantee full mass conservation, the mixed finite element methods are used not only for the Darcy-Forchheimer flow equations but also for the solute transport equation by introducing an auxiliary flux variable. Based on the cut-off operators and an auxiliary function of velocity, we prove the stability and a priori error estimates for velocity, pressure, concentration and porosity in different norms by using the coupled analysis approach. Numerical results are given to verify theoretical analysis of the proposed

Table 6.4: Numerical errors of pressures on different rectangular meshes at different time steps

Mesh \ Time Steps	25	50	75	100
20 × 20	0.00038845	0.0010679	0.0015855	0.001986
40 × 40	0.00020249	0.00048864	0.00066225	0.0008296
80 × 80	6.0701e-05	0.00014087	0.00019473	0.00026732

Table 6.5: Numerical errors of velocities on different rectangular meshes at different time steps

Mesh \ Time Steps	25	50	75	100
20 × 20	0.4557	0.6968	0.7424	0.7506
40 × 40	0.3077	0.3876	0.4488	0.5181
80 × 80	0.1517	0.2000	0.2489	0.3110

methods.

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Fig. 6.1: The contours of pressures in the error analysis example at the 25th(a), 50th(b), 75th(c) and 100th(d) time step respectively.

Table 6.6: The properties of acid flow and porous medium in dissolution experiment

Property	Value
Viscosity of acid flow (μ)	$10^{-2}\text{Pa}\cdot\text{s}$
Density of acid flow (ρ)	$10^3\text{kg}/\text{m}^3$
Longitudinal dispersion coefficient	10^{-3}
Transverse dispersion coefficient	10^{-4}
Molecular diffusion coefficient	10^{-9}
Surface reaction rate constant (k_s)	$10\text{m}/\text{s}$
Local mass-transfer coefficient (k_c)	$1\text{m}/\text{s}$
Dissolving power of acid (α)	$0.1\text{kg}/\text{mole}$
Density of the rock (ρ_s)	$2500\text{kg}/\text{m}^3$
Initial interfacial area (a_0)	0.5m^{-1}
Injection concentration of acid flow ($c_I \times 10\text{mole}/\text{m}^3$)	$10\text{mole}/\text{m}^3$

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Fig. 6.2: The streamlines in the error analysis example at the 25th(a), 50th(b), 75th(c) and 100th(d) time step respectively.

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Fig. 6.3: The configurations of concentrations in the error analysis example at the 25th(a), 50th(b), 75(c) and 100th(d) time step respectively.

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Fig. 6.4: The configurations of porosity in the error analysis example at the 25th(a), 50th(b), 75th(c) and 100th(d) time step respectively.

Fig. 6.5: The contours of pressures in dissolution experiment at the 25th(a), 50th(b) and 100th(c) time step respectively.

(a)

(b)

(c)

Fig. 6.6: The streamlines in dissolution experiment at the 25th(a), 50th(b) and 100th(c) time step respectively.

Fig. 6.7: The configurations of concentrations in dissolution experiment at the 25th(a), 50th(b) and 100th(c) time step respectively.

Fig. 6.8: The configurations of porosity in dissolution experiment at the 1th(a), 25th(b), 50th(c) and 100th(d) time step respectively.