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**Multiphase flow simulation with gravity effect in anisotropic porous media using
multipoint flux approximation**

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Abstract

Numerical investigations of two-phase flows in anisotropic porous media have been conducted. In the flow model, the permeability has been considered as a full tensor and is implemented in the numerical scheme using the multipoint flux approximation within the framework of finite difference method. In addition, the experimenting pressure field approach is used to obtain the solution of the pressure field, which makes the matrix of coefficient of the global system easily constructed. A number of numerical experiments on the flow of two-phase system in two-dimensional porous medium domain are presented. In this work, the gravity is included in the model to capture the possible buoyancy-driven effects due to density differences between the two phases. Different anisotropy scenarios have been considered. From the numerical results, interesting patterns of the flow, pressure, and saturation fields emerge, which are significantly influenced by the anisotropy of the absolute permeability field. It is found that the two-phase system moves along the principal direction of anisotropy.

Furthermore, the effects of anisotropy orientation on the flow rates and the cross flow index are also discussed in the paper.

Keywords: anisotropic porous media, full-tensor permeability, gravity, multiphase flow, multipoint flux approximation.

1. Introduction

Multiphase flows in porous media are ubiquitous at all scales from large scale applications as in oil and gas subsurface reservoirs to small scale applications as in porous membranes, metallic foams, filters and many others. No wonder, therefore, the great deal of motivations among researchers to adapt a framework that could handle the complexity of the flow of multiphase systems in porous media. Bearing in mind that researchers are still unable to fully comprehend to a great satisfaction the flow of multiphase systems at pore scale, it is, therefore, expected that formulating the problem of multiphase flows in porous media is even harder. One notices that even if it might be possible to resolve all the details down to pore scale, it will still be challenging to consider large-scale domains, which is a consequence of the enormous computing power that would be required. This necessitates that a coarser framework be developed based on the continuum hypothesis. The mathematical techniques that are used to rigorously develop such framework have to start from the equations that are applicable at pore scale and scale it up to porous media continuum scale. Unfortunately, since the governing equations at the pore scale for multiphase system are complex enough and there are still unresolved issues (e.g., the moving contact line problem), it is extremely difficult to rigorously derive upscaled equations starting with equations defined at pore scale. Moreover, even if this is possible they are going to be unclosed and a set of tedious exercises will need to be carried out to suggest

closures [1]. Therefore, there is a great deal of motivation among researchers to extend Darcy's law such that an expression for phase velocity may be obtained. Researches, in this field, suggest that all the complexities of multiphase flow in porous media are lumped into a single multiplicative scalar parameter; namely relative permeability.

In the context of subsurface rock formations, another level of complexity arise as a result of anisotropy of media properties, which is due to the geological processes that took place over the longer geologic time scale. As a consequence, the fluid flow direction will not be only dependent on the pressure gradient but also on the principal directions of anisotropy. All the geological features mentioned above deliver the challenges on the development of robust reservoir simulators. The standard two-point flux approximation (TPFA) finite difference method is unfortunately not capable of handling full-tensor permeability, which limits the use of this method in many of porous media applications. This has led to the emergence of what is called the multipoint flux approximation (MPFA). Two approaches have been introduced to arrive at the finite difference stencil associated with the MPFA. The first was proposed by Aavatsmark et al. [2] and Aavatsmark [3] who introduced MPFA O-method based on the finite volume method for which the pressure at the mid edges (face centers in 3-D) are used to ensure the continuity of pressure and flux. There are several other types of MPFA such as MPFA L-method [4-6], MPFA U-method [7] and MPFA Z-method [8]. In addition, Edwards and Rogers [9] and Lee et al. [10] introduced the flux-continuous approach based on the framework of finite difference, which is equivalent to the work of Aavatsmark et al. [2] and Aavatsmark [3]. The second approach of MPFA is based on the mixed finite element point of view. Wheeler and Yotov [11] proved that MPFA could be derived from the lowest-order

Brezzi-Douglas-Marini (BDM) mixed finite element through special quadrature rule. A trapezoidal quadrature rule is employed such that it allows the local velocity elimination and leads to cell-centered stencil for the pressure (cell-centered finite difference). Employing quadrature rules in evaluating integrals alleviates the needs to care about the specific form of approximating shape functions and instead the values of the functions at nodal points are used. The resulting algebraic system is symmetric and positive definite. This approach is so-called multipoint flux mixed finite element (MPFMFE). In the last decade, the implementation of MPFA into single-phase or multiphase flow models for 2-D and 3-D problems has been deeply discussed [12-16]. As indicated in its name, MPFA requires more point stencil than TPFA. For example, the divergence operator requires 9-point and 27-point stencil in 2-D and 3-D problems, respectively. This makes the construction of the matrix of coefficient a difficult task and prone to errors in terms of coding. Therefore, we apply a newly developed technique, the so-called experimenting pressure field approach. This technique generates the matrix of coefficient automatically within the solver routine. This is, tremendously, beneficial particularly when there are long expressions of discretized algebraic equations. This technique has been implemented successfully in many engineering applications involving anisotropy of media properties [14-19].

The purpose of this paper is to simulate the flow of multiphase system in anisotropic porous media by using the MPFA method combined with the experimenting pressure field approach. In addition, we are also interested in investigating the effect of gravitational force in driving fluid migration in anisotropic porous media. In this work, numerical experiments for different anisotropy scenarios consider the inflow of a fluid phase that has lower density than the existing fluid inside the domain. The density

difference between the two phases induces buoyancy-driven upward flow of the lower density fluid.

This paper consists of five main sections: section 1 discusses the background and motivation of the study including the literature review on the development of MPFA and the scope of this work. Section 2 presents the governing equations of the two-phase flow model as well as the effect of gravitational force on the horizontal flux. Section 3 describes the fundamental concept of the MPFA method followed by the numerical discretization. Section 4 demonstrates the numerical results of the considered scenarios. The relation between the anisotropy orientation and the cross flow index is also discussed. Finally, we conclude the study in the last section.

2. Modeling equations and workflow of simulation

The governing equations that describe two-phase flow in porous media include the mass conservation equation and Darcy's law. The mass conservation equation describes the balance between the inflow and outflow of mass through a specified domain. The balance principle of physical quantities in the form of partial differential equations is a consequence of adapting the continuum hypothesis in porous media [20-22]. Meanwhile, Darcy's law describes the relationship among the total volumetric flow rate of each phase with the potential (pressure) gradient [23]. The mass conservation equation is described by

$$\frac{\partial(\phi\rho_{\alpha}S_{\alpha})}{\partial t} + \nabla \cdot (\rho_{\alpha}\mathbf{u}_{\alpha}) = q_{\alpha}, \quad \alpha = w, n \quad (1)$$

Here, ϕ , ρ_{α} , S_{α} , q_{α} , and $\mathbf{u}$ are the porosity, the fluid density (kg/m^3), the saturation, the sources/sinks ($\text{kg/m}^3\text{s}$), and the flux (m/s) of α -phase, respectively. The index of w refers to the wetting phase and n for the non-wetting phase. Darcy's law for the flux of each phase is given by

$$\mathbf{u}_\alpha = -\frac{k_{r\alpha}}{\mu_\alpha} \mathbf{K}(\nabla p_\alpha - \rho_\alpha \mathbf{g}), \quad \alpha = w, n \quad (2)$$

where μ_α and p_α are the fluid viscosity (Pa·s) and the pressure (Pa) of α -phase, respectively, $\mathbf{g}$ is the gravitational acceleration (m/s²), $\mathbf{K}$ is the absolute permeability tensor (m²), and $k_{r\alpha}$ is the relative permeability. The permeability tensor, $\mathbf{K}$, is given by

$$\mathbf{K} = \begin{bmatrix} K_{xx} & K_{xy} \\ K_{yx} & K_{yy} \end{bmatrix} \quad (3)$$

The full-tensor permeability is assumed to be symmetric ($K_{xy} = K_{yx}$) and positive definite ($K_{xx}K_{yy} > K_{xy}^2$). The relative permeability, which is a function of phase saturation, describes how a fluid phase interferes the flow behavior of another fluid phase and vice versa [24]. There are several models for the relative permeability such as Brooks-Corey [25] and van Genuchten [26]. In this work, we use the relative permeability model of Brook-Corey, which may be written in the form:

$$\begin{aligned} k_{rw}(S_\alpha) &= (1 - S_\alpha)^{\frac{2+3\lambda}{\lambda}} \\ k_{rn}(S_\alpha) &= S_\alpha^2 \left[1 - (1 - S_\alpha)^{\frac{2+\lambda}{\lambda}} \right] \end{aligned} \quad (4)$$

where λ is the pore size distribution index and its values vary depending on the heterogeneity of the reservoir rock. The greater the value of λ , the more homogeneous the reservoir rock will be. The typical values for λ range between 0.2 and 3. The Brooks-Corey's model has been used in many two-phase flow studies [27-30].

The governing equations as given above are coupled and there is no explicit equation for the pressure. In this work, the Implicit Pressure-Explicit Saturation (IMPES) scheme is adapted in which an equation for the pressure is obtained as follows: let's define the total flux as the summation of the wetting phase flux and the non-wetting phase flux,

$$\mathbf{u}_t = \mathbf{u}_w + \mathbf{u}_n \quad (5)$$

Rewriting the total flux in (5) based on the Darcy velocity for each phase in (2) and neglecting capillarity, one obtains

$$\mathbf{u}_t = -\lambda_t \mathbf{K} \nabla p + (\lambda_w \rho_w + \lambda_n \rho_n) \mathbf{K} \mathbf{g} \quad (6)$$

where $\lambda_\alpha = \frac{k_{r\alpha}}{\mu_\alpha}$ is the phase mobility and it is a function of phase saturation.

Substituting the total flux (6) into the summation of the two mass conservation equations, we obtain the pressure equation as follows

$$\nabla \cdot (-\lambda_t \mathbf{K} \nabla p + (\lambda_w \rho_w + \lambda_n \rho_n) \mathbf{K} \mathbf{g}) = q_t \quad (7)$$

By ignoring the capillary pressure in the model then the pressure is equal for both phases (i.e., $p = p_w = p_n$).

The saturation equation of incompressible immiscible two-phase flow for each phase is given by

$$\phi \frac{\partial S_\alpha}{\partial t} + \nabla \cdot \mathbf{u}_\alpha = q_\alpha, \quad \alpha = w, n \quad (8)$$

Considering the saturation equation of the non-wetting phase so we have the following

$$\phi \frac{\partial S_n}{\partial t} + \nabla \cdot \mathbf{u}_n = q_n \quad (9)$$

where

$$\mathbf{u}_n = f_n \mathbf{u}_t - f_n \lambda_w (\rho_w - \rho_n) \mathbf{K} \mathbf{g} \quad (10)$$

Here, $f_n = \frac{\lambda_n}{\lambda_t}$ is the fractional flow function and λ_t is the total mobility, which is the summation of the mobility of both phases, $\lambda_t = \sum_\alpha \lambda_\alpha$. Looking at (10), it can be seen that the gravity is not only affecting the movement of the phase flux in the vertical direction but also affecting the horizontal flux due to the off-diagonal elements of the permeability tensor (will be discussed more details in section 2.1).

In the two-phase flow model, the phase saturations are constrained by

$$S_w + S_n = 1 \quad (11)$$

In this work, the computational domain is rectangular and subjected to the boundary, $\partial\Omega = \Gamma_D \cup \Gamma_N$ where Γ_D is the Dirichlet boundary and Γ_N is the Neumann boundary.

The boundary conditions are considered as follows

$$p_\alpha = p_B \text{ on } \mathbf{x} \in \Gamma_D, \quad \alpha = w, n \quad (12)$$

$$\mathbf{u}_\alpha \cdot \mathbf{n} = u_B \text{ on } \mathbf{x} \in \Gamma_N, \quad \alpha = w, n \quad (13)$$

where p_B is the given pressure on the boundary Γ_D and u_B is the given velocity on the boundary Γ_N , and $\mathbf{n}$ is the unit normal vector pointing out on the boundary $\partial\Omega$. The boundary condition for the phase saturation is defined as

$$S_\alpha = S_B \text{ on } \Gamma_N, \quad \alpha = w, n \quad (14)$$

with the initial condition,

$$S_\alpha = S_\alpha^0 \text{ in } \Omega, \quad \alpha = w, n \quad (15)$$

We employ the IMPES scheme [31-33] to solve the two-phase flow model. As indicated in its name, the pressure equation is solved implicitly and the saturation equation is evaluated explicitly. First, we solve the pressure equation by implementing the experimenting pressure field approach and applying the MPFA method to obtain the flux. Then, we solve the saturation equation to obtain the fluid phase saturation. Since the saturation equation is solved explicitly then the time step must satisfy the CFL condition to ensure numerical stability.

2.1. Effect of gravitational force on horizontal flux and upwinding

In this section, we address the effect of gravitational force on horizontal flux due to the existence of the off-diagonal elements in the permeability tensor. In the two-point flux approximation, which is well suited when the permeability is considered as scalar or diagonal tensor, such cross flow phenomenon is not observed. The gravitational

force only survives along the vertical direction. In the case of multipoint flux approximation with full permeability tensor, it is clear that the off-diagonal elements, K_{xy} , will have an effect on the horizontal component of the flow. To clarify, let (10) be written as

$$\mathbf{u}_n = f_n(\mathbf{u}_t + \mathbf{G}) \quad (16)$$

where

$$\mathbf{G} = \lambda_w(\rho_w - \rho_n)\mathbf{K}\mathbf{g} \quad (17)$$

which may be expand as

$$\begin{bmatrix} G_x \\ G_y \end{bmatrix} = \lambda_w(\rho_w - \rho_n) \begin{bmatrix} K_{xx} & K_{xy} \\ K_{yx} & K_{yy} \end{bmatrix} \begin{bmatrix} 0 \\ g_y \end{bmatrix} \quad (18)$$

From (18), it can be seen that G_x has contribution to the horizontal flux ($G_x \neq 0$). Likewise, this gravitational force affects the upwinding in the horizontal direction because we evaluate the upwinding according to (16) in which the term $\mathbf{G}$ exists. Typically, the gravity term may cause countercurrent flow between the phases if the upwinding is taken based on the total flux [34]. Hence, instead of applying the upwinding directly to the total flux, we suggest to include the effect of (17) then performing the upwinding afterwards. In other words, we calculate the term inside the bracket in (16) and apply the fractional flow upwinding.

3. Multipoint flux approximation and numerical discretization

In this section, we discuss the fundamental concept of multipoint flux approximation (MPFA) and its numerical discretization based on the framework of finite difference. Aavatsmark et al. [2] and Aavatsmark [3] have discussed the MPFA O-method in which the interface midpoints are introduced as continuity points. The MPFA method is designed to have a correct discretization of the flow equations for non-orthogonal grids as well as for general orientation of the principal directions of the permeability

tensor, which cannot be managed by the two-point flux approximation (TPFA). TPFA performs well in terms of discretization on non-orthogonal grids only if the grid orientations are aligned with the principal directions of the permeability tensor [35]. The non-orthogonal grids are necessitated in the model when the geological structure in the model is geometrically complex. There are several other types of MPFA such as MPFA L-method [4-6], MPFA U-method [7] and MPFA Z-method [8]. In this paper, however, we only discuss and apply the MPFA O-method. The convergence of MPFA O-method has been examined either theoretically as in [36,37] or numerically as in [38-40].

MPFA requires more point stencils than TPFA as depicted in Fig. 1. In MPFA method the calculation of one flux component involves six adjacent cells for 2-D problem while it involves only two adjacent cells in TPFA. MPFA method introduces the surface midpoints as continuity points to ensure the linear variation and continuity of the pressure and the flux.

MPFA method shapes an interaction region (dashed lines) surrounding each grid node and involving four neighboring cells (Fig. 2). If a domain has $m \times n$ grids then there are $(m+1) \times (n+1)$ interaction regions. The flux in the TPFA is approximated by

$$u_i \approx T_i(p_2 - p_1) \quad (19)$$

while in MPFA the flux is approximated by

$$u_i \approx \sum_{j \in \Omega} T_{ij} \Phi_j \quad (20)$$

Here, T_{ij} are the transmissibility coefficients, Φ_j is the pressure gradient, and Ω is the domain that depends on the dimension of the problem. In 2-D problem, the set of Ω comprises of 6 cells while in 3-D case it contains 18 cells. Referring to Fig. 2, we have each flux in each interaction region calculated by

$$\begin{aligned} u_x^- &\approx -T_{11}^x(p_x^- - p_{11}) - T_{11}^x(p_y^- - p_{11}) \\ &\approx -T_{21}^x(p_{21} - p_x^-) - T_{21}^x(p_y^+ - p_{21}) \end{aligned} \quad (21)$$

$$\begin{aligned} u_x^+ &\approx -T_{12}^x(p_x^+ - p_{12}) - T_{12}^x(p_{12} - p_y^-) \\ &\approx -T_{22}^x(p_{22} - p_x^+) - T_{22}^x(p_{22} - p_y^+) \end{aligned} \quad (22)$$

$$\begin{aligned} u_y^- &\approx -T_{11}^y(p_y^- - p_{11}) - T_{11}^y(p_x^- - p_{11}) \\ &\approx -T_{21}^y(p_{12} - p_y^-) - T_{12}^y(p_x^+ - p_{12}) \end{aligned} \quad (23)$$

$$\begin{aligned} u_y^+ &\approx -T_{21}^y(p_y^+ - p_{21}) - T_{21}^y(p_{21} - p_x^-) \\ &\approx -T_{22}^y(p_{22} - p_y^+) - T_{22}^y(p_{22} - p_x^+) \end{aligned} \quad (24)$$

The transmissibility coefficients above contain information about the permeability and the grids. The total mobility is evaluated at the center of the cell and is multiplied with the transmissibility coefficients. From (21)-(24), we have the numerical discretization for our work as follows

$u_{t,x}^-$:

$$u_{t,x}^- = -\lambda_{t,11} K_{11}^{xx} \frac{p_x^- - p_{11}}{\frac{1}{2} \Delta x_{11}} - \lambda_{t,11} K_{11}^{xy} \frac{p_y^- - p_{11}}{\frac{1}{2} \Delta y_{11}} + (\lambda_{w,11} \rho_w + \lambda_{n,11} \rho_n) K_{11}^{xy} g \quad (25)$$

$$u_{t,x}^- = -\lambda_{t,21} K_{21}^{xx} \frac{p_{21} - p_x^-}{\frac{1}{2} \Delta x_{21}} - \lambda_{t,21} K_{21}^{xy} \frac{p_y^+ - p_{21}}{\frac{1}{2} \Delta y_{21}} + (\lambda_{w,21} \rho_w + \lambda_{n,21} \rho_n) K_{21}^{xy} g$$

$u_{t,x}^+$:

$$u_{t,x}^+ = -\lambda_{t,12} K_{12}^{xx} \frac{p_x^+ - p_{12}}{\frac{1}{2} \Delta x_{12}} - \lambda_{t,12} K_{12}^{xy} \frac{p_{12} - p_y^-}{\frac{1}{2} \Delta y_{12}} + (\lambda_{w,12} \rho_w + \lambda_{n,12} \rho_n) K_{12}^{xy} g \quad (26)$$

$$u_{t,x}^+ = -\lambda_{t,22} K_{22}^{xx} \frac{p_{22} - p_x^+}{\frac{1}{2} \Delta x_{22}} - \lambda_{t,22} K_{22}^{xy} \frac{p_{22} - p_y^+}{\frac{1}{2} \Delta y_{22}} + (\lambda_{w,22} \rho_w + \lambda_{n,22} \rho_n) K_{22}^{xy} g$$

$u_{t,y}^-$:

$$u_{t,y}^- = -\lambda_{t,11} K_{11}^{yy} \frac{p_y^- - p_{11}}{\frac{1}{2} \Delta y_{11}} - \lambda_{t,11} K_{11}^{yx} \frac{p_x^- - p_{11}}{\frac{1}{2} \Delta x_{11}} + (\lambda_{w,11} \rho_w + \lambda_{n,11} \rho_n) K_{11}^{yy} g \quad (27)$$

$$u_{t,y}^- = -\lambda_{t,12} K_{12}^{yy} \frac{p_{12} - p_y^-}{\frac{1}{2} \Delta y_{12}} - \lambda_{t,12} K_{12}^{yx} \frac{p_x^+ - p_{12}}{\frac{1}{2} \Delta x_{12}} + (\lambda_{w,12} \rho_w + \lambda_{n,12} \rho_n) K_{12}^{yy} g$$

$u_{t,y}^+$:

$$u_{t,y}^+ = -\lambda_{t,21} K_{21}^{yy} \frac{p_y^+ - p_{21}}{\frac{1}{2} \Delta y_{21}} - \lambda_{t,21} K_{21}^{yx} \frac{p_{21} - p_x^-}{\frac{1}{2} \Delta x_{21}} + (\lambda_{w,21} \rho_w + \lambda_{n,21} \rho_n) K_{21}^{yy} g \quad (28)$$

$$u_{t,y}^+ = -\lambda_{t,22} K_{22}^{yy} \frac{p_{22} - p_y^+}{\frac{1}{2} \Delta y_{22}} - \lambda_{t,22} K_{22}^{yx} \frac{p_{22} - p_x^+}{\frac{1}{2} \Delta x_{22}} + (\lambda_{w,22} \rho_w + \lambda_{n,22} \rho_n) K_{22}^{yy} g$$

The unknowns from the equations (25)-(28) above are the cell-centered pressures, the interface pressures, and the fluxes. From these equations we eliminate the interface pressures through mathematical manipulation so that the remaining unknowns are the cell-centered pressures and the fluxes. In each interaction region, we have the system of equations that consist of four equations and written in the matrix form as the following

$$\mathbf{R} \mathbf{u} = \mathbf{C} \mathbf{p} + \mathbf{G} \quad (29)$$

where $\mathbf{R}$ is constructed based on the permeability and the total mobility. The matrix of

$\mathbf{R}$ is expressed as

$$\mathbf{R} = \frac{1}{2} \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \quad (30)$$

where

$$R_{11} = \begin{bmatrix} (R_{11}^{xx} \Delta x_1 + R_{21}^{xx} \Delta x_2) & 0 \\ 0 & (R_{12}^{xx} \Delta x_1 + R_{22}^{xx} \Delta x_2) \end{bmatrix} \quad (31)$$

$$R_{22} = \begin{bmatrix} (R_{11}^{yy} \Delta y_1 + R_{12}^{yy} \Delta y_2) & 0 \\ 0 & (R_{21}^{yy} \Delta y_1 + R_{22}^{yy} \Delta y_2) \end{bmatrix} \quad (32)$$

$$R_{21} = \begin{bmatrix} R_{11}^{yx} \Delta y_1 & R_{12}^{yx} \Delta y_2 \\ R_{21}^{yx} \Delta y_1 & R_{22}^{yx} \Delta y_2 \end{bmatrix} \quad (33)$$

$$R_{12} = \begin{bmatrix} R_{11}^{xy} \Delta x_1 & R_{21}^{xy} \Delta x_2 \\ R_{12}^{xy} \Delta x_1 & R_{22}^{xy} \Delta x_2 \end{bmatrix} \quad (34)$$

$\mathbf{C}$ is the matrix of coefficient,

$$\mathbf{C} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \quad (35)$$

$\mathbf{u}$ is the vector of unknowns velocities,

$$\mathbf{u} = [u_x^- \quad u_x^+ \quad u_y^- \quad u_y^+]^T \quad (36)$$

$\mathbf{p}$ is the vector of unknown pressures,

$$\mathbf{p} = [p_{11} \quad p_{21} \quad p_{12} \quad p_{22}]^T \quad (37)$$

$\mathbf{G}$ is the vector containing the gravity term and some other parameters,

$$\mathbf{G} = g \begin{bmatrix} 0 \\ 0 \\ -\frac{\frac{1}{2} \Delta y_{11} (\lambda_{w,11} \rho_w + \lambda_{n,11} \rho_n)}{\lambda_{t,11}} - \frac{\frac{1}{2} \Delta y_{12} (\lambda_{w,12} \rho_w + \lambda_{n,12} \rho_n)}{\lambda_{t,12}} \\ -\frac{\frac{1}{2} \Delta y_{21} (\lambda_{w,21} \rho_w + \lambda_{n,21} \rho_n)}{\lambda_{t,21}} - \frac{\frac{1}{2} \Delta y_{22} (\lambda_{w,22} \rho_w + \lambda_{n,22} \rho_n)}{\lambda_{t,22}} \end{bmatrix} \quad (38)$$

We obtain the fluxes in each node by solving (29). Then we take the average of the two neighboring fluxes in the same edge to obtain the flux at the midpoint of the edge.

In other words, we return back to the framework of TPFA. We loop over $(m+1) \times (n+1)$ grid nodes for a domain with $m \times n$ cells. This section is only demonstrating the derivation for 2-D problem but implementation in 3-D problems should be straightforward. In 3-D problems, the interaction region consists of 8 adjacent cells and the size of $\mathbf{R}$, $\mathbf{u}$, $\mathbf{C}$, $\mathbf{p}$, and $\mathbf{G}$ matrices are 12×12 , 12×1 , 12×8 , 8×1 , and 12×1 , respectively.

The experimenting pressure field algorithm is used to generate the global system for solving the pressure equation. In this approach, a set of experimenting predefined fields are used to operate on the local system of discretized equations to automatically obtain the global matrix of coefficients [17]. This greatly simplifies the coding particularly when solving complex stencils like those encountered in MPFA [14-19].

4. Numerical results

In this section, we introduce a set of numerical experiments that demonstrate the effects of gravity and anisotropy on the flow in porous media. As mentioned previously, we employ the IMPES scheme in this work. The experimenting pressure field approach is combined with MPFA method (Algorithm 1). We present comparison of numerical results between anisotropic (with different rotation angles) and isotropic cases to highlight the anisotropy effect. The values of permeability tensor elements are generated by

$$\mathbf{K} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \quad (39)$$

Equation (39) ends up to the following formulation,

$$\mathbf{K} = \begin{bmatrix} K_1 \cos^2 \theta + K_2 \sin^2 \theta & (K_1 - K_2) \sin \theta \cos \theta \\ (K_1 - K_2) \sin \theta \cos \theta & K_1 \sin^2 \theta + K_2 \cos^2 \theta \end{bmatrix} \quad (40)$$

where K_1 and K_2 are the permeabilities in the horizontal and vertical directions while θ is the rotation angle. The value of K_1 is typically greater than K_2 since the deposit of the sediments, which forms the stratification (rock layers), induces the permeability to be dominant in one direction than in the others. The permeability tensor formulation in (40) is equivalent to the one in Fanchi [41]. Fig. 3 illustrates the coordinate systems associated with the principle direction of permeability tensor.

However, before presenting the results of the numerical experiments on the effect of anisotropy, a couple of verification exercises are performed to build confidence in our numerical approach. We compare our method with the analytical solution of the Buckley-Leverett problem [42], which describes the one-dimensional displacement of two immiscible fluids. We also compare with the finite difference solution of Hoteit and Firoozabadi [43] for the same problem. From Figs. 4a,b it is clear that our methodology compares very well with both the analytical solution and that of Hoteit and Firoozabadi. The parameters used in these comparison exercises are taken from the work of Hoteit and Firoozabadi [43].

For anisotropic media, the example introduced in this work presents a 2-D domain in which an anisotropic layer in the middle (Fig. 5) is considered. The size of the domain is $50 \times 50 \text{ m}^2$ and it is discretized into 50×50 meshes ($\Delta x = \Delta y = 1 \text{ m}$). The porous medium has uniform porosity of 0.2. The density and viscosity of the non-wetting phase are 700 kg m^{-3} and 0.7 cP, respectively. Meanwhile, the density and viscosity of the wetting phase are 1000 kg m^{-3} and 1 cP, respectively. The reservoir rock permeability is assumed 35 millidarcy (md) and K_1 is set to be ten times greater than K_2 . The porous medium is initially saturated with the wetting fluid. We imposed a Neumann boundary condition, i.e., incoming flux of the non-wetting phase from the

middle of the bottom of the reservoir (representing a horizontal well). There are two hydrostatic pressure (Dirichlet) boundary conditions at the top left and right sides of the top region with the pressure on the left side slightly greater than on the right side.

We attempt to simulate the flow migration for different combinations of anisotropy orientation from $\theta = 0^\circ$ up to $\theta = 90^\circ$ with 30° interval and $\theta = 135^\circ$. We run all the cases for 60 years simulation time. The permeability tensor formulation in (40) generates several permeability tensors corresponding to each rotation angle as presented in Table 1. In this work, the rotation angle of $\theta = 0^\circ$ is still considered as anisotropic case as long as the main diagonal elements have different values, although the off-diagonal elements are zero. This is different with the isotropic case, which the values of the main diagonal elements are uniform.

Figs. 6a-f depicts the saturation contours of the non-wetting phase for the different anisotropy scenarios. Fig. 6-b shows the non-wetting phase saturation profile when the anisotropy of the middle layer is $\theta = 30^\circ$. From this figure, we may observe that the flow migrates upwards in the isotropic bottom layer, which is mainly driven by buoyancy force due to density differences between the two phases. The flow begins to move aligned with the principal directions of anisotropy on penetrating the middle anisotropic layer. The flow would rather migrate along the direction of K_1 than K_2 because K_1 is ten times larger than K_2 . The non-wetting phase fluid sweeps the wetting-phase fluid towards the open boundary on the right side of the top layer.

Similarly, for the cases of $\theta = 60^\circ$, $\theta = 90^\circ$, and $\theta = 135^\circ$ (Figs. 6c-e), the flow shapes a certain flow path based on the rotation angles of anisotropy. When $\theta = 0^\circ$ the flow spreads more horizontally than vertically because K_1 (which is on the horizontal direction) is larger than K_2 . This implies that the flow encounters higher resistance to move upwards in the anisotropic layer and consequently making some accumulation

of phase saturation in this layer (highlighted by red colors in Fig. 6a). Saturation profiles along the vertical centerline, $x = 26$ m is shown in Fig. 7. It is clear now that the case with $\theta = 0^\circ$ has the highest saturation in the anisotropic layer due to the accumulation as mentioned above. From Fig. 7 we can see that the higher the rotation angle the lower the saturation in anisotropic layer, except for $\theta = 135^\circ$. The cross section of saturation profile for the isotropic case is also plotted and its graph is close to the case with $\theta = 60^\circ$, except in the middle of the domain where the anisotropic layer exists. Therefore, we observe similar contour profile of saturations for both cases as depicted in Figs. 6c,d.

Fig. 8 on the other hand shows the history of the flux of the nonwetting phase at the right boundary for different anisotropy scenarios. In this figure one notices that the breakthrough time is different for each anisotropy scenario. It may be observed that it takes longer to reach the exit boundary when the anisotropy angle is zero. This is a consequence of the larger resistance in the direction of the flow that this scenario shows. The flow for the case with $\theta = 30^\circ$ reach the boundary earlier than the other cases (followed by the cases with $\theta = 60^\circ$, $\theta = 90^\circ$). We do not include the case of $\theta = 135^\circ$, which has flow rate much smaller than the other cases. This is indicated by the lower saturation (compared to others) around the open boundary on the right side as depicted in Fig. 6e. Meanwhile, the flow in the isotropic case is relatively quick to reach the steady state condition compared to the anisotropic cases. We plot the graphs only for 30 years simulation time because the flow rates have reached the steady state condition within less than ten years, except the case for $\theta = 0^\circ$ that needs approximately 30 years to reach the steady state condition.

Fig. 9 shows the correlation between the permeability tensor elements against the rotation angle. We observe that the permeability along the principal direction of x -axis

declines when the rotation angle increases. On the contrary, the permeability along the principal direction of y-axis has linear relationship with the rotation angle, i.e., it increases when the rotation angle becomes larger. In the meantime, the off-diagonal elements plot shows symmetric graph in which the value is increasing from $\theta = 0^\circ$ up to $\theta = 30^\circ$ and decreasing from $\theta = 60^\circ$ to $\theta = 90^\circ$. It may has maximum value between $\theta = 30^\circ$ and $\theta = 60^\circ$, which is not the considered case here.

Furthermore, we are also interested in observing the relationship between the cross flow index and the rotation angle. According to Hassanpour [44], the cross flow index is defined as the ratio between the off-diagonal and main diagonal elements of the permeability tensor, which is given by

$$cross\ flow\ index = \frac{(K_{xy} + K_{yx})}{(K_{xx} + K_{yy})} \quad (41)$$

The cross flow index describes the variation of the off-diagonal elements of permeability tensor. From Fig. 10, it can be seen that the shape of the cross flow index graph is similar to the one discusses above, which is increasing from $\theta = 0^\circ$ up to $\theta = 30^\circ$ and decreasing from $\theta = 60^\circ$ to $\theta = 90^\circ$ by having symmetric graph due to the symmetric tensor.

5. Summary and conclusions

In this work, modeling and simulation of immiscible incompressible two-phase flow in anisotropic porous media have been carried out. The anisotropy of porous media properties in subsurface is inevitable given the complex geologic processes. While the TPFA finite difference method is not capable of managing the full permeability tensor, we have applied the MPFA that can accommodate full tensors. However, the MPFA discretization stencil is complex and the experimenting field approach has been

adapted to handle such complexities. In this approach the local problems are solved to construct the global system.

Gravitational force is also considered in this work. Due to density differences between phases an induced buoyancy-driven flow is expected. For the case of anisotropic porous media, gravity influences cross flow along normal direction to the gravity. Several numerical experiments have been considered to highlight the role of permeability anisotropy on the flow of a two phase system in porous media. It is observed that the buoyancy force drives the flow migration upwards and the anisotropy align the flow direction closer to the principal direction of anisotropy.

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Algorithm 1. Algorithm for implementing experimenting pressure field approach combined with MPFA method.

```

Given the size of the domain,  $L_x, L_y, n_x, n_y$ 
Given the rock and fluid properties,  $\mathbf{K}, \phi, \rho_\alpha, \mu_\alpha$ 
Given the initial and boundary conditions,  $S_\alpha^0, p_{bdry}, u_{bdry}$ 
For  $t = 1:nt$ 
    % Construct the matrix of coefficients
    Generating experimenting pressure fields,  $p_\alpha^E$ 
    Solve equation (29) to obtain the nodal fluxes,  $u_{x_\alpha}^+, u_{x_\alpha}^-, u_{y_\alpha}^+, u_{y_\alpha}^-$ 
    Averaging the nodal fluxes to obtain the mid-edge fluxes,  $u_{x_\alpha}, u_{y_\alpha}$ 
    Calculate the residual,  $R = \nabla \cdot \mathbf{u}_\alpha - q$ 
    Construct the matrix of coefficient,  $A$ , based on  $R$ 

    % Construct the RHS
    Generating zeros pressure field except at the boundary,  $p_{bdry}$ 
    Solve equation (29) to obtain the nodal fluxes,  $u_{x_\alpha}^+, u_{x_\alpha}^-, u_{y_\alpha}^+, u_{y_\alpha}^-$ 
    Averaging the nodal fluxes to obtain the mid-edge fluxes,  $u_{x_\alpha}, u_{y_\alpha}$ 
    Calculate the residual,  $R = \nabla \cdot \mathbf{u}_\alpha - q$ 
    Construct the RHS,  $b$ , based on  $R$ 

```

Table 1. Permeability tensor values with different rotation angles of anisotropy.

Rotation angles	Kxx (md)	Kyy (md)	Kxy (md)	Kyx (md)
$\theta = 0^\circ$	35	3.5	0	0
$\theta = 30^\circ$	27.13	11.38	13.64	13.64
$\theta = 60^\circ$	11.38	27.13	13.64	13.64
$\theta = 90^\circ$	3.5	35	0	0
$\theta = 135^\circ$	19.25	19.25	-15.75	-15.75

Fig. 1. Two-point flux (a) and the multipoint flux (b) approximations.

Fig. 2. Interaction region of multipoint flux approximation in 2-D problem.

Fig. 3. Rotation angle of the principal direction of anisotropy.

(a)

(b)

Fig. 4. Verification of numerical solution with analytical solution (Buckley-Leverett [42]) and Hoteit-Firoozabadi solution [43].

Fig. 5. Problem setup with anisotropic layer exists in the middle of the domain.

(a)

(b)

Fig. 6. Non-wetting phase saturation profiles and flux fields at 60 years with different anisotropy orientation, (a) $\theta = 0$, (b) $\theta = 30$, (c) $\theta = 60$, (d) $\theta = 90$, (e) $\theta = 135$, and (f) isotropic porous media.

Fig. 7. Cross section saturation profiles in the middle of the domain ($x = 26$ m).

Fig. 8. Correlation between the flow rate and the rotation angle.

Fig. 9. Correlation between the permeability tensor elements and the rotation angle.

Fig. 10. Correlation between the cross flow index and the rotation angle.

Highlights:

- Two-phase flow simulation with gravity effect in anisotropic porous media.
- Combining MPFA method with experimenting pressure field approach.
- Comparison of anisotropic and isotropic cases is highlighted.
- Influences of buoyancy forces on the flow of a two-phase system are investigated.

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