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A novel numerical flux for the 3D Euler equations with general equation of state

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Abstract

Here we extend the flux vector splitting approach recently proposed in (*E F Toro and M E Vázquez-Cendón. Flux splitting schemes for the Euler equations. Computers and Fluids. Vol. 70, Pages 1-12, 2012*). The scheme was originally presented for the 1D Euler equations for ideal gases and its extension presented in this paper is threefold: (i) we solve the three-dimensional Euler equations on general meshes; (ii) we use a general equation of state; and (iii) we achieve high order of accuracy in both space and time through application of the semi-discrete ADER methodology on general meshes. The resulting methods are systematically assessed for accuracy, robustness and efficiency on a carefully selected suite of test problems. Formal high accuracy is assessed through convergence rates studies for schemes of up to 4th order of accuracy in both space and time on unstructured meshes.

Keywords: Hyperbolic systems, upwinding, flux vector splitting, Euler equations, general equation of state, ADER method.

1. Introduction

Advanced computational methods for compressible flow rely on a first-order, monotone (for the scalar case) scheme as the building block. Though there seems to be an endless proliferation of publications aiming at the *ultimate numerical flux*, identifying a single scheme that is optimal for all cases of practical interest seems to be an impossible task. And hence the labouring goes on. Proposed methods fall essentially into two classes: upwind methods and centred or symmetric methods. In spite of their increased complexity, upwind methods tend to be preferred when having to resolve fine features, particularly those associated with intermediate characteristic fields. Then two possible choices are the

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Godunov approach [19] and associated Riemann solvers [38], and the Flux Vector Splitting (FVS) approach. FVS schemes provide upwinding for fast waves at a lower computational effort and algorithm complexity than the Godunov approach with a good Riemann solver. Intermediate fields may also be well represented in the FVS approach if one adopts the more recent versions, such as that of Liou and Steffen [26]. Earlier FVS schemes, such as those reported in [34], [43] and [45], suffered from excessive numerical dissipation for contacts, shear waves and shear layers. For a review of FVS schemes see Chapter 8 of [38]. For background and reviews on the various classes of methods available see, for example, [18], [38] and [25].

In this paper we extend the flux vector splitting method recently proposed by Toro and Vázquez [42]. The scheme, called the TV flux here after, was first presented for the one-dimensional Euler equations, for ideal gases. A distinctive feature of the TV flux is that it separates completely advection terms from pressure terms, thus providing the possibility of taking advantage of their diverse speeds of propagation. Toro and Vázquez [42] also proposed a numerical approach that emerges from two separate systems produced by their flux vector splitting, the advection system and the pressure system. It turns out that the pressure system is hyperbolic and a very simple solution of the associated Riemann problem provides all the items required for flux evaluation. The resulting scheme proved to be simple and very robust, two ingredients that are usually difficult to reconcile.

Here we present a threefold extension of the TV flux: (i) we solve the three-dimensional Euler equations on general meshes; (ii) we use a general equation of state (EOS); and (iii) we achieve high order of accuracy in both space and time through application of the fully-discrete ADER methodology in its finite volume setting. The resulting methods are systematically assessed for accuracy, robustness and efficiency on a carefully selected suit of test problems. Formal high accuracy is assessed through convergence rates studies. Schemes of up to 4th order of accuracy in both space and time on unstructured meshes are implemented and assessed.

The rest of this paper is structured as follows: Background material is briefly reviewed in Section 2. In Section 3 we extend the TV flux to the 3D Euler equations for general equations of state. In Section 4 we extend the methods to high order of accuracy on general meshes. Numerical results are shown in Section 5 and a summary and concluding remarks are found in Section 6.

2. Background

Here we briefly review the background strictly necessary to present the scheme for the three-dimensional Euler equations.

2.1. Review of the TV Splitting

Consider the one-dimensional Euler equations of gas dynamics

$$\partial_t \mathbf{Q} + \partial_x \mathbf{F}(\mathbf{Q}) = \mathbf{0} , \quad (1)$$

with

$$\mathbf{Q} = \begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix}, \quad \mathbf{F}(\mathbf{Q}) = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ u(E + p) \end{bmatrix}, \quad (2)$$

where ρ is density, u is particle velocity, p is pressure and E is total energy given as

$$E = \rho \left(\frac{1}{2} u^2 + e \right). \quad (3)$$

To close system (1) we need the caloric equation of state, whereby the specific internal energy e is related to pressure and density as

$$e = e(\rho, p) \quad \text{or} \quad p = p(\rho, e). \quad (4)$$

For ideal gases

$$e(\rho, p) = \frac{p}{\rho(\gamma - 1)}. \quad (5)$$

We assume system (1) is solved with a conservative method

$$\mathbf{Q}_i^{n+1} = \mathbf{Q}_i^n - \frac{\Delta t}{\Delta x} [\mathbf{F}_{i+\frac{1}{2}} - \mathbf{F}_{i-\frac{1}{2}}], \quad (6)$$

where $\mathbf{Q}_i^n$ is the cell average and $\mathbf{F}_{i+\frac{1}{2}}$ is the numerical flux, which is to be determined. Following the flux vector splitting approach one writes

$$\mathbf{F}(\mathbf{Q}) = \mathbf{A}(\mathbf{Q}) + \mathbf{P}(\mathbf{Q}) \quad (7)$$

with corresponding numerical flux as

$$\mathbf{F}_{i+\frac{1}{2}} = \mathbf{A}_{i+\frac{1}{2}} + \mathbf{P}_{i+\frac{1}{2}}. \quad (8)$$

In the flux vector splitting method of Toro and Vázquez [42] the flux is decomposed thus

$$\mathbf{F}(\mathbf{Q}) = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ u(\frac{1}{2}\rho u^2 + \rho e + p) \end{bmatrix} = \begin{bmatrix} \rho u \\ \rho u^2 \\ \frac{1}{2}\rho u^3 \end{bmatrix} + \begin{bmatrix} 0 \\ p \\ u(\rho e + p) \end{bmatrix}, \quad (9)$$

with the respective advection and pressure fluxes defined as

$$\mathbf{A}(\mathbf{Q}) = u \begin{bmatrix} \rho \\ \rho u \\ \frac{1}{2}\rho u^2 \end{bmatrix}, \quad \mathbf{P}(\mathbf{Q}) = \begin{bmatrix} 0 \\ p \\ u(\rho e + p) \end{bmatrix}. \quad (10)$$

The TV flux has the following features: (i) the advection flux $\mathbf{A}(\mathbf{Q})$ contains no pressure terms, that is all pressure terms are included in the pressure flux $\mathbf{P}(\mathbf{Q})$; (ii) the advection operator expresses advection of mass, momentum and kinetic energy; (iii) the flux allows immediate inclusion of a general equation of state in the pressure vector in (10) via the general function $e(\rho, p)$ in (4).

2.2. Three Dimensional Euler Equations with General Equation of State

The time-dependent, inhomogeneous, compressible Euler equation in three space dimensions are

$$\partial_t \mathbf{Q} + \partial_x \mathbf{F}(\mathbf{Q}) + \partial_y \mathbf{G}(\mathbf{Q}) + \partial_z \mathbf{H}(\mathbf{Q}) = \mathbf{0} , \quad (11)$$

where $\mathbf{Q}$ is the vector of conserve variable, the unknowns of the problem; $\mathbf{F}(\mathbf{Q})$, $\mathbf{G}(\mathbf{Q})$ and $\mathbf{H}(\mathbf{Q})$ are the fluxes in the x , y and z directions, all given as follows

$$\left. \begin{aligned} \mathbf{Q} &= \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ E \end{bmatrix} & \mathbf{F}(\mathbf{Q}) &= \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho uw \\ u(E + p) \end{bmatrix} \\ \mathbf{G}(\mathbf{Q}) &= \begin{bmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ \rho vw \\ v(E + p) \end{bmatrix} & \mathbf{H}(\mathbf{Q}) &= \begin{bmatrix} \rho w \\ \rho uw \\ \rho vw \\ \rho w^2 + p \\ w(E + p) \end{bmatrix} \end{aligned} \right\} \quad (12)$$

E is the total energy given as

$$E = \rho \left[\frac{1}{2} V^2 + e \right] , \quad V^2 = u^2 + v^2 + w^2 , \quad (13)$$

with the specific internal energy $e(p, \rho)$ given in terms of the general caloric equation of state (4), which can also be expressed in different forms. The form of the general equation of state determines the expression for the speed of sound a , for example

$$p = p(\rho, e) \rightarrow a = \sqrt{\frac{p}{\rho^2} p_e + p_\rho} , \quad e = e(\rho, p) \rightarrow a = \sqrt{\frac{p}{\rho^2 e_p} - \frac{e_\rho}{e_p}} , \quad (14)$$

with partial derivatives of $e(\rho, p)$ defined as

$$e_\rho \equiv \frac{\partial e}{\partial \rho} , \quad e_p \equiv \frac{\partial e}{\partial p} . \quad (15)$$

Several EOS can be written in the Mie-Grüneisen form

$$\left. \begin{aligned} p &= \Gamma \frac{e}{v} + f(v) , \\ \Gamma &= v \frac{\partial p}{\partial e} |_v : \text{Grüneisen coefficient} , \end{aligned} \right\} \quad (16)$$

where $v = 1/\rho$ is the specific volume and the function $f(v)$ determines the particular EOS of interest. Three prominent examples included here are the Jones-Wilkins-Lee (JWL) EOS

$$f(v) = A \left(1 - \frac{\Gamma}{R_1} \frac{v_0}{v} \right) \exp \left(-R_1 \frac{v}{v_0} \right) + B \left(1 - \frac{\Gamma}{R_2} \frac{v_0}{v} \right) \exp \left(-R_2 \frac{v}{v_0} \right) , \quad (17)$$

the Cochran-Chan EOS

$$f(v) = A \left(\frac{1 - R_1 + \Gamma}{1 - R_1} \right) \left(\frac{v}{v_0} \right)^{-R_1} - B \left(\frac{1 - R_2 + \Gamma}{1 - R_2} \right) \left(\frac{v}{v_0} \right)^{-R_2} \quad (18)$$

and the van der Waals equation of state

$$f(v) = \frac{R}{C_v} \left(\frac{\alpha}{v(v - \beta)} \right) - \frac{\alpha}{v^2}, \quad \Gamma = \frac{R}{C_v} \left(\frac{v}{v - \beta} \right). \quad (19)$$

The constant quantities v_0 , α , β , A , B , C_v , R_1 and R_2 must be specified, see [24], for example. Further references regarding the Riemann problem for the Euler equations with general equation of state are [27], [32] and [24], for example. Regarding numerical methods see [4], [17] and [24], amongst others.

3. TV Flux for the 3D Euler Equations with General EOS

As the Euler equations are rotationally invariant [38], un-split finite volume methods require a *numerical flux in the normal direction across each face of a computational element* in three space dimensions. Without loss of generality it is sufficient to consider the flux in the x -direction, in Cartesian coordinates.

3.1. The TV Flux and the Pressure System

Following Toro and Vázquez [42] the x -flux (the normal flux in the x -direction) is decomposed thus

$$\mathbf{F}(\mathbf{Q}) = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho v \\ \rho w \\ u(\frac{1}{2}\rho V^2 + \rho e + p) \end{bmatrix} = \begin{bmatrix} \rho u \\ \rho u^2 \\ \rho uv \\ \rho uw \\ \frac{1}{2}\rho u V^2 \end{bmatrix} + \begin{bmatrix} 0 \\ p \\ 0 \\ 0 \\ u(\rho e + p) \end{bmatrix}, \quad (20)$$

with the advection and pressure fluxes respectively defined as

$$\mathbf{A}(\mathbf{Q}) = u \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \frac{1}{2}\rho V^2 \end{bmatrix}, \quad \mathbf{P}(\mathbf{Q}) = \begin{bmatrix} 0 \\ p \\ 0 \\ 0 \\ u(\rho e + p) \end{bmatrix}. \quad (21)$$

Again, following [42] we may consider two systems, the advection system and the pressure system

$$\partial_t \mathbf{Q} + \partial_x \mathbf{A}(\mathbf{Q}) = \mathbf{0}, \quad \partial_t \mathbf{Q} + \partial_x \mathbf{P}(\mathbf{Q}) = \mathbf{0}. \quad (22)$$

Actually, only the pressure system is needed to completely determine the numerical fluxes for both the advection and the pressure operators. In terms of primitive variables the pressure system becomes

$$\partial_t \mathbf{V} + \mathbf{B}(\mathbf{V}) \partial_x \mathbf{V} = \mathbf{0}, \quad (23)$$

with

$$\mathbf{V} = \begin{bmatrix} \rho \\ u \\ v \\ w \\ p \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/\rho \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{u(\rho e_p + e)}{\rho e_p} & \frac{h}{e_p} & 0 & 0 & u \end{bmatrix}, \quad (24)$$

where h is the specific enthalpy given as

$$h = e + p/\rho. \quad (25)$$

The eigenvalues of the pressure system (23) are found to be

$$\lambda_1 = \frac{1}{2}(u - A) < \lambda_2 = \lambda_3 = \lambda_4 = 0 < \lambda_5 = \frac{1}{2}(u + A), \quad (26)$$

with

$$A = \sqrt{u^2 + 4h/(\rho e_p)}. \quad (27)$$

The corresponding right eigenvectors are linearly independent and are given as

$$\left. \begin{aligned} \mathbf{R}_1 &= \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \\ \rho(u - A) \end{bmatrix}, \quad \mathbf{R}_2 = \begin{bmatrix} 1 \\ -u(\rho e_p + e) \\ \rho h \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{R}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \\ \mathbf{R}_4 &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{R}_5 = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \\ \rho(u + A) \end{bmatrix}. \end{aligned} \right\} \quad (28)$$

Therefore the pressure system (23) is hyperbolic, though not strictly hyperbolic, as the zero eigenvalue has multiplicity 3. In the case of multicomponent flow with m species we would have m advection equations added to the 3D Euler equations. In that case, the multiplicity of the zero eigenvalue would be $m + 3$ and the TV flux would still resolve all intermediate characteristic fields.

3.2. The TV Numerical Flux

The TV numerical flux is based on the splitting of the form (7) with advection and pressure vectors given by (21) and corresponding fluxes as

$$\mathbf{F}_{i+\frac{1}{2}} = \mathbf{A}_{i+\frac{1}{2}} + \mathbf{P}_{i+\frac{1}{2}}, \quad (29)$$

where the advection flux is

$$\mathbf{A}_{i+\frac{1}{2}} = \begin{cases} u_{i+\frac{1}{2}}^* \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \frac{1}{2}\rho V^2 \end{bmatrix}_i & \text{if } u_{i+\frac{1}{2}}^* \geq 0, \\ u_{i+\frac{1}{2}}^* \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \frac{1}{2}\rho V^2 \end{bmatrix}_{i+1} & \text{if } u_{i+\frac{1}{2}}^* < 0 \end{cases} \quad (30)$$

and the pressure flux is

$$\mathbf{P}_{i+\frac{1}{2}} = \begin{bmatrix} 0 \\ p_{i+\frac{1}{2}}^* \\ 0 \\ 0 \\ u_{i+\frac{1}{2}}^* \left[\rho_K e(\rho_K, p_{i+\frac{1}{2}}^*) + p_{i+\frac{1}{2}}^* \right] \end{bmatrix}. \quad (31)$$

The pressure $p_{i+\frac{1}{2}}^*$, velocity $u_{i+\frac{1}{2}}^*$ and densities ρ_K are to be found at each interface by solving a local Riemann problem for the pressure system (23), which in terms of primitive variables reads

$$\left. \begin{aligned} \partial_t \mathbf{V} + \mathbf{B}(\mathbf{V}) \partial_x \mathbf{V} &= \mathbf{0}, \\ \mathbf{V}(x, 0) &= \begin{cases} \mathbf{V}_L \equiv \mathbf{V}_i^n & \text{if } x < 0, \\ \mathbf{V}_R \equiv \mathbf{V}_{i+1}^n & \text{if } x > 0, \end{cases} \end{aligned} \right\} \quad (32)$$

the solution of which is

$$\left. \begin{aligned} u_{i+\frac{1}{2}}^* &= \frac{C_R u_R - C_L u_L}{C_R - C_L} - \frac{2}{C_R - C_L} (p_R - p_L), \\ p_{i+\frac{1}{2}}^* &= \frac{C_R p_L - C_L p_R}{C_R - C_L} + \frac{1}{2} \frac{C_R C_L}{C_R - C_L} (u_R - u_L), \\ \rho_K &= \begin{cases} \rho_i^n & \text{if } u_{i+\frac{1}{2}}^* \geq 0, \\ \rho_{i+1}^n & \text{if } u_{i+\frac{1}{2}}^* < 0, \end{cases} \end{aligned} \right\} \quad (33)$$

with

$$C_L = \rho_L (u_L - A_L), \quad C_R = \rho_R (u_R + A_R). \quad (34)$$

A_K is computed from (27) with arguments from the left (L) or the right (R).

Solution (33) is obtained by applying the simplest of approximations to the solution of a Riemann problem, namely linearised Riemann invariants. Such solver is useless when used in conventional formulations, it is entropy violating, produces early vacuum, well before is due, and crashes easily in the presence of strong shocks. It is highly surprising that with the proposed flux vector approach such simple Riemann solver works so well, as will be demonstrated. However, the reason for such robustness is unclear to us. In the original paper [42] we showed that the proposed procedure was exact for the linear advection equation. But for the full non-linear Euler equations the procedure is not identical to that for the linear advection equation, but analogous, relying on the formulated pressure system to compute normal velocity and pressure, and on the advection system to compute density and advected mass, momentum and kinetic energy. We are currently unable to state whether the proposed procedure constitutes a very generalisable principle.

Remark: The computation of the numerical flux requires two EOS-function evaluations, namely $e(\rho, p)$ and $e_p(\rho, p)$.

3.3. Properties of the scheme

The TV flux enjoys a number of properties that are worth enumerating:

1. *Subsonic pressure system.* Note that from (26) we have that the eigenvalues associated to the pressure waves satisfy the following property

$$\lambda_1 = \frac{1}{2}(u - A) < \lambda_2 = \lambda_3 = \lambda_4 = 0 < \lambda_5 = \frac{1}{2}(u + A). \quad (35)$$

Therefore the solution of the Riemann problem for the pressure system is always subsonic. This means that no sampling of the solution is necessary to identify the Godunov state in order to compute the flux. In particular, no transonic states arise explicitly that might require a special treatment to avoid entropy-violating shocks.

2. *Positivity of density.* It is easy to verify that the update formula for density becomes

$$\rho_i^{n+1} = \rho_i^n - \frac{\Delta t}{\Delta x} [u_{i+\frac{1}{2}}^* \rho_{K,i+\frac{1}{2}} - u_{i-\frac{1}{2}}^* \rho_{K,i-\frac{1}{2}}], \quad (36)$$

where $u_{i-\frac{1}{2}}^*$, $u_{i+\frac{1}{2}}^*$, $\rho_{K,i-\frac{1}{2}}$ and $\rho_{K,i+\frac{1}{2}}$ are given by (33). Introducing Courant numbers

$$c_{i-\frac{1}{2}} = \frac{\Delta t}{\Delta x} u_{i-\frac{1}{2}}^*, \quad c_{i+\frac{1}{2}} = \frac{\Delta t}{\Delta x} u_{i+\frac{1}{2}}^* \quad (37)$$

and analysing the four possible cases it is found that the most restrictive condition for positivity of density ρ_i^{n+1} is

$$\max_i \left\{ c_{i+\frac{1}{2}} \right\} \leq \frac{1}{2}. \quad (38)$$

This is a kind of Courant condition on the particle speed of the pressure system, which is a very mild restriction indeed.

3. *Exact recognition of stationary contacts and shear waves.* For an isolated stationary contact wave this property was proved in [42]. Consider now the case of isolated stationary shear waves, starting from the case of a shear wave in the y -direction. Proceeding in an analogous manner to that of a contact wave in [42] we find that an isolated stationary shear wave in y -direction is unperturbed by the scheme. The same can be proved for an isolated stationary shear wave in the z -direction. This property is very useful for resolving vortices and shear layers, particularly if the schemes are to be applied to the compressible Navier-Stokes equations. Whether this property of our scheme extends to other types of waves in other systems remains to be explored. We note that this will largely depend on the associated pressure system and the particular numerical approach to compute the pressure-flux component of the total flux.

The TV scheme has so far been described for the 3D Euler equations for a general equations of state. As presented, the resulting method is first-order accurate. In the next section we extend the method to high order of accuracy in space and time on unstructured meshes.

4. High-order Extension: the ADER Approach

The proposed TV flux can be used in any of the existing frameworks for constructing high-order approximations, such as semi-discrete ENO/WENO Runge-Kutta finite volume methods and Discontinuous Galerkin finite element methods. The TV flux is also applicable to fully discrete, one-step high order finite volume and DG formulations in the ADER framework.

4.1. The ADER approach

Here we briefly describe the fully discrete one-step ADER approach to construct numerical schemes of arbitrarily high order of accuracy in space and time for solving hyperbolic equations. ADER (Arbitrary Accuracy DERivative Riemann problem method) was first put forward by Toro et al. [39], in the finite volume framework, for solving linear hyperbolic equations in one and multiple space dimensions on Cartesian meshes; see also Schwartzkopff, Munz and Toro [33]. The extension of finite volume ADER schemes (ADER-FV) to non-linear equations, due to Titarev and Toro [36], is based on a semi-analytical, explicit solution of the generalized Riemann problem put forward by Toro and Titarev [40]. Since then, ADER has also been extended to the discontinuous Galerkin finite element framework by Dumbser [12], giving rise to ADER-DG schemes. For an elementary introduction to ADER schemes and the generalised Riemann problem, the reader is referred to chapters 19 and 20 of the textbook by Toro [38]. Further generalisations were put forward by Dumbser et al. [12], setting ADER-FV and ADER-DG in a generalised framework. The ADER approach has undergone numerous extensions and applications, examples include [2, 21, 23, 22, 35, 36, 37, 40, 41, 12, 5, 6, 10, 11, 3, 9, 7, 1, 28, 29, 31, 13, 14, 30].

Figure 1: Finite volume scheme in one space dimension. New cell average $\mathbf{Q}_i^{n+1}$ requires old cell average $\mathbf{Q}_i^n$, intercell numerical fluxes $\mathbf{F}_{i-1/2}$, $\mathbf{F}_{i+1/2}$ and numerical source $\mathbf{S}_i$.

A succinct review of the ADER approach, in the frame of the finite volume method, is presented here, in terms of a one-dimensional system of hyperbolic balance laws

$$\partial_t \mathbf{Q}(x, t) + \partial_x \mathbf{F}(\mathbf{Q}(x, t)) = \mathbf{S}(\mathbf{Q}(x, t)) , \quad (39)$$

where $\mathbf{Q} \in \mathbf{R}^m$ is the vector of conserved variables, the unknowns of the problem, $\mathbf{F}(\mathbf{Q})$ is the physical flux and $\mathbf{S}(\mathbf{Q})$ is the source term. Direct integration of (39) in the space-time control volume $V_i^n = [x_{i-1/2}, x_{i+1/2}] \times [t^n, t^{n+1}]$ yields

$$\mathbf{Q}_i^{n+1} = \mathbf{Q}_i^n - \frac{\Delta t}{\Delta x} [\mathbf{F}_{i+1/2} - \mathbf{F}_{i-1/2}] + \Delta t \mathbf{S}_i , \quad (40)$$

with

$$\left. \begin{aligned} \mathbf{Q}_i^n &= \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \mathbf{Q}(x, t^n) dx , \\ \mathbf{F}_{i+1/2} &= \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \mathbf{F}(\mathbf{Q}(x_{i+1/2}, t)) dt , \\ \mathbf{S}_i &= \frac{1}{\Delta t \Delta x} \int_{t^n}^{t^{n+1}} \int_{x_{i-1/2}}^{x_{i+1/2}} \mathbf{S}(\mathbf{Q}(x, t)) dx dt . \end{aligned} \right\} \quad (41)$$

Formula (40) is exact if integrals (41) are exact. Finite volume methods depart from an approximate interpretation of (40), with approximations for (41). We shall still use formula (40) but understood as a finite volume formula for numerical purposes, see Fig. 1. ADER type methods aim at evaluating the integrals (41) to high order of accuracy, resulting in explicit, one-step high order numerical methods (40) to solve (39). Note that only the second and

third integrals in (41) are required throughout the space time domain, while the first integral is computed only at the initial time. ADER schemes are a generalisation of Godunov's method and are based on two building blocks: (i) a non-linear reconstruction procedure and (ii) solution of a local generalised Riemann problem at each cell interface for flux evaluation. In the presence of source terms one requires an additional procedure to evaluate the volume integral in (41), for which additional Cauchy problem solutions are required within the space-time control volume. In next section we introduce these Cauchy problems.

4.2. The Generalized Riemann Problem: GRP

The Generalized Riemann Problem (GRP) for (39) is the Cauchy problem

$$\left. \begin{aligned} \partial_t \mathbf{Q}(x, t) + \partial_x \mathbf{F}(\mathbf{Q}(x, t)) &= \mathbf{S}(\mathbf{Q}(x, t)) , \\ \mathbf{Q}(x, 0) &= \begin{cases} \mathbf{P}_L(x), & x < 0 , \\ \mathbf{P}_R(x), & x > 0 , \end{cases} \end{aligned} \right\} \quad (42)$$

with $\mathbf{P}_L(x)$ and $\mathbf{P}_R(x)$ smooth functions, which for the ADER-FV come from the reconstruction procedure. Note that one only requires the solution along the interface position $x = 0$, as a function of time. We denote this solution by $\mathbf{Q}_{LR}(\tau)$

- ADER numerical flux:

$$\mathbf{F}_{i+\frac{1}{2}} = \frac{1}{\Delta t} \int_0^{\Delta t} \mathbf{F}(\mathbf{Q}_{LR}(\tau)) d\tau , \quad (43)$$

where $\mathbf{Q}_{LR}(\tau)$ is the time-dependent solution of the GRP (42). One usually applies a quadrature rule to evaluate (43) to the appropriate order of accuracy.

- Numerical source:

$$\mathbf{S}_i = \sum_{j=1}^K \frac{w_j}{\Delta t} \int_0^{\Delta t} \mathbf{S}(\mathbf{Q}_j(\tau)) d\tau . \quad (44)$$

This results from quadrature rule approximation to the volume integral in (41) in the space-time control volume $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [t^n, t^{n+1}]$.

In (44) w_j are the weights associated to the spatial quadrature points $x = \xi_j$ and $\mathbf{Q}_j(\tau)$ is the time-dependent solution, at $x = \xi_j$, of the initial-value problem

$$\left. \begin{aligned} \partial_t \mathbf{Q}(x, t) + \partial_x \mathbf{F}(\mathbf{Q}(x, t)) &= \mathbf{S}(\mathbf{Q}(x, t)) , \\ \mathbf{Q}(x, 0) &= \mathbf{P}_L(x) . \end{aligned} \right\} \quad (45)$$

Note that the strategy to solve the Generalized Riemann Problem (42) is also used to solve (45) to compute $\mathbf{Q}_j(\tau)$ in (44).

In three space dimensions the numerical flux is

$$\mathcal{D}_{i+\frac{1}{2}} = \frac{1}{\Delta t} \int_0^{\Delta t} \left(\int \int_{A_k} \mathbf{D} \cdot \mathbf{n}_k d\mathbf{A} \right) d\tau \quad (46)$$

with $\mathbf{D} = (\mathbf{F}, \mathbf{G}, \mathbf{H})$.

5. Numerical results

In order to illustrate the performance of the proposed methods we select examples that include one-dimensional problems with exact or a reliable reference solution to assess robustness, accuracy and efficiency. In 3D we assess the schemes against a reference solution on Cartesian meshes and accuracy for smooth solutions on unstructured meshes is assessed via a convergence rates study for schemes of upto 4th order in space and time.

5.1. RAE: robustness, accuracy and efficiency

In this section we solve selected 1D test problems with the aim of assessing three important features of numerical methods, namely robustness, accuracy and efficiency.

Robustness. In order to test robustness of the TV numerical scheme we solve the classical blast wave test problem proposed by Woodward and Colella [44] for the Euler equations for ideal gases. This test is enhanced here by adopting the Jones-Wilkins-Lee equation of state for a complex material.

The initial conditions in the spatial domain $[0, 1]$ at time $t = 0$ consist of three constant states, namely $\mathbf{W}_L = [\rho_L, u_L, p_L] = [1.0, 0.0, 10^3]$ for $x < 0.1$ m; $\mathbf{W}_M = [\rho_M, u_M, p_M] = [1.0, 0.0, 10^{-2}]$ for $0.1 < x < 0.9$ and $\mathbf{W}_R = [\rho_R, u_R, p_R] = [1.0, 0.0, 10^2]$ for $x > 0.9$. We assume the Jones-Wilkins-Lee equation of state with parameters $A = 6.321 \times 10^2$, $B = -4.472$, $R_1 = 11.3$, $R_2 = 1.13$, $\Gamma = 0.8938$ and $v_0 = 1.0$.

The domain $0 < x < 1$ is discretised with the standard mesh of 3000 cells and we plot the solution at the standard output time $t_{out} = 0.038$ s. For these computations we used $C_{cfl} = 0.9$ for both methods. The initial conditions are very severe; from the left discontinuity there emerges a very strong shock wave. In fact, for the ideal gas equation of state it is known that this shock has shock Mach number 198. During the time period considered in this test, there have been multiple wave-wave and wave-boundary interactions, making the problem an exceedingly demanding test problem for robustness, as well as accuracy. There is no exact solution for this test problem and thus we compare the present solution with that of the reference numerical solution, namely that of the first-order Godunov method used in conjunction with the exact Riemann solver.

In Fig. 2 we compare the numerical solution for density obtained with the present TV flux (circles) and the reference solution (full line) obtained with the Godunov method in conjunction with the exact Riemann solver. The resolution of discontinuities is identical in both methods, though in the smooth parts one just about sees that the peak values of the Godunov method with the exact Riemann solver are slightly higher, a sign of better accuracy. It is nonetheless safe to state that the present TV flux is robust and accurate for this very demanding test problems.

Accuracy. Here we solve a shock-tube problem [24] for the Euler equations with a complex EOS and compare the numerical solution of the present method against the exact solution and the reference solution from the Godunov method used in conjunction with the exact Riemann solver. Fig. 3 shows the results.

The initial condition at time $t = 0$ in the spatial domain $[0, 1]$ is $\mathbf{W}_L = [\rho_L, u_L, p_L] = [1895, 0, 10^{11}]$ to the left of the interface located at $x_0 = 0.43$ m and $\mathbf{W}_R = [\rho_R, u_R, p_R] = [236.875, 0, 10^{10}]$ to the right of the interface. The fluid material is PBX-9502, for which the Jones-Wilkins-Lee equation of state is adopted, with parameters $A = 1.362 \times 10^{12}$, $B = 7.199 \times 10^{10}$, $R_1 = 6.2$, $R_2 = 2.2$, $\Gamma = 0.5$ and $v_0 = 5.27704 \times 10^{-4}$. The computational domain is discretized with 100 cells and the CFL coefficient used is $CFL = 0.9$. The numerical solution for density and velocity is shown at the output time $t_{out} = 34 \times 10^{-6}$ s in Fig. 3, where comparison is made against the exact solution (line) and the corresponding numerical solution using the Godunov first order method in conjunction with the exact Riemann problem (triangles). Overall, the solution from the Godunov method is more accurate than that of the present TV flux. Recall that the Godunov method is the reference first-order method, as for the scalar case it can be proved to be the most accurate first-order monotone scheme. For the discontinuities, shock and contact, the schemes are comparable, but for the rarefaction wave the Godunov method is clearly more accurate. However, the Godunov method, as is well known, shows slight overshoots at the tail of the rarefaction and behind the shock. The TV solution looks perfectly *monotone* and, as compared against an exact solution, we may state that the scheme is accurate.

Efficiency. To test efficiency we solve the same shock tube problem as for the accuracy test above and compare the present TV flux against the flux from the exact Riemann solver and the flux from the Dumbser-Osher-Toro Riemann (DOT) solver [15], [16], for a sequence of meshes. In Fig. 4 we show the *efficiency plot*, that is error versus CPU time for seven meshes, for three schemes (three straight lines). We observe that given a mesh there are three corresponding errors and three CPU times. For example, the top circle on the bottom curve, the top triangle on top curve and the top square on the middle curve correspond respectively to the errors and CPU times of the TV flux, the DOT solver and the exact solver, for the coarsest mesh used. It is seen that the error of the present TV scheme is somewhat higher than those of the DOT scheme and the exact solver. The same is true for the second mesh, the third and so on. However, even though the errors of the DOT and exact schemes are lower, their CPU costs are

Figure 2: Robustness: the Woodward and Colella test problem for the Euler equations with the Jones-Wilkins-Lee equation of state. Comparison between the present TV flux (circles) and the first-order Godunov method in conjunction with the exact Riemann solver (full line) at time $t_{out} = 0.038$.

much higher. A precise comparison is made by first fixing an error (a horizontal line), which will be the same for all schemes, and then finding the intercepts with the TV flux curve (CPU time for TV), the DOT curve (CPU time for DOT) and the curve for the exact solver (CPU for the flux from the exact Riemann solver). For example, for Error=0.02, Fig. 4 shows that the present TV flux attains that error 15 times faster than the DOT solver and 9 times faster than the exact solver. The huge difference in efficiency between the present TV scheme and the other two solvers is mainly due to the complexity of the equation of state. It is however paradoxical that the approximate DOT solver is 1.68 times more expensive than the exact Riemann solver, although it is known that for the ideal gas case the original approximate Osher-Solomon solver is several times more expensive than an efficient exact Riemann solver [38]. The DOT solver, being a numerical version of the analytical Osher-Solomon solver, carries some of the inherent expense of the approach. The huge advantage, however, of DOT over the exact solver (and the present solver) is that DOT is an all-purpose universal scheme, it can be applied to any hyperbolic system, as long as the eigenstructure is available, exactly or numerically; the exact Riemann solver is always *ad hoc* and very cumbersome for real materials.

5.2. Three dimensional test: spherical explosion

Here we solve the three-dimensional Euler equations with the van der Waals equation of state on a cubic domain $[0, 2] \times [0, 2] \times [0, 2]$ discretised with a Cartesian mesh. As initial condition at time $t = 0$ we consider two constant states separated by the surface of a sphere of radius $r = 0.4$ centred at $[1, 1, 1]$. The inner state is $\rho_{ins} = 1.0$, $\mathbf{V}_{ins} = (0, 0, 0)$, $p_{ins} = 1.0$. The outer state is $\rho_{ins} = 0.1$, $\mathbf{V}_{ins} = (0, 0, 0)$, $p_{ins} = 0.125$. For the van der Waals equation of state we take the parameters $\alpha = 0.14$, $\beta = 3.258 \times 10^{-5}$, $R = 286.90$ and $Cv = 577.80$. The three-dimensional numerical solution is obtained with the TV flux with a mesh of 400^3 hexahedra and a CFL coefficient of $CFL = 0.3$.

Fig. 5 shows computed results for density at time $t_{out} = 0.25$. The top frame depicts a density surface for $z = 0$. The bottom frame shows a 1D-cut profile in the radial direction, compared with the reference solution obtained from a one-dimensional inhomogenous formulation of the problem. See [38] for details. The performance of the present numerical scheme is judged to be satisfactory for this three-dimensional problem.

5.3. Convergence rates for the 2D Euler equations on unstructured meshes

Here we assess the accuracy of the high-order ADER scheme in conjunction with the present TV flux as the building block. We solve the two dimensional ideal gas Euler equations on unstructured triangular meshes and assess empirical convergence rates. To this end we use the classical two-dimensional test presented in [20]. The computational domain is $\Omega = [-5, 5] \times [-5, 5]$ and periodic boundary conditions are imposed. The initial condition consists of a mean constant flow modified by an isentropic perturbation. The initial mean flow is

Table 1: Convergence rates. The two-dimensional Euler equations on unstructured meshes are solved with the ADER method with the present TV flux as building block. Expected convergence rates of three and four are attained.

Order 3	1/h	L_1	O_1	L_2	O_2	L_∞	O_∞
	1	2.80E+00	0.00	6.28E-01	0.00	4.73E-01	0.00
	2	1.61E+00	0.80	4.23E-01	0.57	3.53E-01	0.42
	4	3.20E-01	2.33	9.26E-02	2.19	7.47E-02	2.24
	8	4.92E-02	2.70	1.32E-02	2.81	8.68E-03	3.11
	16	6.63E-03	2.89	1.78E-03	2.89	1.18E-03	2.88
Order 4	1/h	L_1	O_1	L_2	O_2	L_∞	O_∞
	2	8.95E-01	0.00	1.84E-01	0.00	1.30E-01	0.00
	4	3.98E-02	4.49	1.11E-02	4.05	1.63E-02	2.99
	8	2.76E-03	3.85	6.54E-04	4.08	8.01E-04	4.35
	16	1.68E-04	4.04	3.77E-05	4.12	4.18E-05	4.26

given by $\rho = 1$, $p = 1$, $(u, v) = (1, 1)$ and the perturbation is given by

$$\left. \begin{aligned} \delta u &= -\frac{\epsilon}{2\pi} e^{\frac{1}{2}(1-r^2)} y, \\ \delta v &= \frac{\epsilon}{2\pi} e^{\frac{1}{2}(1-r^2)} x, \\ \delta \rho &= (1 + \delta T)^{\frac{1}{\gamma-1}} - 1, \\ \delta p &= (1 + \delta T)^{\frac{\gamma}{\gamma-1}} - 1, \\ \delta T &= -\frac{(\gamma-1)\epsilon^2}{8\gamma\pi^2} e^{1-r^2}, \end{aligned} \right\} \quad (47)$$

where $r^2 = x^2 + y^2$ and $\epsilon = 5$ is the vortex strength. This test is defined for an ideal fluid with $\gamma = 1.4$.

Table 1 shows errors and convergence rates for orders three and four, all measured in the L_1 , L_2 and L_∞ norms. It is seen that the theoretically expected convergence rates are attained, for this smooth solution test problem.

5.4. Collision of a shock wave with a triangle in two space dimensions

Here we solve the two-dimensional Euler equations with the van der Waals EOS on the squared domain $[-0.65, 0.5] \times [-0.5, 0.5]$ with a triangular solid body inside, defined by its vertices $v_1 = (-0.2, 0.0)$, $v_2 = (0.1, -1/6)$ and $v_3 = (0.1, 1/6)$. We apply the TV flux method extended to high order of accuracy using the ADER approach as described earlier. For this problem we use the method of third order of accuracy in both space and time, for smooth solutions.

The bottom and top boundaries at $y = -0.5$, $y = 0.5$ are solid impermeable walls and so are the edges of the triangular body. At the left boundary $x = -0.65$ we apply an inflow boundary condition and at the right boundary $x = 0.5$ we

apply transmissive boundary conditions. As initial condition we take a right-facing shock positioned at $x = -0.55$ separating two constant states. Ahead of the shock on the right we choose the initial state $\mathbf{W}_R = [\rho_R, V_R, p_R] = [1.225, 0, 1.01325 \times 10^5]$. Quantities are in SI units. To obtain the state behind the shock on the left we apply the Rankine-Hugoniot conditions imposing a shock wave of shock Mach number=1.3. For details see Chapter 2.4 in [38] or [2]. The van der Waals EOS parameters are $\alpha = 0.14$, $\beta = 3.258 \times 10^{-5}$, $R = 286.90$ and $Cv = 577.80$. This time-dependent problem involves the interaction between the initial shock wave and the triangular solid structure, followed by multiple wave-wave and wave-boundary interactions.

The computational domain is discretised with 10^6 triangles and the equations are solved with the third order (in space and time) ADER-FV method, with the TV flux as the building block. Fig. 6 shows density contour plots for two times, namely time $t = 1.6 \times 10^{-3}$ s (top) and $t = 2.2 \times 10^{-3}$ s (bottom). The wave patterns emerging from the present numerical solution are consistent with those of existing numerical as well as experimental results.

6. Summary and Conclusions

In this paper we have extended the TV flux of Toro and Vázquez, first present for the 1D Euler equations for ideal gases [42], to the three-dimensional Euler equations with general equation of state. We have also extended their numerical method that relies entirely on the pressure system to find the numerical flux. Moreover, we have set the entire framework as the building block for the higher-order ADER methodology to compute high-order approximations in space and time on unstructured meshes. The schemes have been assessed on several carefully chosen test problems to demonstrate robustness, accuracy and efficiency, three key features of numerical methods. Formal high-order of accuracy on a problem with smooth solution has been demonstrated through a convergence rates study for schemes of up to fourth-order of accuracy in space and time on two-dimensional unstructured meshes. The approach can be implemented for any desired order of accuracy.

The TV flux offers a viable and attractive approach for computing solutions to the equations of compressible media. The scheme reconciles simplicity and efficiency with accurate resolution of fast and slow waves. The extensions presented here are set in the finite volume framework, but the whole building block can also be used in the ADER discontinuous Galerkin framework, in which the solution of the generalised Riemann problem allows arbitrary time and space accuracy. An attractive, potential extension of the TV flux concerns the full use of the total separation of advection from the pressure terms. This means separating slow advection waves from fast pressure waves, so that the former could be treated explicitly with CFL coefficient close to unity and the latter could be dealt with implicitly. Encouraging results in this direction are reported in the very recent work [8], in which use is made of a staggered-mesh approach. The design of more sophisticated methods for implicit treatment of the pressure terms remains an open problem.

Figure 3: Accuracy: shock-tube problem for the Euler equations with the Jones-Wilkins-Lee equation of state. Comparison between the exact (line) and first-order numerical solutions: TV flux (circles) denotes the present scheme and God+Ex.RS (triangles) denotes the Godunov method in conjunction with the exact solution of the Riemann problem.

Figure 4: Efficiency plot *Error* versus *CPUtime* for a sequence of 7 meshes, for a shock-tube problem for the Euler equations with the Jones-Wilkins-Lee equation of state. Two approximate methods are compared against an exact Riemann solver: the present TV splitting scheme and the DOT scheme. It is seen that for the same *Error* = 0.02 the present TV scheme is 15 times faster than the DOT scheme and 9 times faster than the exact solver.

Figure 5: Spherical explosion test problem for the Euler equations with the van der Waals equation of state on a Cartesian mesh. Top frame: first-order TV solution for density at $z = 0$. Bottom frame: First-order TV solution (symbol) along radial direction compared to reference numerical solution (line) along the radial direction.

Figure 6: Shock wave impinging on stationary triangular solid body. Incident shock Mach number is $= 1.3$. The 2D Euler equations with van der Waals equation of state are solved on unstructured mesh with $\sim 1 \times 10^6$ triangles. Numerical method: ADER third order with TV flux. Density contours shown at times $t = 1.6 \times 10^{-3}$ (top) and $t = 2.2 \times 10^{-3}$ (bottom).

- [1] D. S. Balsara, C. Meyer, M. Dumbser, H. Du, and Z. Xu. Efficient implementation of ADER schemes for Euler and magnetohydrodynamical flows on structured meshes - speed comparisons with Runge-Kutta methods. *J. Comput. Phys.*, 235:934–969, 2013.
- [2] C. E. Castro. *High-Order ADER FV/DG Numerical Methods for Hyperbolic Equations*. PhD thesis, Department of Civil and Environmental Engineering, University of Trento, Italy, 2007.
- [3] C. E. Castro and E. F. Toro. Solvers for the High-Order Riemann Problem for Hyperbolic Balance Laws. *J. Comput. Phys.*, 227:2481–2513, 2008.
- [4] P. Colella and H. H. Glaz. Efficient Solution Algorithms for the Riemann Problem for Real Gases. *J. Comput. Phys.*, 59:264–289, 1985.
- [5] M. Dumbser. *Arbitrary High Order Schemes for the Solution of Hyperbolic Conservation Laws in Complex Domains*. PhD thesis, PhD Thesis, Institut für Aero- und Gasdynamik, Universität Stuttgart, Germany, 2005.
- [6] M. Dumbser. Building Blocks for Arbitrary High Order Discontinuous Galerkin Schemes. *Journal of Scientific Computing*, 27:215–230, 2006.
- [7] M. Dumbser, D. Balsara, E. F. Toro, and C. D. Munz. A Unified Framework for the Construction of One-Step Finite-Volume and Discontinuous Galerkin Schemes. *J. Comput. Phys.*, 227:8209–8253, 2008.
- [8] M. Dumbser and V. Casulli. A conservative, weakly nonlinear semi-implicit finite volume scheme for the compressible Euler equations with general equation of state. *J. Applied Mathematics and Computation.*, Submitted:–, 2015.
- [9] M. Dumbser, C. Enaux, and E. F. Toro. Finite Volume Schemes of Very High Order of Accuracy for Stiff Hyperbolic Balance Laws. *J. Comput. Phys.*, 227(8):3971–4001, 2008.
- [10] M. Dumbser and M. Käser. Arbitrary High Order Non-Oscillatory Finite Volume Schemes on Unstructured Meshes for Linear Hyperbolic Systems. *J. Comput. Phys.*, 221(2):693–723, 2007.
- [11] M. Dumbser, M. Käser, and E. F. Toro. An Arbitrary High Order Discontinuous Galerkin Method for Elastic Waves on Unstructured Meshes V: Local Time Stepping and p -Adaptivity. *Geophysical Journal International*, 171:695–717, 2007.
- [12] M. Dumbser and C. D. Munz. ADER Discontinuous Galerkin Schemes for Aeroacoustics. *Comptes Rendus Mécanique*, 333:683–687, 2005.
- [13] M. Dumbser, O. Zanotti, A. Hidalgo, and D. S. Balsara. ADER-WENO finite volume schemes with space-time adaptive mesh refinement. *Commun. Comput. Phys.*, 248:257–286, 2013.

- [14] M. Dumbser, O. Zanotti, R. Loubère, and S. Diot. A posteriori subcell limiting of the discontinuous Galerkin finite element method for hyperbolic conservation laws. *J. Comput. Phys.*, 278:47–75, 2014.
- [15] Michael Dumbser and Eleuterio F. Toro. A simple extension of the Osher Riemann solver to general non-conservative hyperbolic systems. *Journal of Scientific Computing*, 48:70–88, 2011.
- [16] Michael Dumbser and Eleuterio F. Toro. On universal Osher-type schemes for general nonlinear hyperbolic conservation laws. *Commun. Comput. Phys.*, 10:635–671, 2011.
- [17] P. Glaister. An Approximate Linearised Riemann Solver for the Euler Equations of Gas Dynamics. *J. Comput. Phys.*, 74:382–408, 1988.
- [18] E. Godlewski and P. A. Raviart. *Numerical Approximation of Hyperbolic Systems of Conservation Laws*. Applied Mathematical Sciences, Vol. 118, Springer–Verlag, New York, 1996.
- [19] S. K. Godunov. A Finite Difference Method for the Computation of Discontinuous Solutions of the Equations of Fluid Dynamics. *Mat. Sb.*, 47:357–393, 1959.
- [20] C. Hu and C. W. Shu. Weighted Essentially Non-oscillatory Schemes on Triangular Meshes. *J. Comp. Phys.*, 150:97–127, 1999.
- [21] M. Käser. *Adaptive Methods for the Numerical Simulation of Transport Processes*. PhD thesis, Institute of Numerical Mathematics and Scientific Computing, University of Munich, Germany, 2003.
- [22] M. Käser. ADER Schemes for the Solution of Conservation Laws on Adaptive Triangulations. Mathematical Methods and Modelling in Hydrocarbon Exploration and Production. Editors: Dr. Armin Iske, Dr. Trygve Randen. Springer–Verlag. ISBN: 978-3-540-22536-2, 2005.
- [23] Martin Käser and Armin Iske. ADER schemes on adaptive triangular meshes for scalar conservation laws. *J. Comput. Phys.*, 205:486–508, 2005.
- [24] B. J. Lee, C. E. Castro, E. F. Toro, and N. Nikiforakis. Adaptive Osher-type scheme for the Euler equations with highly nonlinear equations of state. *J. Comput. Physics*, 246:165–183, 2013.
- [25] R. J. LeVeque. *Finite Volume Methods for Hyperbolic Problems*. Cambridge University Press, 2002.
- [26] M. S. Liou and C. J. Steffen. A New Flux Splitting Scheme. *J. Comput. Phys.*, 107:23–39, 1993.
- [27] R. Menikoff and B. J. Plohr. The Riemann Problem for Fluid Flow of Real Materials. *Reviews of Modern Physics*, 61:75–130, 1989.

- [28] G. I. Montecinos, C. E. Castro, M. Dumbser, and E. F. Toro. Comparison of solvers for the generalized Riemann problem for hyperbolic systems with source terms. *Journal of Computational Physics*, 231:6472–64943, 2012.
- [29] G. I. Montecinos, L. O. Müller, and E. F. Toro. Hyperbolic reformulation of a 1D viscoelastic blood flow model and ADER finite volume schemes. *Journal of Computational Physics*, 266:101–123, 2014.
- [30] G. I. Montecinos and E. F. Toro. Reformulations for general advection-diffusion-reaction equations and locally implicit ADER schemes. *Journal of Computational Physics*, 275:415–442, 2014.
- [31] Lucas O. Müller and Eleuterio F. Toro. Enhanced global mathematical model for studying cerebral venous blood flow. *Journal of Biomechanics*, 47(13):3361–3372, 2014.
- [32] L. Quartapelle, L. Castelletti, A. Guardone, and G. Quaranta. Solution of the Riemann Problem of Classical Gasdynamics. *J. Comput. Phys.*, 190:118–140, 2003.
- [33] T. Schwartzkopff, C. D. Munz, and E. F. Toro. ADER: High-Order Approach for Linear Hyperbolic Systems in 2D. *J. Scientific Computing*, 17:231–240, 2002.
- [34] J. L. Steger and R. F. Warming. Flux Vector Splitting of the Inviscid Gasdynamic Equations with Applications to Finite Difference Methods. *J. Comput. Phys.*, 40:263–293, 1981.
- [35] Y. Takakura and E. F. Toro. Arbitrarily Accurate Non-Oscillatory Schemes for a Non-Linear Conservation Law. *J. Computational Fluid Dynamics*, 11:7–18, 2002.
- [36] V. A. Titarev and E. F. Toro. ADER: Arbitrary High Order Godunov Approach. *J. Scientific Computing*, 17:609–618, 2002.
- [37] V. A. Titarev and E. F. Toro. ADER Schemes for Three-Dimensional Hyperbolic Systems. *J. Comput. Phys.*, 204:715–736, 2005.
- [38] E. F. Toro. *Riemann Solvers and Numerical Methods for Fluid Dynamics, Third Edition*. Springer-Verlag, 2009.
- [39] E. F. Toro, R. C. Millington, and L. A. M. Nejad. Towards Very High-Order Godunov Schemes. In *Godunov Methods: Theory and Applications. Edited Review, E. F. Toro (Editor)*, pages 905–937. Kluwer Academic/Plenum Publishers, 2001.
- [40] E. F. Toro and V. A. Titarev. Solution of the Generalised Riemann Problem for Advection-Reaction Equations. *Proc. Roy. Soc. London A*, 458:271–281, 2002.

- [41] E. F. Toro and V. A. Titarev. ADER Schemes for Scalar Hyperbolic Conservation Laws with Source Terms in Three Space Dimensions. *J. Comput. Phys.*, 202(1):196–215, 2005.
- [42] E. F. Toro and M. E. Vázquez-Cendón. Flux splitting schemes for the Euler equations. *Computers and Fluids*, 70:1–12, 2012.
- [43] B. van Leer. Flux-Vector Splitting for the Euler Equations. Technical Report ICASE 82-30, NASA Langley Research Center, USA, 1982.
- [44] P. Woodward and P. Colella. The Numerical Simulation of Two-Dimensional Fluid Flow with Strong Shocks. *J. Comput. Phys.*, 54:115–173, 1984.
- [45] G-C. Zha and E. Bilgen. Numerical Solution of Euler Equations by a New Flux Vector Splitting Scheme. *Int. J. Numer. Meth. Fluids*, 17:115–144, 1993.