Fully Digital 1-D, 2-D and 3-D Multiscroll Chaos as Hardware Pseudo Random Number Generators

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Abstract—This paper introduces the first fully digital implementation of 1-D, 2-D and 3-D multiscroll chaos using the sawtooth nonlinearity in a 3rd order ODE with the Euler approximation. Systems indicate chaotic behaviour through phase space boundedness and positive Lyapunov exponent. Low-significance bits form a PRNG and pass all tests in the NIST SP. 800-22 suite without post-processing. Real-time control of the number of scrolls allows distinct output streams with 2-D and 3-D multiscroll chaos enabling greater controllability. The proposed PRNGs are experimentally verified on a Xilinx Virtex 4 FPGA with logic utilization less than 1.25%, throughput up to 5.25 Gbits/s and up to 512 distinct output streams with low cross-correlation.

Keywords—Chaos; nonlinear systems; digital circuits; pseudo random number generator; field programmable gate array;

I. INTRODUCTION

Chaotic oscillators have been emphasized for applications in random number generation [1]–[6] wherein the repeatable, reliable and high-performance chaotic sources required for these applications [7] are difficult to realize using analog circuits with chaos degradation due to established issues of PVT-sensitivity [8]. In particular, multi-scroll chaos generators have been exhaustively studied [9]–[12], wherein the limitations on the dynamic range of analog components and low supply voltages has restricted the number of scrolls that can be realized due to complex circuitry needed for control [13] unless mixed-signal approaches are used [14]. Register transfer level design has been shown to enable reliable and fully digital chaotic oscillators [15]–[17] for direct application in digital systems. Analog limitations are eliminated and repeatability is enabled through ease of register initialization. However, previous fully digital implementations of the chaotic logistic map had very high hardware requirements [2], [6] and diminished throughput. Jerk-equation based chaotic systems [18] have been digitally implemented and assessed for the effect of bus width and delay elements on the chaotic response [15] and optimized for random number generation [5].

Using a fully digital pipelined architecture, this paper implements for the first time, 1-D, 2-D and 3-D multiscroll chaos using sawtooth nonlinearities with controllable bounds. Particularly, this is the first digital implementation of 2-D and 3-D grid-scroll chaos. The Euler approximation is used to

TABLE I
1-D, 2-D AND 3-D MULTISCROLL CHAOS: SYSTEM ODES AND EQUILIBRIUM POINTS.

	System ODE	Equilibrium Points
1D	$\dot{X} = Y$ $\dot{Y} = Z$ $\dot{Z} = -\alpha[F(X) + Y + Z]$	$(i + 0.5, 0, 0)$ $i \in [L_x, U_x]$
2D	$\dot{X} = F(Y)$ $\dot{Y} = Z$ $\dot{Z} = -\alpha[F(X) + F(Y) + Z]$	$(i + 0.5, j + 0.5, 0)$ $i \in [L_x, U_x]$ $j \in [L_y, U_y]$
3D	$\dot{X} = F(Y)$ $\dot{Y} = F(Z)$ $\dot{Z} = -\alpha[F(X) + F(Y) + F(Z)]$	$(i + 0.5, j + 0.5, k + 0.5)$ $i \in [L_x, U_x]$ $j \in [L_y, U_y]$ $k \in [L_z, U_z]$

numerically solve the system of 3rd order ordinary differential equations (ODEs). These are then implemented as pseudo random number generators (PRNGs) without any additional post-processing by truncating high-significance bits. This eliminates short-term predictability [5] and here also suppresses cross-correlation and enables passage of all NIST SP. 800-22 randomness tests. Extension of scrolls to higher dimensions enables enhanced programmability and higher nonlinear complexity. The PRNGs are experimentally verified on a Xilinx Virtex 4 FPGA with logic utilizations not exceeding 1.25%, throughput verified up to 5.25 Gbits/s, and 3 times greater area-efficiency than previous work while providing high nonlinear complexity through multiscroll chaos and up to 512 controllable output streams with low cross-correlation.

II. DIGITAL IMPLEMENTATION

Multiscroll chaos can be described using a 3rd order ODE using a sawtooth nonlinearity $F(X)$ with integer lower bound L , upper bound U and floor function $\lfloor X \rfloor$ given by:

$$F(X) = \begin{cases} X - U - 0.5 & X \in (U + 0.5, \infty) \\ X - L - 0.5 & X \in (-\infty, L + 0.5) \\ X - \lfloor X \rfloor - 0.5 & X \in [L + 0.5, U + 0.5] \end{cases} \quad (1)$$

Essentially, multidimensional scrolls arise from expanding the nonlinearity in X [14] to Y and Z . Table I specifies the ODEs for 1-D, 2-D and 3-D multiscroll chaos. At each equilibrium point $(\dot{X}, \dot{Y}, \dot{Z}) = (0, 0, 0)$, all three systems have the same Jacobian and characteristic equations:

$$J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha & -\alpha & -\alpha \end{bmatrix} \quad (2a)$$

$$s^3 + \alpha s^2 + \alpha s + \alpha = 0 \quad (2b)$$

giving eigenvalues of $(-0.9687, 0.0156 \pm 0.9836i)$ for $\alpha = 0.9375$, with one negative real eigenvalue and a pair of complex conjugate eigenvalues with positive real parts, a *saddle point of index 2* that is necessary for the formation of scrolls at each equilibrium point, verified through numerical simulation. The flow is dissipative as the sum of eigenvalues is negative. The Euler approximation (with step size h) gives the numerical solution according to Fig. 1(a) for 3-D multiscroll chaos:

$$X_{t+h} = X_t + h[\dot{X}_t] \quad (3a)$$

$$Y_{t+h} = Y_t + h[\dot{Y}_t] \quad (3b)$$

$$Z_{t+h} = Z_t + h[\dot{Z}_t] \quad (3c)$$

Registers $\{X, Y, Z\}$ store the system state while combinational logic determines the next state. A 32-bit fixed-point two's complement representation is used with 5 bits for the sign and integer part and 27 bits for the fraction part. The step-size is $h = 2^{-3}$, simplifying to a hardcoded right-shift. The

system parameter ($\alpha = 0.9375$) is realized by noting that $0.9375 = 1 - 2^{-4}$. For 1-D and 2-D multiscroll chaos, $F(Z)$ and $F(Y)$ modules are eliminated as required by the ODEs and replaced by wires. While many parameters can be made controllable in a digital multiscroll system, varying only the number of scrolls incurs the smallest performance penalty [16]. The hardware implementation of the sawtooth function in (1) is shown in Fig. 1(b) with the resulting output in Fig. 1(c). L and U are integer-valued, with $L = -6$ fixed and a controllable 3-bit U ($U \in [-4, 3]$), two less than the integer width to prevent arithmetic overflow. Sign-extension by 2 bits matches U with the integer width and one-padding to the right by 1 bit adds 0.5. This is fully scalable and more control can be added simply by using a wider integer width (thus wider L and U) to accommodate more scrolls. The number of scrolls in each dimension $d \in \{x, y, z\}$ is given as:

$$N_d = U_d - L_d + 1 = U_d + 7, \quad U_d \in [-4, 3] \quad (4)$$

ranging between 3 and 10 and dependent only on the upper bound of the sawtooth nonlinearity in that dimension.

III. CHAOTIC RESPONSE

Attractors (X - Y or Y - Z phase plots) from the digitally implemented 1-D, 2-D and 3-D multiscroll systems are shown in Fig. 2 with all of them showing excellent phase-space boundedness. It is important to note that since this paper introduces finite precision numerical solutions of what are infinite-precision continuous-time ODEs, the trajectories produced by the digital implementations are, by definition, pseudo-chaotic and approximate the chaotic behavior expected from

Fig. 1. (a) Schematic for 3-D multiscroll chaos: 2-D and 1-D multiscrolls can be obtained by replacing $F(Z)$ and $F(Y)$ with wires based on the ODEs, (b) Digital realization of the sawtooth function and (c) the bounded sawtooth function from (1) with $L = -6$ and $U = \{-3, -1, 1\}$.

Fig. 2. Oscilloscope trace of experimentally obtained (a) X-Y attractor with $U_x = -1$ [1-D : 6 scrolls], (b) X-Y attractor with $(U_x, U_y) = (1, -2)$ [2-D : 8×5 scrolls] and (c)-(d) X-Y and Y-Z attractors with $(U_x, U_y, U_z) = (-2, -2, -4)$ [3-D : $5 \times 5 \times 3$ scrolls].

Fig. 3. Oscilloscope trace of 1-D multiscroll (a)-(c) X-Y, Y-Z and X-Z attractors after truncation of defective bits, time series and FFT of X for (c)-(d) original output and (e)-(f) after truncation of defective bits.

analog systems. However, the pseudo-chaos can be verified by calculating the maximum Lyapunov exponent (MLE), wherein a positive MLE verifies the existence of chaotic dynamics. Mathematically, this value expresses the effect of an arbitrarily small change in initial conditions on the overall divergence in the output solution over time, according to $\delta S(t) \approx e^{\lambda t} S(\delta t)$. Using software based on [19], the MLEs are 0.138, 0.230, and 0.259 for 1-D (6 scrolls), 2-D (8×5 scrolls) and 3-D ($5 \times 5 \times 3$ scrolls) multiscroll systems and thus indicate chaotic dynamics. The MLE generally increases for a larger number of scrolls [16] and remains positive.

TABLE II
NIST SP. 800-22 TEST RESULTS AND AREA/PERFORMANCE RESULTS ON A XILINX VIRTEX 4 FPGA

NIST SP. 800-22 Results						
	1-D		2-D		3-D	
	PV	PP	PV	PP	PV	PP
Monobits	✓	0.96	✓	1.00	✓	0.98
Block Frequency	✓	0.93	✓	0.98	✓	1.00
Cumulative Sums	✓	0.98	✓	1.00	✓	0.98
Runs	✓	1.00	✓	1.00	✓	1.00
Longest Run	✓	0.98	✓	1.00	✓	1.00
Rank	✓	1.00	✓	1.00	✓	0.98
FFT	✓	0.98	✓	1.00	✓	0.98
N. O. Template	✓	0.99	✓	0.99	✓	0.99
O. Template	✓	0.98	✓	0.98	✓	1.00
Universal	✓	1.00	✓	0.96	✓	0.96
Approx. Entropy	✓	1.00	✓	0.96	✓	1.00
Random Excursion	✓	1.00	✓	0.99	✓	0.98
Random Excursion V.	✓	1.00	✓	0.98	✓	0.99
Serial	✓	0.98	✓	0.99	✓	0.99
Linear Complexity	✓	0.98	✓	1.00	✓	1.00
Overall Result	Pass		Pass		Pass	

Experimental Results on the Xilinx Virtex 4 FPGA

Total LUTs	251	271	295
Total FFs	96	96	96
Gate Count (Gc)	2776	2936	3128
PRNG Bits (N_{PRNG})	45	45	45
Frequency [MHz]	116.73	113.69	111.29
Throughput [Mb/s]	5253	5116	5008
FOM [Gc/Th.]	1.89	1.74	1.60

TABLE III
STATISTICS OF THE CROSS-CORRELATIONS BETWEEN OUTPUT STREAMS FOR DIFFERENT CONTROL PARAMETERS FOR 1-D, 2-D AND 3-D MULTISCROLL SYSTEMS.

		Max.	Min.	Mean	Median	Std. Dev.
1-D	X	0.1742	$9.85e-04$	0.0347	0.0285	0.03642
	Y	0.0226	$2.58e-06$	0.0074	0.0077	0.00541
	Z	0.0240	$1.51e-04$	0.0085	0.0067	0.00669
	R	0.0035	$8.75e-05$	0.0011	0.0008	0.00099
2-D	X	0.1120	$1.01e-05$	0.0198	0.0152	0.01753
	Y	0.1173	$4.67e-06$	0.0207	0.0172	0.01625
	Z	0.0278	$5.61e-06$	0.0055	0.0047	0.00417
	R	0.0051	$9.04e-07$	0.0013	0.0011	0.00100
3-D	X	0.1442	$8.30e-09$	0.0090	0.0059	0.01025
	Y	0.1467	$6.32e-08$	0.0097	0.0061	0.01120
	Z	0.1665	$4.94e-08$	0.0110	0.0067	0.01305
	R	0.0098	$5.32e-10$	0.0013	0.0011	0.00095

IV. IMPLEMENTATION AS PRNGS

The positive MLE of the digitally implemented multiscroll systems indicates that the output is unpredictable over the long term and thus is suitable to be implemented as a pseudo random number generator. Discarding the high-significance bits greatly suppresses short-term predictability [5] and enables the passage of all tests in the NIST SP. 800-22 statistical test suite [20]. In the 1-D, 2-D and 3-D case, the high 17 bits from each of the outputs X, Y and Z are discarded. The low 15 bits from X, Y and Z are concatenated and form a PRNG for which the NIST SP. 800-22 test results, area utilization

TABLE IV
COMPARISON WITH PREVIOUSLY REPORTED CHAOS-BASED PRNGs.

System		Area (Gc)	T.put (Mb/s)	FOM	NIST
Addabbo, 2007 [1]	Rényi Map	3988	200	0.05	Pass
Chen, 2010 [2]	Log. Map	9622	200	0.02	Pass
Li, 2010 [3]	Log. Map	9136	200	0.02	Fail
Chen, 2010 [4]	Log. Map	31655	3200	0.10	Pass
Zidan, 2011 [5]	ODE	2464	1180	0.48	Pass
Li, 2012 [6]	Log. Map	11903	6400	0.54	Pass
This Work					
1-D Multiscroll	ODE	2776	5253	1.89	Pass
2-D Multiscroll	ODE	2936	5116	1.74	Pass
3-D Multiscroll	ODE	3128	5008	1.60	Pass

and throughput on a Xilinx Virtex 4 XC4VSX35-10FF668 FPGA (30,720 LUTs, 30,720 FFs) are shown in Table II. All three systems exhibit excellent statistical properties with experimentally verified throughput up to 5.25 Gbits/s and LUT utilization not exceeding 1.25% and FF utilization not exceeding 0.45% of total resources available on the FPGA. A gate count is estimated as $G_c = 8 \times (LUTs + FFs)$ and the area efficiency is assessed through a figure of merit determined as $FOM = (N_{PRNG} \times f_{CLK})/G_c$. As expected, there is an overhead in area when extending to 2-D and 3-D multiscroll chaos. However, this extension to multi-dimensions gives a multifold increase in the number of distinct output streams from 8 to 64 and 512 respectively on the basis of the 3-bit control parameter assigned to the upper bound in each dimensions. Oscilloscope traces of the X-Y, Y-Z and X-Z attractors after truncation and time series and frequency response of the X-variable before and after truncation are shown in Fig. 3. Clearly, the discarding defective bits gives a far more random output and a uniform spectrum. Table III summarizes the statistics of the magnitude of cross-correlation coefficients from each system for different control parameters (8 streams for 1-D, 64 streams for 2-D and 512 streams for 3-D). The cross-correlations of the native X, Y and Z outputs of each system show very high correlation that is completely suppressed in the corresponding PRNG output (R). The average cross-correlation is on the order of 10^{-3} . The three multiscroll chaos-based PRNGs implemented in this paper are compared to previous work in Table IV. Clearly, the implemented systems have lower area than all the implementations of 1-D chaotic maps. On that basis, the proposed systems outperform previous literature from an area-efficiency perspective given the ~ 5 Gbits/s throughput and are approximately 3 times more area efficient than the best previous implementation.

V. CONCLUSION

Fully digital implementations of 1-D, 2-D and 3-D multiscroll chaos through the sawtooth nonlinearity are introduced using the Euler approximation. The output indicates chaotic behaviour through positive maximum Lyapunov exponent and phase-space boundedness. Furthermore, the number of scrolls

in all three dimensions can be manipulated in real-time through a controllable upper bound. The low-significant bits from each dimension are concatenated to yield PRNGs that pass the NIST SP. 800-22 testing suite with throughput up to 5.25 Gbits/s and can produce up to 512 distinct streams with low cross-correlation. These have useful applications in chaos-based digital communication and encryption systems.

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