The Effect of Numerical Techniques on Differential Equation Based Chaotic Generators

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Abstract—In this paper, we study the effect of the numerical solution accuracy on the digital implementation of differential chaos generators. Four systems are built on a Xilinx Virtex 4 FPGA using Euler, mid-point, and Runge-Kutta fourth order techniques. The twelve implementations are compared based on the FPGA used area, maximum throughput, maximum Lyapunov exponent, and autocorrelation confidence region. Based on circuit performance and the chaotic response of the different implementations, it was found that less complicated numerical solution has better chaotic response and higher throughput.

I. INTRODUCTION

Chaotic circuits find many applications in cryptography [1], [2] and random number generation [3]. Moreover, chaos is used in digital communication systems as chaos shift keying (CSK) [4]. Chaos generators can be implemented using operational amplifiers [5], discrete MOS transistors [6], and fully integrated [7].

The analog chaotic generators are sensitive to the running conditions, process variations, and the operating temperature. Also, tolerances in analog realization degrade the quality of the chaos output [8]. Moreover, the initial conditions cannot be set precisely in the analog generator, and the circuit capacitors require a huge on-chip area. In addition, using analog generators in digital systems requires digitization of the output. On the other hand, the digital chaotic systems overcome these problems, and can be easily synchronized in communications and encryption systems.

Digital chaos is mostly based on logistic maps [9]. However Differential equation based chaos provides more nonlinearity and is of higher dimensional order compared to the logistic maps. The increased complexity enables building more secured chaos based systems.

Recently an all digital implementation of differential equation based systems has been introduced [3], [10]. Also a differential equation based multi-scroll system is presented in [11]. The differential equation based chaos was implemented by realizing its numerical solution. In this paper we study different numerical solutions (Euler, mid-point, and Runge-Kutta fourth) and compare their performance. We also show that the procedure is universal by studying four different systems with different nonlinearities and comparing their results. The four used systems are given in Table I.

All the systems implemented on a Xilinx Virtex 4 FPGA, using three different numerical techniques. The twelve differ-

ent implementations are compared based on their areas, the maximum throughput. The chaotic behavior is also compared based on the Maximum Lyapunov Exponent (MLE) and the autocorrelation confidence region. We show that the Euler technique provides the best performance compared to midpoint and Runge-Kutta 4th-order, based of the mentioned comparison technique. Also, The midpoint express better response with respect to Runge-Kutta 4th-order.

II. DIGITAL IMPLEMENTATION AND NUMERICAL TECHNIQUES

Digital generators for differential equation based chaos are implemented by realizing the numerical solution of the system equations. The state variables of the system are realized as registers, and the numerical solution of each variable are implemented as combinational block, as shown in Fig. 1.

Many circuit optimizations can be introduced to improve the performance of the combinational part. One of these optimizations is the elimination of constant multiplications with the aid of suitable selection of the system parameters to be in the powers of two. What make this optimization possible is the fact that chaos behavior are expressed for interval of parameters, not a single value. Another important optimization is using fixed point representation scheme for realizing chaotic circuit. This is based on that the outputs of the chaotic generator are within bounded intervals.

Fig. 1. The general block diagram of the digital implementation of 3D differential based chaos generator.

One of the main design point of the digital chaos generator is the selection of numerical solution. The adopted numerical technique will affect the chaotic response of the realization. We select three known numerical solution, with different complexities and accuracies, to study their effect on the response

TABLE I
THE CHAOTIC SYSTEMS USED WITHIN THE COMPARISONS.

	System 1 Lorenz [12]	System 2 Chen [13]	System 3 Elwakil [5]	System 4 Elwakil [5]
Equations	$\dot{X} = \alpha(Y - X)$ $\dot{Y} = (\rho - Z)X - Y$ $\dot{Z} = XY - \beta Z$	$\dot{X} = a(Y - X)$ $\dot{Y} = \text{sign}(x)(b - Z) + cy$ $\dot{Z} = y - dZ$	$\dot{X} = a(Y - X)$ $\dot{Y} = \text{sign}(x)(b - Z)$ $\dot{Z} = x - cZ$	$\dot{X} = Y$ $\dot{Y} = Z$ $\dot{Z} = -Z - YB(Y) - X$
Nonlinearity	Multiplication	Sign, Modules	Sign, Modules	Piecewise

of the circuit. Assume the following differential equation:

$$\dot{A} = F_A(A, B, \dots) \quad (1)$$

This equation is solved using the following techniques:

Euler Method:

It is a simple technique which required one calculation per numerical iteration, in return it produces less accurate results compared with other more complex technique. Euler method is defined as,

$$A_{i+1} = A_i + h \cdot F_A(A_i, B_i, \dots) \quad (2)$$

Fig. 2. Block diagram of the digital circuit used to evaluate numerical solution using Euler technique. $h' = \log_2(h)$ represents shift in wires for implementing the arithmetic shift.

Fig. 3. Block diagram of the digital circuit used to evaluate numerical solution using midpoint technique. $h' = \log_2(h)$ represents shift in wires for implementing the arithmetic shift.

Fig. 4. Block diagram of the digital circuit used to evaluate numerical solution using Runge-Kutta 4th-order technique. $h' = \log_2(h)$ represents shift in wires for implementing the arithmetic shift.

where A_i and A_{i+1} are the solutions of the variable A at steps i and $i + 1$ respectively, and h is the time step. Fig. 2 shows the schematic of the Euler numerical solution.

Midpoint Method: This method is defined as,

$$K_A = F_A(A_i, B_i, \dots) \quad (3a)$$

$$A_{i+1} = A_i + h \cdot F_A(A_i + 0.5hK_A, B_i + 0.5hK_B, \dots) \quad (3b)$$

Compared to Euler, this technique produces more accurate numerical iterations. On the other hand, this technique requires more hardware, and longer calculation path as shown in Fig. 3

Runge-Kutta 4th-order: This method have the longest calculation path and requires large area, but it produces the most accurate numerical approximation compared to the previous technicians. Runge-Kutta 4th-order is defined as,

$$K_{A1} = F_A(A_i, B_i, \dots) \quad (4a)$$

$$K_{A2} = F_A(A_i + 0.5hK_{A1}, B_i + 0.5hK_{B1}, \dots) \quad (4b)$$

$$K_{A3} = F_A(A_i + 0.5hK_{A2}, B_i + 0.5hK_{B2}, \dots) \quad (4c)$$

$$K_{A4} = F_A(A_i + hK_{A3}, B_i + hK_{B3}, \dots) \quad (4d)$$

$$A_{i+1} = A_i + h(K_1 + 2K_2 + 2K_3 + K_4)/6 \quad (4e)$$

Fig. 4 shows that evaluating the solution using the Runge-Kutta 4th-order method which have the longest pipeline.

III. EXPERIMENTAL RESULTS

The twelve circuits implemented using Verilog VHDL on Xilinx Virtex[®] 4 XC4VSX35 FPGA. We used 32 bits fixed-point scheme for representing the numbers. GNU iVerilog software was used for circuit simulation, and Xilinx ISE[®] 11, with optimized for time (speed) option selected, was used for synthesizing the circuit and calculating the used area and maximum frequency. The attractors and the time diagrams are captured using TekTronix[™] MSO 414 oscilloscope.

TABLE II
DETAILED COMPARISON FOR THE FOUR SYSTEMS. THE CIRCUIT DATA IS GIVEN IMPLEMENTATION ON XILINX VIRTEX 4[®] XC4VSX35 FPGA.

System	Numerical Method	Slices (15,360)	Slice FF (30,720)	LUTs (30,720)	DSPs (192)	Freq. (MHz)	Throughput (Gb/s)	Autocorrelation (95% Conf. Range)	MLE (250k iter.)
1 Lorenz [12]	Euler	145	96	287	8	53.53	1.71	23	1.667
	MP	222	96	440	16	28.00	0.90	36	1.179
	RK 4O	658	97	1,216	32	13.17	0.42	38	1.167
2 Chen [13]	Euler	144	97	284	0	122.01	3.90	183	0.299
	MP	286	96	568	0	72.01	2.30	188	0.267
	RK 4O	860	98	1,611	0	28.02	0.90	202	0.236
3 Elwakil [5]	Euler	137	97	265	0	126.73	4.06	1,137	0.027
	MP	295	96	547	0	77.56	2.48	1,139	0.022
	RK 4O	916	96	1,650	0	33.51	1.07	1,219	0.020
4 Elwakil [5]	Euler	109	97	215	0	129.52	4.14	2,982	0.159
	MP	195	96	385	0	62.95	2.01	6,540	0.146
	RK 4O	597	97	1,140	0	29.73	0.95	117,262	0.106

The detailed FPGA utilization and maximum frequency for the different implementations are given in Table II. In general, system 1 requires more area and express lower throughput because it uses multiplication as nonlinear element. As shown in Fig 2, 3 and 4, the main block F_A is serially repeated two and four times for the midpoint and Runge-Kutta 4th-order methods compared to one appearance in the Euler technique. This repetition will have an impact on the area and delay with almost the same factor. Based on the number of FPGA slices used, Euler and midpoint implantations require 15-22% and 32-34% of the area required by Runge-Kutta 4th-order respectively. The same approximate factor appears for the number of DSPs used in system 1, which is the only system that uses this blocks due to its multiplication nonlinear element. In the same direction, Euler and midpoint implantations expresses 3.8-4.4x and 2.1-2.5x higher throughput compared to Runge-Kutta 4th-order respectively. The number of the used registers is almost independent of the numerical method.

Fig. 5 shows the oscilloscope snapshots for the X - Z projection of the different implementations attractors. It appear from the figure that the used numerical technique did not

affect the pattern of the attractor. In some cases, Euler circuit produces more spread attractor compared to midpoint and Runge-Kutta 4th-order, as shown in the case of system 1 in Fig. 5(a,b,c). In the time domain response and under the same initial conditions, the circuit behavior are totally changed for the three different techniques. Fig. 6 shows that the circuit time waveform of system 1, where the Euler output jumps between scrolls more than midpoint which is more than Runge-Kutta 4th-order. This property is useful for the chaos based random number generators that uses the sign of the output signal as a source of randomness.

IV. CHAOTIC RESPONSE

The chaotic response of the different implementations is compared using the MLE, autocorrelation 95% confidence region, and some statistical measurement. MLE is a sufficient measurement for chaotic behavior. Table II shows the calculated MLE for the different implementations using the software provided in [14]. Based on the calculated values of all systems, Euler implementation have better chaotic response than midpoint, which is also better than Runge-Kutta 4th-order. Euler and midpoint implantations expresses 27-50% and

Fig. 5. Oscilloscope snapshots of the attractors projection on the X - Z plane for different implementations, (a-c) system 1, (d-f) system 2, (g-i) system 3 and (j-l) system 4. The attractors in (a,d,g,j), (b,e,h,k), and (c,f,i,l) are for the Euler, midpoint, and Runge-Kutta 4th-order implementations, respectively.

Fig. 6. Oscilloscope snapshots of the circuit output of system 1 for the same time span and circuit frequency for the different implementations.

1-37% higher MLE compared to Runge-Kutta 4th-order respectively. Fig. 7 shows the MLE of system 1 implementations versus the number of iterations, for the three implementations.

Another technique used to compare the output is its autocorrelation. The less correlation between the successive outputs expresses better random properties of the system. The 95% confidence region of the autocorrelation is used to compare the different techniques. The confidence region is defined by the largest lag after which the correlation value did not go out of the region bounds. The lower lag value means less correlation. Table II shows lag value for the 95% confidence region for the variable X for the different implementations. In all of the cases the correlation response of Euler implementation is better chaotic response than midpoint, which is again better than Runge-Kutta 4th-order. The autocorrelation is calculated for data sample of 900k. Fig. 8 shows the autocorrelation of the variable X of system 4 different implementations.

From the previous comparison it is found that Euler implementations express better chaotic response than midpoint, which in turn is better than Runge-Kutta 4th-order, because the numerical solution error adds an extra nonlinearity to the chaotic system.

V. CONCLUSION

In this paper we studied the effect of the numerical solution techniques on the performance of the digital differential based chaotic systems. Four different system are implemented on FPGA using three different numerical techniques. By comparing the four systems, It was found that Euler technique have a

Fig. 7. The MLE iterations for the different implementations of system 1.

Fig. 8. Autocorrelation for the variable X of system 4, for different lag scans. The dashed lines bounded the 95% confidence region.

better response than the midpoint, and the midpoint is better than the Runge-Kutta 4th-order. The systems are compared based on the required area, maximum throughput, maximum Lyapunov exponent, and autocorrelation confidence region.

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