Supplemental Material for “Elastic metamaterials with simultaneously negative effective shear modulus and mass density”

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There can be four cases of total s-p or p-s mode conversions on the interface of a double-negative metamaterial and a normal solid under negative refraction:

Case 1: Incident shear waves from a double negative metamaterial.
Case 2: Incident longitudinal waves from a double negative metamaterial.
Case 3: Incident shear waves from a normal solid.
Case 4: Incident longitudinal waves from a normal solid.

It is obvious that Case 3 can be obtained from the time reversal of Case 2 and Case 4 can be obtained from the time reversal of Case 1. Below we derive the general conditions of total mode conversions under negative refraction.

Fig. S1. A schematic graph of s-p mode conversion during the negative refraction on the interface of a double-negative elastic metamaterial of Case 1 below.
For $s$ waves, the displacement is $\vec{u}_i = u_i (\sin \alpha \hat{x} + \cos \alpha \hat{y}) e^{ik_i (-\cos ax + \sin ay)}$

For $p$ waves, the displacement is $\vec{u}_i = u_i (\cos \beta \hat{x} + \sin \beta \hat{y}) e^{ik_i (\cos bx + \sin by)}$

The displacements must match on the interface $x = 0$, i.e.,

$$\begin{align*}
  u_i \sin \alpha e^{ik_i \sin ay} &= u_i \cos \beta e^{ik_i \sin by}, \\
  u_i \cos \alpha e^{ik_i \sin ay} &= u_i \sin \beta e^{ik_i \sin by}.
\end{align*}$$

(S1)

From Eq. (S1), we find $u_i \sin \alpha = u_i \cos \beta$ and $u_i \cos \alpha = u_i \sin \beta$, which means that $u_i = u_i$ and $\alpha + \beta = \pi/2$. In the case of normal refraction, for incident angle $0 < \alpha < \pi/2$ we have $\beta < 0$, thus $\alpha + \beta < \pi/2$, the displacements are not possible to match on the interface. Total conversion is only possible in the case of negative refraction.

From Eq. (S1), we also find $k_i \sin \alpha = k_i \sin \beta$, which means the wave vector parallel to the interface must conserve. Substituting $\alpha + \beta = \pi/2$, we obtain $k_i \sin \alpha = k_i \cos \alpha$.

From $k_i = \frac{\omega}{v_i} = \frac{\omega}{\sqrt{\mu_i/\rho_i}}$ and $k_i = \frac{\omega}{v_i} = \frac{\omega}{\sqrt{(\kappa_2 + \mu_2)/\rho_2}}$, we find

$$\tan \alpha = \frac{k_i}{k_i} = \frac{\sqrt{\mu_i/\rho_i}}{(\kappa_2 + \mu_2)/\rho_2}.$$ 

We also need to consider the matching condition of stresses.

For $s$ waves, $\vec{u}_i = u_i (\sin \alpha \hat{x} + \cos \alpha \hat{y}) e^{ik_i (-\cos ax + \sin ay)}$. Thus, the strains are

$$\begin{align*}
  S_{xx}' &= \frac{\partial u_x}{\partial x} = u_i \sin \alpha (-ik_i \cos \alpha) e^{ik_i (-\cos ax + \sin ay)} = -iu_i k_i \sin \alpha \cos \alpha e^{ik_i (-\cos ax + \sin ay)}, \\
  S_{yy}' &= \frac{\partial u_y}{\partial y} = u_i \cos \alpha (ik_i \sin \alpha) e^{ik_i (-\cos ax + \sin ay)} = iu_i k_i \sin \alpha \cos \alpha e^{ik_i (-\cos ax + \sin ay)}, \\
  S_{xy}' &= \frac{1}{2} \left( \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) \\
  &= \frac{1}{2} \left( u_i \sin \alpha (ik_i \sin \alpha) e^{ik_i (-\cos ax + \sin ay)} + u_i \cos \alpha (-ik_i \cos \alpha) e^{ik_i (-\cos ax + \sin ay)} \right) \\
  &= \frac{1}{2} \left( iu_i k_i \sin^2 \alpha - iu_i k_i \cos^2 \alpha \right) e^{ik_i (-\cos ax + \sin ay)}. \\
\end{align*}$$

And the stress can be written as
\[\sigma'_{xx} = (\kappa' + \mu') S'_{xx} + (\kappa' - \mu') S'_{xy}\]
\[= - (\kappa' + \mu') iu_i k_i \sin \alpha \cos \alpha e^{ik_i(- \cos \alpha x + \sin \alpha y)} + (\kappa' - \mu') iu_i k_i \sin \alpha \cos \alpha e^{ik_i(- \cos \alpha x + \sin \alpha y)}\]
\[= -2i \mu u_i k_i \sin \alpha \cos \alpha e^{ik_i(- \cos \alpha x + \sin \alpha y)}\]
\[= -i \mu u_i k_i \sin 2 \alpha e^{ik_i(- \cos \alpha x + \sin \alpha y)}\],
\[\sigma'_{yy} = (\kappa' + \mu') S'_{yy} + (\kappa' - \mu') S'_{yx}\]
\[= (\kappa' + \mu') iu_i k_i \sin \alpha \cos \alpha e^{ik_i(- \cos \alpha x + \sin \alpha y)} - (\kappa' - \mu') iu_i k_i \sin \alpha \cos \alpha e^{ik_i(- \cos \alpha x + \sin \alpha y)}\]
\[= 2i \mu u_i k_i \sin \alpha \cos \alpha e^{ik_i(- \cos \alpha x + \sin \alpha y)}\]
\[= i \mu u_i k_i \sin 2 \alpha e^{ik_i(- \cos \alpha x + \sin \alpha y)},\]
\[\sigma'_{xy} = 2 \mu S_{xy}\]
\[= \mu (iu_i k_i \sin^2 \alpha - iu_i k_i \cos^2 \alpha) e^{ik_i(- \cos \alpha x + \sin \alpha y)}\]
\[= -i \mu u_i k_i \cos 2 \alpha e^{ik_i(- \cos \alpha x + \sin \alpha y)}.\]

For \textit{p} waves, \(\bar{u}_i = u_i (\cos \beta \hat{x} + \sin \beta \hat{y}) e^{ik_i(\cos \beta x + \sin \beta y)}\). Thus, the strains are
\[S'_{xx} = \frac{\partial u_i}{\partial x} = u_i \cos \beta (ik_i \cos \beta) e^{ik_i(\cos \beta x + \sin \beta y)} = iu_i k_i \cos^2 \beta e^{ik_i(\cos \beta x + \sin \beta y)},\]
\[S'_{yy} = \frac{\partial u_i}{\partial y} = u_i \sin \beta (ik_i \sin \beta) e^{ik_i(\cos \beta x + \sin \beta y)} = iu_i k_i \sin^2 \beta e^{ik_i(\cos \beta x + \sin \beta y)},\]
\[S'_{xy} = \frac{1}{2} \left( \frac{\partial u_i}{\partial y} + \frac{\partial u_i}{\partial x} \right)\]
\[= \frac{1}{2} \left( u_i \cos \beta (ik_i \sin \beta) e^{ik_i(\cos \beta x + \sin \beta y)} + u_i \sin \beta (ik_i \cos \beta) e^{ik_i(\cos \beta x + \sin \beta y)} \right)\]
\[= iu_i k_i \sin \beta \cos \beta e^{ik_i(\cos \beta x + \sin \beta y)}.\]

And the stress can be written as
\[
\sigma_{xx}' = (\kappa_2 + \mu_2) S_{xx}' + (\kappa_2 - \mu_2) S_{yy}' \\
= (\kappa_2 + \mu_2) i u_k \kappa_i \cos \beta e^{i k_x (\cos \beta x + \sin \beta y)} + (\kappa_2 - \mu_2) i u_k \kappa_i \sin \beta e^{i k_x (\cos \beta x + \sin \beta y)} \\
= i (\kappa_2 (\cos^2 \beta + \sin^2 \beta) + \mu_2 (\cos^2 \beta - \sin^2 \beta)) u_k \kappa_i e^{i k_x (\cos \beta x + \sin \beta y)} \\
= i (\kappa_2 + \mu_2 \cos 2\beta) u_k \kappa_i e^{i k_x (\cos \beta x + \sin \beta y)}, \\
\sigma_{yy}' = (\kappa_2 + \mu_2) S_{yy}' + (\kappa_2 - \mu_2) S_{xx}' \\
= (\kappa_2 + \mu_2) i u_k \kappa_i \sin^2 \beta e^{i k_x (\cos \beta x + \sin \beta y)} + (\kappa_2 - \mu_2) i u_k \kappa_i \cos^2 \beta e^{i k_x (\cos \beta x + \sin \beta y)} \\
= i (\kappa_2 (\cos^2 \beta + \sin^2 \beta) + \mu_2 (-\cos^2 \beta + \sin^2 \beta)) u_k \kappa_i e^{i k_x (\cos \beta x + \sin \beta y)} \\
= i (\kappa_2 - \mu_2 \cos 2\beta) u_k \kappa_i e^{i k_x (\cos \beta x + \sin \beta y)}, \\
\sigma_{xy}' = 2\mu_2 S_{xy}' \\
= 2\mu_2 i u_k \kappa_i \sin \beta \cos \beta e^{i k_x (\cos \beta x + \sin \beta y)} \\
= i \mu \kappa_i u_k \sin 2\beta e^{i k_x (\cos \beta x + \sin \beta y)}.
\]

\(\sigma_{xx}\) and \(\sigma_{xy}\) must also be continuous on the interface \(x = 0\), i.e.,

\[
-i \mu \kappa_i u_k \sin 2\alpha e^{i k_x \sin \beta y} = i (\kappa_2 + \mu_2 \cos 2\beta) u_k \kappa_i e^{i k_x \sin \beta y}, \\
-i \mu \kappa_i u_k \cos 2\alpha e^{i k_x \sin \beta y} = i \mu \kappa_i u_k \sin 2\beta e^{i k_x \sin \beta y}.
\]

(S2)

By substituting \(\alpha + \beta = \pi/2\), \(u_i = u_j\) and \(k_i \sin \alpha = k_j \cos \alpha\) in Eq. (S2), we find

\(\kappa_2 = \mu_2 \cos 2\alpha - \mu_1 \sin 2\alpha / \tan \alpha\) and \(-\mu_1 = \mu_2 \tan 2\alpha \tan \alpha\).
Conclusion: total conversion is possible under the case of negative refraction.

The total conversion matching conditions are:

\[\begin{align*}
\alpha + \beta &= \pi/2, \\
\kappa_2 &= \mu_2 \cos 2\alpha - \mu_1 \sin 2\alpha / \tan \alpha, \\
-\mu_i &= \mu_2 \tan 2\alpha \tan \alpha, \\
\tan \alpha &= \frac{k_i}{k_i} = \frac{\sqrt{\mu_i/\rho_i}}{\sqrt{(\kappa_2 + \mu_2)/\rho_2}}. \\
\end{align*}\] (S3)

Since Case 3 can be obtained from the time reversal of Case 2 and Case 4 can be obtained from the time reversal of Case 1, here we discuss only Cases 1 and 3.

Case 1: we consider a transverse plane wave incident from the left medium of \(\mu_1 < 0, \rho_1 < 0\) with an incident angle of \(0 < \alpha < \pi/2\) as shown in Fig. S1. First, we can obtain \(\mu_2\) by using \(-\mu_i = \mu_2 \tan 2\alpha \tan \alpha\). Then, we can obtain \(\kappa_2\) by using \(\kappa_2 = \mu_2 \cos 2\alpha - \mu_1 \sin 2\alpha / \tan \alpha = \mu_2 / \cos 2\alpha\). At last, we can obtain \(\rho_2\) by using

\[\tan \alpha = \frac{k_i}{k_i} = \frac{\sqrt{\mu_i/\rho_i}}{\sqrt{(\kappa_2 + \mu_2)/\rho_2}}.\]

Therefore, \(\mu_2\), \(\kappa_2\) and \(\rho_2\) are all obtained. It is worth mentioning that when \(\alpha > \pi/4\), we have \(\mu_2 < 0\), but \(\kappa_2 > 0\) and \(\kappa_2 + \mu_2 = \mu_2 (1 + \cos 2\alpha)/\cos 2\alpha > 0\), which indicates a double positive medium for refracted longitudinal waves on the right (together with \(\rho_2 > 0\)). In this case the medium on the right is not a normal solid.

Case 3: we consider a transverse plane wave incident from the left medium of \(\mu_1 > 0, \rho_1 > 0\) with an incident angle of \(0 < \alpha < \pi/2\). \(\mu_2\), \(\kappa_2\) and \(\rho_2\) can also be obtained from Eq. (S3). Note that we have \(\kappa_2 + \mu_2 = \mu_2 (1 + \cos 2\alpha)/\cos 2\alpha < 0\) and \(\rho_2 < 0\), which indicate a double negative medium for refracted longitudinal waves on the right.