Nonlinear Analysis of Ring Oscillator and Cross-Coupled Oscillator Circuits

Xiaoqing Ge∗ Murat Arcak† Khaled N. Salama‡

Abstract

Hassan Khalil’s research results and beautifully written textbook on nonlinear systems have influenced generations of researchers, including the authors of this paper. Using nonlinear systems techniques, this paper analyzes ring oscillator and cross-coupled oscillator circuits, which are essential building blocks in digital systems. The paper first investigates local and global stability properties of an n-stage ring oscillator by making use of its cyclic structure. It next studies global stability properties of a class of cross-coupled oscillators which admit the representation of a dynamic system in feedback with a static nonlinearity, and presents sufficient conditions for almost global convergence of the solutions to a limit cycle when the feedback gain is in the vicinity of a bifurcation point. The result are also extended to the synchronization of interconnected identical oscillator circuits.

1 Introduction

Due to their integrated nature, voltage-controlled oscillators (VCOs) are widely used in commercial applications. Ring oscillators and cross-coupled oscillators are two important types of VCOs in many electronic systems. Ring oscillators are used in applications such as clock recovery circuits [1] and disk-drive read channels [2], while cross-coupled oscillators can be employed in function generators, frequency synthesizers, etc. Some of the early stability studies were based on experimental results and linear analysis tools. Reference [3] used the Nyquist stability criterion to analyze the stability of a linearized oscillator. Reference [4] applied linear theory to the analysis of oscillation-frequency and oscillation-amplitude stability of nonlinear feedback oscillators. The global characteristics of oscillators, however, can be understood only with nonlinear systems techniques.

In addition to nonlinear analysis of individual oscillators, synchronization in systems of identical or nearly identical coupled oscillators has long been a topic of interest. Networks of
synchronized oscillator circuits are being increasingly employed in digital communications to account for the frequency drift of individual oscillators and to accommodate more than one digital standard. A key problem in the study of oscillator networks is to characterize parameter regimes of individual oscillators and properties of their interconnection structure that guarantee synchronization.

In the first part of the paper, we investigate the stability properties of an n-stage ring oscillator. By using the cyclic structure of the interconnection and by adapting the “secant criterion” [5] developed for nonlinear systems with a cyclic structure, we obtain a sufficient condition for global asymptotic stability of the origin. We show that this condition is also necessary for stability when the inverters comprising the circuit are identical, and make use of a recently proven Poincaré-Bendixson Theorem for cyclic systems with arbitrary order [6] to conclude the presence of periodic orbits when the stability of the equilibrium is lost. We then study a network of identical ring oscillators and derive a synchronization condition that consists of a modification of the secant criterion as in [7].

In the second part, we represent the class of cross-coupled oscillators as a Lurie system [8], which consists of a linear block in feedback with a static nonlinearity, and analyze its stability properties within the absolute stability framework. We first prove boundedness of the trajectories using the “stiffening” property of the static nonlinearity [9]. By making the linear block passive with a Popov multiplier, we conclude following the tools in [10] that the system experiences either a supercritical pitchfork bifurcation or a supercritical Hopf bifurcation. In the case of the Hopf bifurcation we further show that a unique almost1 globally asymptotically stable limit cycle exists. Finally, we consider an interconnection of identical cross-coupled oscillators through a linear, symmetric input-output coupling. By appropriately selecting the interconnection matrix and key parameters of each oscillator, we achieve synchronization of the oscillators.

2 Stability Analysis of an n-Stage Ring Oscillator

2.1 System Model

A typical ring oscillator consists of an odd number of digital-inverters connected in a feedback loop. Figure 1 shows the proposed model for a digital-inverter composed of a cascaded - tanh(·) nonlinearity and a low-pass filter. The nonlinearity is characterized by \( V_n = -V_{sat}\tanh(V_i/V_s) \), where \( V_{sat} \) is a saturation voltage and \( V_s \) is used to adjust the internal slope. Thus, the behavior of an inverter is described by the following differential equation [11]:

\[
V_o + TV_o = -V_{sat}\tanh(V_i/V_s).
\]

Introducing the variables \( x = V_i, \ y = V_o, \ \alpha = 1/V_s, \ \beta = V_{sat}/T \) and \( \eta = 1/T \), we transform equation (1) into:

\[
\dot{y} = -\eta y - \beta \tanh(\alpha x)
\]

where \( \alpha \cdot \beta = V_{sat}/(V_s \cdot T) > 0 \) defines the internal gain of the inverting amplifier.

1By “almost” we mean convergence from all initial conditions except for those on a set of measure zero.
Figure 1: Proposed nonlinear model of a digital-inverter.

\[ V_i \overset{\text{Nonlinearity}}{\rightarrow} -V_{sat} \cdot \tanh(V_i) \overset{\text{Low-pass filter}}{\rightarrow} V_n \overset{\frac{1}{1+T}}{\rightarrow} V_o \]

Figure 2: n-stage ring oscillator.

Figure 2 shows an n-stage ring oscillator formed of n cascaded inverters connected in a feedback loop. Each inverter can be modeled by (2) and, hence, the feedback system is described by:

\[
\begin{align*}
\dot{x}_1 &= -\eta_1 x_1 - \beta_1 \tanh(\alpha_1 x_n) \\
\dot{x}_2 &= -\eta_2 x_2 - \beta_2 \tanh(\alpha_2 x_1) \\
\dot{x}_3 &= -\eta_3 x_3 - \beta_3 \tanh(\alpha_3 x_2) \\
\vdots & \quad \vdots \\
\dot{x}_n &= -\eta_n x_n - \beta_n \tanh(\alpha_n x_{n-1}).
\end{align*}
\]

This system admits a cyclic structure studied in [5] in the context of biochemical reaction networks, where each oscillator in the ring is driven by the previous one. Because each of the n blocks possesses a negative gain, the interconnection serves as a negative feedback system when n is an odd number.

### 2.2 Stability Analysis

In this section we derive a global stability condition for the origin of (3). This condition is important because, when negated, it serves as a necessary condition for the existence of limit cycles.

When n is odd, the diagonal similarity transformation \( P = \text{diag}(p_1, \ldots, p_n) \) with \( p_i = (-1)^{i+1} \) brings (3) to the form

\[
\begin{align*}
\dot{\tilde{x}}_1 &= -\eta_1 \tilde{x}_1 - \beta_1 \tanh(\alpha_1 \tilde{x}_n) \\
\dot{\tilde{x}}_2 &= -\eta_2 \tilde{x}_2 + \beta_2 \tanh(\alpha_2 \tilde{x}_1) \\
\dot{\tilde{x}}_3 &= -\eta_3 \tilde{x}_3 + \beta_3 \tanh(\alpha_3 \tilde{x}_2) \\
\vdots & \quad \vdots \\
\dot{\tilde{x}}_n &= -\eta_n \tilde{x}_n + \beta_n \tanh(\alpha_n \tilde{x}_{n-1}).
\end{align*}
\]
Local stability of the n-stage ring oscillator system can be evaluated by linearization of (4) at the origin:

\[
\begin{pmatrix}
\dot{\tilde{x}}_1 \\
\dot{\tilde{x}}_2 \\
\dot{\tilde{x}}_3 \\
\vdots \\
\dot{\tilde{x}}_n
\end{pmatrix} =
\begin{pmatrix}
-\eta_1 & 0 & 0 & \cdots & -k_1 \\
k_2 & -\eta_2 & 0 & \cdots & 0 \\
0 & k_3 & -\eta_3 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & k_n & -\eta_n
\end{pmatrix}
\begin{pmatrix}
\tilde{x}_1 \\
\tilde{x}_2 \\
\tilde{x}_3 \\
\vdots \\
\tilde{x}_n
\end{pmatrix}
\] (5)

where

\[
k_i := \alpha_i \beta_i > 0.
\] (6)

It follows from the secant criterion in [12] that the origin is asymptotically stable if:

\[
\frac{k_1 \cdots k_n}{\eta_1 \cdots \eta_n} < \sec\left(\frac{\pi}{n}\right)^n.
\] (7)

Furthermore, when \(\eta_i\)'s are identical, (7) is also necessary for asymptotic stability [5, 12].

In [5], Arcak and Sontag extended the secant criterion to a class of cyclic systems with sector nonlinearities using output strict passivity concepts [13], and provided sufficient conditions for global asymptotic stability of the origin:

**Lemma 1** [5] Consider the system

\[
\begin{align*}
\dot{x}_1 &= -a_1(x_1) - b_n(x_n) \\
\dot{x}_2 &= -a_2(x_2) + b_1(x_1) \\
\vdots &= \vdots \\
\dot{x}_n &= -a_n(x_1) + b_{n-1}(x_{n-1})
\end{align*}
\] (8)

where \(a_i(\cdot)\) and \(b_i(\cdot)\) are continuous functions satisfying

\[
x_i a_i(x_i) > 0, \quad x_i b_i(x_i) > 0 \quad \forall x_i \neq 0,
\] (9)

and suppose there exits constants \(\gamma_i > 0\) such that

\[
\frac{b_i(x_i)}{a_i(x_i)} \leq \gamma_i \quad \forall x_i \neq 0.
\] (10)

If these \(\gamma_i\)'s satisfy

\[
\gamma_1 \cdots \gamma_n < \sec(\pi/n)^n,
\] (11)

then the equilibrium \(x = 0\) is asymptotically stable. If, further, the functions \(b_i(\cdot)\) are such that

\[
\lim_{|x| \to \infty} \int_0^{x_i} b_i(\sigma) d\sigma = \infty
\] (12)

i = 1, \cdots, n, then \(x = 0\) is globally asymptotically stable.

In our model (3), inequality (10) of Lemma 1 holds with \(\gamma_i = k_i+1/\eta_i\) for \(i = 1, \cdots, n - 1\), and with \(\gamma_n = k_1/\eta_n\). This means that the local stability condition (7) given above coincides with (11) and also guarantees global asymptotic stability:
Theorem 1  Consider the n-stage ring oscillator (3) with n representing an odd number. If (7) holds, then the origin is globally asymptotically stable. Furthermore, if \( \eta_i \)'s are identical, (7) is a necessary and sufficient condition for global asymptotic stability.

Proof In (4), \( a_i(\tilde{x}_i) = \eta_i \tilde{x}_i \), \( b_i(\tilde{x}_i) = \beta_{i+1} \tanh(\alpha_{i+1} \tilde{x}_i) \) for \( i = 1, \cdots, n-1 \) and \( b_n(\tilde{x}_n) = \beta_1 \tanh(\alpha_1 \tilde{x}_n) \). Both \( a_i(\cdot) \) and \( b_i(\cdot) \) belong to the sector \([0, +\infty)\) and, thus, condition (9) holds for any nonzero \( \tilde{x}_i \). Condition (10) is satisfied with \( \gamma_i = k_{i+1}/\eta_i \) for \( i = 1, \cdots, n-1 \), and with \( \gamma_n = k_1/\eta_n \). Moreover, for \( i = 1, \cdots, n-1 \),

\[
\lim_{|x_i| \to \infty} \int_0^{x_i} b_i(\sigma)d\sigma = \lim_{|x_i| \to \infty} \int_0^{x_i} \beta_{i+1} \tanh(\alpha_{i+1} \sigma)d\sigma = \lim_{|x_i| \to \infty} \frac{\beta_{i+1}}{\alpha_{i+1}} \ln|\cosh(\alpha_{i+1} x_i)| = \infty \tag{13}
\]

and, for \( i = n \),

\[
\lim_{|x_n| \to \infty} \int_0^{x_n} b_n(\sigma)d\sigma = \lim_{|x_n| \to \infty} \int_0^{x_n} \beta_1 \tanh(\alpha_1 \sigma)d\sigma = \lim_{|x_n| \to \infty} \frac{\beta_1}{\alpha_1} \ln|\cosh(\alpha_1 \tilde{x}_n)| = \infty. \tag{14}
\]

Thus, if (7) holds, then the origin of system (4) is globally asymptotically stable from Lemma 1. Furthermore, if \( \eta_i \)'s are identical and (7) fails, then the origin is unstable from the discussion of the linearization above. Thus, (7) is also a necessary condition for global asymptotic stability when \( \eta_i \)'s are identical.

The necessity statement of Theorem 1 is important because a typical construction of a ring oscillator consists of identical or nearly identical inverter elements and, thus, (7) becomes a necessary and sufficient condition for global asymptotic stability of the origin. Since the solutions of (3) are bounded due to the bounded nonlinearity \( \tanh(\cdot) \), a Poincaré-Bendixson Theorem for cyclic systems in [6] is applicable and states that the \( \omega \)-limit set of the solutions must be an equilibrium or a periodic orbit. The secant condition (7) thus gives a boundary between stability of the origin and the emergence of limit cycles.

3 Synchronization of Ring Oscillators

We now consider \( N \) identical ring oscillator models of the form (4), rewritten here as:

\[
\begin{align*}
\dot{x}_{1,k} &= -\eta_1 x_{1,k} - \beta_1 \tanh(\alpha_1 x_{n,k}) + u_{1,k} \\
\dot{x}_{2,k} &= -\eta_2 x_{2,k} - \beta_2 \tanh(\alpha_2 x_{1,k}) + u_{2,k} \\
\dot{x}_{3,k} &= -\eta_3 x_{3,k} - \beta_3 \tanh(\alpha_3 x_{2,k}) + u_{3,k} \\
&\quad \vdots \\
\dot{x}_{n,k} &= -\eta_n x_{n,k} - \beta_n \tanh(\alpha_n x_{n-1,k}) + u_{n,k}
\end{align*}
\tag{15}
\]

where \( x_{i,k} \) is the \( i \)th voltage variable \( i = 1, \cdots, n \) in the \( k \)th oscillator circuit \( k = 1, \cdots, N \). The input variable \( u_{i,k} \) represents the coupling between the oscillators, and is of the form:

\[
u_{i,k} = \sum_{j \neq k} d_{jk}(x_{i,j} - x_{i,k}) \tag{16}\
\]
where the parameter \( a_{k,j}^i = d_{j,k}^i \geq 0 \) depends inversely on the resistance that connects \( i \)th nodes of circuits \( j \) and \( k \). In particular, if there is no connection, then \( a_{k,j}^i = d_{j,k}^i = 0 \).

For each node \( i = 1, \cdots, n \), we define a Laplacian matrix \( L_i \in \mathbb{R}^{N \times N} \) that incorporates the parameters \( a_{k,j}^i \) according to:

\[
L_i^j_{k,j} = \begin{cases} 
-a_{k,j}^i & \text{if } k \neq j \\
\sum_{j \neq k} a_{k,j}^i & \text{if } k = j.
\end{cases}
\]

By construction, the Laplacian matrix is symmetric and satisfies \( L_i^1_{1,j} = 0 \) where \( 1_N \) denotes the \( N \)-vector of ones. The eigenvalues of \( L_i \) are denoted as \( \lambda_1^i \leq \lambda_2^i \leq \cdots \leq \lambda_N^i \) where \( \lambda_1^i = 0 \) corresponds to the eigenvector \( 1_N \). The second smallest eigenvalue \( \lambda_2^i \) is strictly positive if the graph corresponding to the Laplacian is connected, and zero otherwise [14].

We first note the solutions of (15)-(16) are bounded because, in the absence of the \( \tanh \) nonlinearities, (15)-(16) consists of \( N \) decoupled linear systems:

\[
\dot{X}_i = -\eta_i X_i - L_i X_i = -(L_i + \eta_i I)X_i \quad i = 1, \cdots, n
\]

where \( X_i := [x_{i,1} \cdots x_{i,N}] \), and since the eigenvalues of \( L_i \) are nonnegative and \( \eta_i > 0 \), the matrix \(-(L_i + \eta_i I)\) is Hurwitz. Boundedness then follows because (15)-(16) consists of this linear Hurwitz system, perturbed by bounded \( \tanh \) nonlinearities. We now use the values \( \lambda_2^i \) calculated from (17) above for each \( i, \cdots, n \) and the circuit parameters in (15) to give a synchronization condition:

**Theorem 2** Consider the interconnected ring oscillators (15)-(16), \( k = 1, \cdots, N \), with \( n \) odd, and let \( \lambda_2^i \) denote the second smallest eigenvalue of the Laplacian matrix (17), \( i, \cdots, n \). If

\[
\frac{k_1 \cdots k_n}{(\eta_1 + \lambda_1^1) \cdots (\eta_n + \lambda_n^N)} < \sec \left( \frac{\pi}{2n} \right)^2,
\]

where \( k_i \) is as defined in (6), then, for any pair \((k, j) \in \{1, \cdots, N\} \times \{1, \cdots, N\}\),

\[
x_{i,j}(t) - x_{i,j}(t) \to 0 \quad i = 1, \cdots, n
\]

as \( t \to \infty \).

**Proof** The result follows from an application of the synchronization condition derived in [7], in combination with a diagonal stability test developed in [5]. As in [7], we view each circuit (15) as an interconnection of the dynamic subsystems:

\[
H_{2i-1} : \quad \dot{x}_i = -\eta_i x_i + u_i + v_{2i-1} \quad y_{2i-1} = x_i, \quad i = 1, \cdots, n,
\]

and static nonlinearities:

\[
\begin{align*}
H_{2i} : \quad y_{2i} &= \beta_{i+1} \tanh(\alpha_{i+1} v_{2i}) \quad i = 1, \cdots, n - 1 \\
H_{2n} : \quad y_{2n} &= \beta_1 \tanh(\alpha_1 v_{2n})
\end{align*}
\]
where we have dropped the index \( k \) for the circuit to simplify the notation and denoted by \( v_i \) and \( y_i \) the input and output of the \( i \)th block, \( i = 1, \cdots, 2n \). Indeed, (15) consists of the dynamic blocks \( H_1, H_3, \cdots, H_{2n-1} \) and the static blocks \( H_2, H_4, \cdots, H_{2n} \) interconnected according to:

\[
v_1 = -y_{2n}, \quad v_{2i+1} = -y_{2i}, \quad i = 1, \cdots, n-1, \quad v_{2i} = y_{2i-1}, \quad i = 1, \cdots, n
\]  

rewritten in vector notation as:

\[
V = \Sigma Y,
\]

where \( V := [v_1 \ v_2 \cdots v_{2n}]^T \), \( Y := [y_1 \ y_2 \cdots y_{2n}]^T \), and

\[
\Sigma = \begin{bmatrix}
0 & \cdots & 0 & -1 \\
1 & 0 & \cdots & 0 \\
0 & -1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & 1 & 0
\end{bmatrix}. 
\]

Following the arguments in [7, Section V.A], it is not difficult to show that the “cocoercivity gain” for the dynamic block \( H_{2i-1} \) is \( \gamma_{2i-1} = \eta_i \), and the gain for the static block \( H_{2i} \) is \( \gamma_{2i} = \frac{1}{k_i} \) where \( k_i \) is as defined in (6). The synchronization test in [7] consists in checking the existence of a diagonal matrix \( P > 0 \) such that:

\[
P(\Sigma - \Gamma) + (\Sigma - \Gamma)^T P < 0,
\]

where \( \Sigma \) is the internal coupling matrix and \( \Gamma \) is a diagonal matrix that consists of the cocoercivity gains of the blocks augmented with the eigenvalues \( \lambda_2 \) calculated from the Laplacian matrices for external coupling. Since there is no coupling for the static blocks, in our application \( \Gamma \) has the form:

\[
\Gamma = \text{diag} \left\{ \eta_1 + \frac{\lambda_2^1}{k_2}, \ \eta_2 + \frac{\lambda_2^2}{k_2}, \ \cdots, \ \eta_n + \frac{\lambda_2^n}{k_2} \right\}.
\]

Because \( \Sigma - \Gamma \) with (26)-(28) has a cyclic structure with an odd number \( n \) of negative entries in the diagonals, the diagonal stability test of [5] is applicable and implies that a diagonal \( P > 0 \) satisfying (27) exists if and only if (19) holds. We thus conclude (20) from Theorem 1 and Corollary 1 in [7].

4 Nonlinear Analysis of a Class of Cross-Coupled Oscillators

4.1 System Model

Cross-coupled oscillators are based on the idea of activating a passive LC tank resonator through a differential negative oscillator. Reference [11] established a fourth-order nonlinear model for a classical cross-coupled sinusoidal oscillator. The modeling procedure is illustrated in Figure 3. In Figure 3(a), each transistor with its resistive load forms one common-source amplifier and two such inverting amplifiers are connected via cross-coupling techniques. Figure 3(b) shows an inverter-based equivalent of the circuit. By looking at the input characteristic
Figure 3: Cross-coupled sinusoidal oscillator and its equivalent: (a) Cross-coupled sinusoidal oscillator. (b) Inverse-based equivalent of the oscillator. (c) Voltage-controlled nonlinear resistor consisting of two inverters. (d) Fourth-order model.

between nodes m and n, we get Figure 3(c) in which two connected inverting amplifiers form a monotone cubic-type differential negative resistor characterized by [15]

$$I_n = g_0 V_n \left( \frac{V_n}{V_0} \right)^2 - 1,$$

(29)

where $I_n$ is the current flowing into the resistor, $V_n$ is the voltage across the resistor, $V_0$ is the zero crossing voltage and $g_0$ is the slope at the origin. The fourth-order nonlinear model in Figure 3(d) is described by the following equations:

$$
\begin{align*}
L_1 I_{L_1} &= V_{C_1} - r_{L_1} I_{L_1} \\
L_2 I_{L_2} &= V_{C_2} - r_{L_2} I_{L_2} \\
C_1 \dot{V}_{C_1} &= -I_n - I_{L_1} \\
C_2 \dot{V}_{C_2} &= I_n - I_{L_2}
\end{align*}
$$

(30)

where $I_n$ is given by (29) with $V_n = V_{C_1} - V_{C_2}$. For a symmetric design we choose $L_1 = L_2 = L$, $C_1 = C_2 = C$, $r_{L_1} = r_{L_2} = r_L$.

Introducing the dimensionless variables $x(t) = V_{C_1}(t)/V_0$, $y(t) = V_{C_2}(t)/V_0$, $z(t) = r_{L_1} I_{L_1}(t)/V_0$, $w(t) = r_{L_2} I_{L_2}(t)/V_0$, $\tau = t/\sqrt{L/C}$, and letting $q = \sqrt{L/C}/r_L$, $A_d = g_0 r_L$, we rewrite (30) as:

$$
\begin{pmatrix}
\dot{x} \\
\dot{y} \\
\dot{z} \\
\dot{w}
\end{pmatrix} = \begin{pmatrix}
0 & 0 & -q & 0 \\
0 & 0 & 0 & -q \\
1/q & 0 & -1/q & 0 \\
0 & 1/q & 0 & -1/q
\end{pmatrix}\begin{pmatrix}
x \\
y \\
z \\
w
\end{pmatrix} + \begin{pmatrix}
q \\
-q \\
0 \\
0
\end{pmatrix}u
$$

(31)

$$
Y = \begin{pmatrix}
1 & -1 & 0 & 0
\end{pmatrix}\begin{pmatrix}
x \\
y \\
z \\
w
\end{pmatrix}^T
$$
where the input $u = -\phi_k(Y)$ depends on the static nonlinearity
\[
\phi_k(Y) = -kY + kY^3. \tag{32}
\]

The parameter $k = A_d > 0$ controls the negative slope at the origin of $\phi_k(\cdot)$.

Note that the linear block (31) can be decomposed into observable and unobservable subsystems, denoted by $G_O$ and $G_{\bar{O}}$, respectively. Using the change of variables $(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{w})' = (x, z, x - y, z - w)'$, we obtain the observable subsystem $G_O$:
\[
\begin{align*}
\begin{pmatrix}
\dot{\tilde{z}} \\
\dot{\tilde{w}}
\end{pmatrix}
&= 
\begin{pmatrix}
0 & -q \\
1/q & -1/q
\end{pmatrix}
\begin{pmatrix}
\tilde{z} \\
\tilde{w}
\end{pmatrix}
+ 
\begin{pmatrix}
2q \\
0
\end{pmatrix}
u \\
Y &= 
\begin{pmatrix}
1 & 0
\end{pmatrix}
\begin{pmatrix}
\tilde{z} \\
\tilde{w}
\end{pmatrix}
\tag{33}
\end{align*}
\]

and the unobservable subsystem $G_{\bar{O}}$:
\[
\begin{align*}
\begin{pmatrix}
\dot{\tilde{x}} \\
\dot{\tilde{y}}
\end{pmatrix}
&= 
\begin{pmatrix}
0 & -q \\
1/q & -1/q
\end{pmatrix}
\begin{pmatrix}
\tilde{x} \\
\tilde{y}
\end{pmatrix}
+ 
\begin{pmatrix}
q \\
0
\end{pmatrix}
u.
\tag{34}
\end{align*}
\]

With this decomposition, we represent the feedback interconnection (31)-(32) as in Figure 4, where $G_{\bar{O}}$ is the stable linear system (34) driven by $u = -\phi_k(Y)$. This cascade structure implies that the stability properties are determined by the feedback loop of $G_O$ and $\phi_k(\cdot)$. The structure and, as we shall see, the behavior of (32)-(33) differs from standard negative-resistance oscillators and calls for the nonlinear analysis presented next.

![Figure 4: Block diagram of the system (31)-(32) decomposed into its observable and unobservable components.](image)

4.2 **Boundedness of Trajectories**

To prove boundedness of trajectories we use the following lemma from [9], which establishes a boundedness mechanism for systems consisting of a linear block $G(s)$ in negative feedback with a static nonlinearity $\phi(\cdot)$:

**Lemma 2** Suppose $G(s)$ is relative degree one and minimum phase, and its high frequency gain $k_p$ is positive. If $\phi(y)$ is stiffening, i.e. for every $m > 0$, there exists $l > 0$ such that
\[
|y| \geq l \Rightarrow \frac{\phi(y)}{y} \geq m,
\tag{35}
\]

then the solutions of the feedback system are ultimately bounded.
In the feedback system in Figure 4, we note from (33) that
\[ G_O(s) = \frac{2q(s + 1/q)}{s^2 + s/q + 1}, \]  
which is relative degree one, minimum phase, and has a positive high frequency gain. Since \( \phi_k(\cdot) \) satisfies the stiffening property, by Lemma 2, the solutions of the feedback system are ultimately bounded.

### 4.3 Hopf Bifurcation and Global Oscillations

In this part we first analyze the nonlinear dynamics of a general class of Lurie systems consisting of a linear block \( G \) and a static stiffening nonlinearity
\[ \phi_k(\cdot) = -ky + \phi(\cdot), \]  
as shown in Figure 5. Then we apply the result to the cross-coupled oscillator in a corollary. We denote by \( G_k \) the positive feedback interconnection of \( G \) and the feedback gain \( k \). Then the feedback system of \( G \) and \( \phi_k(\cdot) \) is equivalently described as the interconnection of \( G_k \) and \( \phi(\cdot) \).

![Figure 5: Equivalent representation of the feedback interconnection of \( G \) and \( \phi_k(\cdot) \).](image_url)

The next theorem follows from a combination of Theorem 3 in [10] and Lemma 4 given in the appendix:

**Theorem 3** Consider the feedback system in Figure 5. Assume that the linear block \( G \) is observable with the transfer function:
\[ G(s) = \frac{d(s + c)}{s^2 + as + b}, \]  
where \( a \geq c > 0, b \geq 0, d > 0, ac \neq b \). The nonlinearity \( \phi(\cdot) \) in (37) is a smooth sector nonlinearity in \( (0, +\infty) \) satisfying \( \phi'(0) = \phi''(0) = 0, \phi'''(0) = K > 0 \). Let \( k^* \geq 0 \) be the minimal value for which \( G_k(s) \) has a pole on the imaginary axis and \( k \geq k^* \) denote a value near the bifurcation, i.e. \( k \in (k^*, \bar{k}] \) for some \( \bar{k} > k^* \).

**Case (1):** If \( G_k(s) \) has a unique pole on the imaginary axis, then the bifurcation is a supercritical pitchfork bifurcation. For \( k \geq k^* \), the origin is a saddle node and its stable manifold \( E_S(0) \) separates the space into two sets, each of which is the basin of attraction of a stable equilibrium.

**Case (2):** If \( G_k(s) \) has a unique pair of conjugated poles on the imaginary axis, then the bifurcation is a supercritical Hopf bifurcation. For \( k \geq k^* \), the origin is unstable and the system has a unique limit cycle which is globally asymptotically stable in \( \mathbb{R}^2 \setminus \{0\} \).
First consider \( a \geq c > 0 \). It follows from Lemma 4 that \( G(s) \) is positive real so the linear block is passive. Because \( G(s) \) is relative degree one, minimum phase and has a positive high frequency gain, by Lemma 2, the trajectories of the feedback system of \( G \) and \( \phi_k(\cdot) \) are ultimately bounded. The closed-loop transfer function of the positive feedback interconnection of \( G \) and \( k \) is:

\[
G_k(s) = \frac{G(s)}{1 - kG(s)} = \frac{d(s + c)}{s^2 + (a - kd)s + (b - kcd)},
\]

so

\[
k^* = \min\left(\frac{a}{d}, \frac{b}{cd}\right).
\]

If \( ac > b \), then \( k^* = \frac{b}{cd} \)

\[
G_k^*(s) = \frac{d(s + c)}{s^2 + (a - b/c)s},
\]

which has a unique pole on the imaginary axis. By Lemma 4, \( G_k^*(s) \) is either positive real or could be made positive real by adding a zero at some \( \alpha > 0 \). Hence, an application of Theorem in [10] shows that the bifurcation is a supercritical pitchfork bifurcation and that all solutions, except for those starting on \( E_S(0) \), converge to one of the two stable equilibria.

If \( ac < b \), then \( k^* = \frac{a}{d} \)

\[
G_k^*(s) = \frac{d(s + c)}{s^2 + (b - ac)},
\]

which has a unique pair of conjugate poles on the imaginary axis. By Lemma 4 and Theorem 3 in [10], \( G_k^*(s) \) can be made positive real by adding a positive zero. Thus, the bifurcation is a supercritical Hopf bifurcation and a unique limit cycle that is almost globally asymptotically stable exists. Now we use Theorem 3 to analyze the feedback loop of \( G_O \) and \( \phi_k(\cdot) \).

**Corollary 1** Consider the feedback system (32)-(33) represented as in Figure 4. When \( q > 1 \), a supercritical Hopf bifurcation occurs at \( k^* = \frac{1}{2q} \). For \( k \geq k^* \), the system possesses a unique limit cycle which is globally asymptotically stable in \( \mathbb{R}^2 \backslash \{0\} \).

**Proof** The closed loop transfer function of \( G_O \) and \( k \) is

\[
G_k(s) = \frac{G_O(s)}{1 - kG_O(s)} = \frac{2q(s + 1/q)}{s^2 + (1/q - 2kq)s + (1 - 2k)}.
\]

Because \( q > 1 \), \( k^* = 1/(2q^2) \). At the bifurcation point,

\[
G_k^*(s) = \frac{2q(s + 1/q)}{s^2 + (1 - 1/q^2)}.
\]

which has a unique pair of conjugated poles on the imaginary axis. By Case (2) in Theorem 3, the bifurcation is a supercritical Hopf bifurcation and there exists a unique limit cycle that is globally asymptotically stable in \( \mathbb{R}^2 \backslash \{0\} \) for \( k \geq k^* \).

As stated at the end of Section II, the unobservable subsystem \( G_{\bar{O}} \) is a stable system driven by \( u = -\phi_k(Y) \). When the supercritical Hopf bifurcation occurs, \( G_O \) has a global asymptotically stable limit cycle in \( \mathbb{R}^2 \backslash \{0\} \) and the trajectories of \( G_{\bar{O}} \) remains bounded. Consequently, the cross-coupled oscillator exhibits a global oscillation.
4.4 Analysis for a Wider Range of Parameters and Simulations

The case we discussed above, where $q > 1$ and $k > k^* = \frac{1}{2q}$, is the range of practical interest. For completeness we further investigate the nonlinear behavior of the feedback system as the parameters $k$ and $q$ vary in a wider range.

The equilibrium points of the system are determined by the equation:

$$q\ddot{z}(2k - 1 - 2k\dot{z}^2) = 0,$$

where $k, q > 0$. First assume that $q > 1$. When $k \leq \frac{1}{2}$, there is only one equilibrium point at the origin: $e_1 = (0,0)^T$. Evaluating the Jacobian matrix at $e_1$, we get

$$A_1 = \begin{pmatrix} 2kq & -q \\ 1/q & -1/q \end{pmatrix}$$

(46)

for which the real parts of the eigenvalues are $2kq^2 - 1$. When $0 < k < \frac{1}{2q}$, the origin is a stable focus. It follows from Popov criterion and Lemma 3 in the appendix that the system is globally asymptotically stable. As $k$ increases, a bifurcation occurs at $k^* = \frac{1}{2q}$. When $\frac{1}{2q} < k < \frac{1}{2}$, the origin is either an unstable node or an unstable focus. Because the trajectories are bounded it follows from the Poincaré-Bendixson Theorem that there must be a limit cycle encircling the origin. When $k > \frac{1}{2}$, the eigenvalues of $A_1$ have opposite signs, so $e_1$ becomes a saddle point. Meanwhile, two new equilibrium points arise: $e_{2,3} = (\pm \sqrt{1-1/2k}, \pm \sqrt{1-1/2k})^T$, which have the same Jacobian matrices:

$$A_2 = A_3 = \begin{pmatrix} 3q - 4kq & -q \\ 1/q & -1/q \end{pmatrix}.$$

(47)

The real parts of the eigenvalues are $\frac{1}{2q}(-4kq^2 + 3q^2 - 1)$. Thus, $e_2$ and $e_3$ are unstable nodes or foci when $\frac{1}{2} < k \leq \frac{3q^2-1}{4q^2}$ and become stable when $k > \frac{3q^2-1}{4q^2}$. Note that when $q \gg 1$, a time-scale separation exists whereby $\dot{z}$ settles faster than $\ddot{w}$. All the trajectories starting in the unstable manifold of the origin approach the nullcline $\dot{z} = 0$. Since $e_2$ and $e_3$ are stable, the trajectories will settle down at either $e_2$ or $e_3$ after reaching the nullcline. Thus, the system is globally bistable in $\mathbb{R}^2\setminus E_S(0)$.

Next we turn to the case $q < 1$. When $0 < k \leq \frac{1}{2}$, there is only one equilibrium point at the origin. Since $\frac{1}{2} < \frac{1}{2q}$, the eigenvalues of $A_1$ always have negative real parts. Thus, the equilibrium point $e_1$ is a stable node or focus. As discussed above, the system is globally asymptotically stable. When $k > \frac{1}{2}$, $e_1$ becomes a saddle point and two new equilibrium points $e_2$ and $e_3$ emerge. It follows from (43) that $k^* = \frac{1}{2} < \frac{1}{2q}$ and

$$G_k^+(s) = \frac{2q(s + 1/q)}{s^2 + (1/q - 2k)s},$$

(48)

which has a unique pole on the imaginary axis. By Case (1) in Theorem 3, the bifurcation is a supercritical pitchfork bifurcation, and for $k \geq k^* = \frac{1}{2}$, the system is globally bistable in $\mathbb{R}^2\setminus E_S(0)$.

We simulate the cross-coupled oscillator in LTspice. We first select $q = 2$. When $k = 0.1$, $e_1$ is a globally asymptotically stable focus, as shown in Figure 6(a). Figure 6(c) shows that when
Figure 6: Phase portraits when $q > 1$. 
$k$ is slightly larger than $k^* = 0.125$, a limit cycle which is almost globally asymptotically stable arises. As $k$ keeps increasing, $e_1$ turns into a saddle point and two unstable equilibria $e_2$ and $e_3$ emerge. All the equilibria are encircled by a limit cycle, as shown in 6(d). When $k = 1.2$, the limit cycle disappears. Trajectories starting in the unstable manifold of $e_1$ settle down at either $e_2$ or $e_3$.

Then we choose $q = 0.5$. When $k = 0.3$, the system has only one equilibrium $e_1$, which is globally asymptotically stable, as shown in Figure 7(a). When $k = 3$, $e_1$ becomes a saddle point and two new stable equilibria $e_2$ and $e_3$ emerge. Almost global bistability is shown in Figure 7(b). In Table 1 we summarize the nonlinear behavior of the feedback system for varying values of $k$ and $q$.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$q$</th>
<th>$e_1$</th>
<th>$e_2$ and $e_3$</th>
<th>System</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>stable focus</td>
<td>saddle point, stable nodes</td>
<td>globally bistable in $\mathbb{R}^2 \setminus E_S(0)$</td>
</tr>
<tr>
<td>$\frac{3q^2 - 1}{4q^2}$</td>
<td>$\frac{1}{2}$</td>
<td>saddle point</td>
<td>saddle point, stable nodes</td>
<td>globally bistable in $\mathbb{R}^2 \setminus E_S(0)$</td>
</tr>
<tr>
<td>$\frac{3q^2 - 1}{4q^2}$</td>
<td>$\frac{1}{2}$</td>
<td>saddle point</td>
<td>saddle point, stable nodes</td>
<td>globally bistable in $\mathbb{R}^2 \setminus E_S(0)$</td>
</tr>
</tbody>
</table>

Table 1: Nonlinear behavior of the feedback system of $G_O$ and $\phi_k(\cdot)$
5 Synchronization of Cross-coupled Oscillator Circuits

5.1 Synchronization Mechanism

Consider an interconnection of \( N \) identical, SISO oscillators, each of which is characterized by

\[
\begin{align*}
\dot{x}_i &= Ax_i - B\phi_k(y_i) + Bu_i, \\
y_i &= Cx_i, \\
\end{align*}
\]

where \( u_i \) and \( y_i \) are the external input and output of oscillator \( i \), respectively. Let \( U = (u_1, \ldots, u_N)^T \) be the input vector and \( Y = (y_1, \ldots, y_N)^T \) be the output vector, then \( U = -\Gamma Y \) indicates a linear input-output coupling through \( \Gamma \). Let \( \Gamma_s \) denote the symmetric part of \( \Gamma \) and \( \lambda_2(\Gamma_s) \) denote the second smallest eigenvalue of \( \Gamma_s \). Let \( G_k(s) \) be the positive feedback interconnection of \( G(s) \) and \( k \), where \( G(s) = C(sI - A)^{-1}B \) is the transfer function of each oscillator. We denote by \( k_{\text{passive}}^* \) the critical value of \( k \) above which \( G_k \) loses passivity.

**Lemma 3** [16] Consider the interconnection described above. Assume that \((A, C)\) is observable, \( \phi(\cdot) \) is monotone increasing and each isolated oscillator \((u_i \equiv 0)\) possesses a globally asymptotically stable limit cycle in \( \mathbb{R}^p \setminus E_s(0) \). If the interconnection matrix \( \Gamma \) is a real, positive semidefinite matrix of rank \( N-1 \) such that \( \Gamma_1 = \Gamma^T 1_N = 0 \), where \( 1_N \) denotes the \( N \)-vector of ones, then for \( \lambda_2(\Gamma_s) > k - k_{\text{passive}}^* > 0 \), the interconnection has a limit cycle which attracts all solutions except those belonging to the stable manifold of the origin, and all the oscillations of the network exponentially synchronize.

5.2 Simulations for Synchronization of Interconnected Cross-coupled Oscillators

We apply Lemma 3 to the synchronization of two interconnected identical cross-coupled oscillators described by (31) and (32). The oscillator in Figure 3(a) is designed to oscillate at 20MHz with \( L = 1\mu H, C = 64pF \) and \( r_L = 25\Omega \). These parameters lead to \( q = 5 \), which implies that the condition to start oscillation is \( k = 25g_0 > \frac{1}{2q^2} = 0.02 \), i.e. \( g_0 > 0.8mA/V \) [11].

![Figure 8: Interconnection of two identical cross-coupled oscillators.](image-url)

From former analysis, when \( q = 5, k = 0.025 \), the oscillator possesses a globally asymptotically stable limit cycle in \( \mathbb{R}^2 \setminus \{0\} \) when the external input is zero. We choose the interconnection matrix \( \Gamma_1 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \), which is rank 1 and has two eigenvalues \( \lambda_1(\Gamma_1) = 0 \) and \( \lambda_2(\Gamma_1) = 2 \).
Figure 9: Synchronization of two identical oscillators through linear, symmetric input-output couplings. (a) coupling through $\Gamma_1$. (b) coupling through $\Gamma_2$.

Figure 10: Synchronization of two unmatched cross-coupled oscillators.
For $G_D(s)$ in (36), $k_{\text{passive}}^* = 0$. By Lemma 3, for $0 < k < 2$, the interconnection has a limit cycle which is globally asymptotically stable in $\mathbb{R}^2 \setminus \{0\}$, and all the oscillations exponentially synchronize. In Figure 8, two identical cross-coupled oscillators are interconnected through the matrix $\Gamma_1$. Simulation of output voltages at $C_1$ and $C_3$ illustrated in Figure 9(a) exhibits a synchronous oscillation. Choosing $\Gamma_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ yields an anti-phase oscillation, as shown in Figure 9(b).

The results above assume that the oscillators are identical. In real circuits, different processing methods, inductive coupling and layout mismatch can cause parameter mismatches between the oscillators, which should be taken into account in the design of synchronization scheme. Here we change the capacitances, inductances and resistances of the second oscillator by 5% - 10% of their original values. The corresponding mismatches in amplitude and phase, as shown in Figure 10, are both within 6%.

Other synchronization schemes can also be used. For example, reference [17] developed a mismatch compensation circuit to alleviate phase errors in LC quadrature voltage-controlled oscillators with a common-mode model. The synchronization frequency and amplitude will be different from those of each oscillator in this case, where one oscillator becomes a master and the others become slaves.

### 6 Conclusions

In this paper, we presented nonlinear analysis for an n-stage ring oscillator and a particular class of cross-coupled oscillators, as well as a synchronization scheme for the interconnection of these oscillators. By representing the n-stage ring oscillator as a cyclic system, we derived a sufficient condition for global asymptotic stability of the origin. For the cross-coupled oscillators, we first presented conditions for the occurrence of supercritical pitchfork or Hopf bifurcations. Then we discussed the nonlinear properties as the parameters vary in a larger range. The robustness of the synchronization scheme for interconnections of nonidentical oscillators is an important issue that deserves further research.

### Appendix

**Lemma 4** Consider the transfer function:

$$G(s) = \frac{s + c}{s^2 + as + b}, \quad (50)$$

where $a, b \geq 0$, $a + b \neq 0$, $c > 0$. If $a \geq c$, then $G(s)$ is positive real. If $a < c$, then there exists a positive constant $\alpha$ such that $\tilde{G}(s) = (s + \alpha)G(s)$ is positive real.

**Proof** A transfer function $G(s)$ is positive real [13] if:

1) All the poles of $G(s)$ have nonpositive real parts. For the poles on the imaginary axis, they should be simple with nonnegative residue;

2) The real part of $G(j\omega)$ is nonnegative.
First consider the case $a \geq c$. If $a^2 \leq 4b$, then $G(s)$ has two conjugate complex poles at $(-a \pm j\sqrt{4b-a^2})/2$, with negative real parts $-a/2$. When $a = 0$, the two poles $\pm j\sqrt{b}$ are on the imaginary axis, with the residue of $G(s)$ equal to 1. If $a^2 > 4b$, then $G(s)$ has two real poles at $(-a \pm \sqrt{a^2-4b})/2$. When $b = 0$, $G(s)$ has only one pole on the imaginary axis, with the residue $c/a > 0$. When $b$ is nonzero, $G(s)$ has two poles in the left half-plane. It follows from (50) that

$$G(j\omega) = \frac{c + j\omega}{b - \omega^2 + j\omega} = \frac{(a-c)\omega^2 + bc + j\omega(b-a-\omega^2)}{(b-\omega^2)^2 + a^2\omega^2},$$  

(51)

so the real part of $G(j\omega)$ is

$$\text{Re}\{G(j\omega)\} = \frac{(a-c)\omega^2 + bc}{(b-\omega^2)^2 + a^2\omega^2},$$  

(52)

which is nonnegative if $a \geq c$. Therefore, when $a \geq c$, $G(s)$ is positive real.

Next we turn to the case $a < c$. Let

$$\tilde{G}(s) = (s+a)G(s) = \frac{(s+c)(s+a)}{s^2 + as + b}.$$  

(53)

If $a^2 \leq 4b$, then $\tilde{G}(s)$ has two conjugate complex poles at $(-a \pm j\sqrt{4b-a^2})/2$, with negative real parts $-a/2$. When $a = 0$, the two poles $\pm j\sqrt{b}$ are on the imaginary axis, with the residue of $\tilde{G}(s) = c + a > 0$. If $a^2 > 4b$, then $G(s)$ has two real poles at $(-a \pm \sqrt{a^2-4b})/2$, which are either in the left half-plane or merge into one pole on the imaginary axis, with the residue $c/a > 0$. Since

$$\text{Re}\{\tilde{G}(j\omega)\} = \frac{\omega^4 + (a\alpha + ac - ca - b)\omega^2 + bca}{(b-\omega^2)^2 + a^2\omega^2},$$  

(54)

the sign of the real part of $\tilde{G}(j\omega)$ is determined by the polynomial $p(\omega) := \omega^4 + (a\alpha - ca + ac - b)\omega^2 + bca$. To make it nonnegative, we choose $\alpha$ such that the roots of the equation

$$\omega^4 + (a\alpha - ca + ac - b)\omega^2 + bca = 0$$  

(55)

are complex, that is,

$$(a\alpha - ca + ac - b)^2 \leq 4bca.$$  

(56)

Let $k_1 = a(c^2 + b - ac)/(a - c)^2$, $k_2 = bc/(a - c)^2$, then for

$$k_1 + k_2 - 2\sqrt{k_1k_2} \leq \alpha \leq k_1 + k_2 + 2\sqrt{k_1k_2},$$

$p(\omega)$ is nonnegative and, thus, $\tilde{G}(s)$ is positive real. Note that $a < c$, so $k_1k_2 > 0$ and $k_1 + k_2 > 2\sqrt{k_1k_2}$, which ensures the positiveness of $\alpha$.

**References**


